QUANTUM LOOP GROUPS

and

CRITICAL K-THEORY

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1) REMINDERS ON NAKAJIMA'S CONSTRUCTION

2) CRITICAL K-THEORY

3) CK-THEORY and QUANTUM LOOP GROUPS

4) CK-THEORY and REPRESENTATIONS of QLGs and MOTIVATIONS

1) REMINDERS on NAKAJIMA's CONSTRUCTION

$Q = (I, Q_1)$ Dynkin quiver, $R = \mathbb{C}[q, q^{-1}]$, $F = \mathbb{C}(q)$

$Q = \circ \xrightarrow{\alpha} \circ$

$$\overline{Q}_f = \begin{tikzcd} \square \arrow[d, shift left, "a_1^*"] & \square \arrow[d, shift left, "a_2^*"] \\ \circ \arrow[u, shift left, "a_1"] \arrow[r, shift left, "\alpha"] & \circ \arrow[u, shift left, "a_2"] \arrow[l, shift left, "\alpha^*"] \end{tikzcd}$$ = the double of the framed quiver

V, W = I-graded $\dim = v, w \in \mathbb{N}I$

$G_V = \prod_{i \in I} GL(V_i)$

$\mathfrak{M}(W) = \bigsqcup_v \mathfrak{M}(v, W)$ = Nakajima quiver variety

= moduli space of stable reps of $\Pi_{\overline{Q}_f} = \dfrac{\mathbb{C}\overline{Q}_f}{([\alpha, \alpha^*] + a^* a)}$

(framed) preprojective alg

= iso classes of reps of $\overline{Q}_f$

$$x = \begin{tikzcd} W_1 \arrow[d, shift left, "x_{a_1^*}"] & W_2 \arrow[d, shift left, "x_{a_2^*}"] \\ V_1 \arrow[u, shift left, "x_{a_1}"] \arrow[r, shift left, "x_\alpha"] & V_2 \arrow[u, shift left, "x_{a_2}"] \arrow[l, shift left, "x_{\alpha^*}"] \end{tikzcd}$$

$+ \ [x_\alpha, x_{\alpha^*}] + x_{a^*} x_a = 0$

$+$ stability condition

$\mathfrak{M}(V)$ smooth, quasi projective variety

$\mathfrak{M}_0(W)$ = categorical (= GIT) quotient of $\{$reps of $\pi_{\bar{Q}_f}\}$

$\pi : \mathfrak{M}(W) \longrightarrow \mathfrak{M}_0(W)$ projective map

$\mathcal{L}(W) = \pi^{-1}(0)$ = moduli space of nilpotent stable reps of $\pi_{\bar{Q}_f}$

$G_W \times \mathbb{C}^\times \curvearrowright \mathfrak{M}(W), \mathfrak{M}_0(W), \mathcal{L}(W)$

$Z(W) = \mathfrak{M}(W) \underset{\mathfrak{M}_0(W)}{\times} \mathfrak{M}(W)$ = Steinberg variety

$D^b \mathrm{Coh}_{G_W \times \mathbb{C}^\times}(\mathfrak{M}^2(W))_{Z(W)}$ = derived category of complexes $\mathcal{E}$ of $G_W \times \mathbb{C}^\times$-equivariant coherent sheaves on $\mathfrak{M}^2(W)$ with $\mathrm{supp}(H^\bullet(\mathcal{E})) \subset Z(W)$ (set theoretic support)

monoidal structure on $D^b \mathrm{Coh}_{G_W \times \mathbb{C}^\times}(\mathfrak{M}^2(W))_{Z(W)}$:

$$\begin{tikzcd}[column sep=small] & \mathcal{M}(W)\times\mathcal{M}(W)\times\mathcal{M}(W) \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ \mathcal{M}(W)\times\mathcal{M}(W) & \mathcal{M}(W)\times\mathcal{M}(W) & \mathcal{M}(W)\times\mathcal{M}(W) \end{tikzcd}$$

$$\mathcal{E} * \mathcal{F} = (R\pi_{13})_*\left((L\pi_{12})^*(\mathcal{E}) \overset{L}{\otimes} (L\pi_{23})^*(\mathcal{F}) \right)$$

COR:

$$\boxed{\text{algebra structure on } K^{G_W\times\mathbb{C}^*}(Z(W))}$$

$$\boxed{\text{action on } K^{G_W\times\mathbb{C}^*}(\mathcal{M}(W)),\ K^{G_W\times\mathbb{C}^*}(\mathcal{L}(W))}$$

THM:

$$\boxed{U_F(Lg) \xrightarrow[\text{homomorphism}]{\text{algebra}} K^{G_W\times\mathbb{C}^*}(Z(W)) \otimes_R F}$$

Q:

What about
- NON SYMMETRIC AND SHIFTED CASE?
- GEOMETRICAL REALIZATION OF SIMPLES?

Use critical K-theory to get both

2) CRITICAL K-THEORY

[Orlov, Efimov-Positselski, Hirano, ...]

$$\begin{cases} X = \text{smooth quasi proj variety } / \mathbb{C}, \\ G \text{ affine algebraic group } \curvearrowright X \\ \phi : X \to \mathbb{C},\ G\text{-invariant regular function} \\ \mathrm{crit}(\phi),\ Z \subseteq \phi^{-1}(0),\ Z \text{ closed} \end{cases}$$

DEF

- $$DCoh_G(X,\phi)_Z := D^b Coh_G(\phi^{-1}(0))_Z \Big/ Perf_G(\phi^{-1}(0))_Z$$

equivariant category of singularities of (X,ϕ)

- $$\boxed{K_G(X,\phi)_Z := K_0\left(DCoh_G(X,\phi)_Z\right)} = \text{critical K-theory}$$

PROPERTIES

① FUNCTORIALITY

② $K_G(X,\phi)_Z$ "supported" on $\mathrm{crit}(\phi)$

i.e. $[\mathcal{E}] \in K_G(X,\phi)_Z \Rightarrow \mathrm{supp}(H^\bullet(\mathcal{E})) \subset Z \cap \mathrm{crit}(\phi)$

$\left(U = \phi^{-1}(0) \setminus \mathrm{crit}(\phi) \text{ smooth} \Rightarrow D^b(\mathrm{Coh}_G(U)) = \mathrm{Perf}_G(U) \right)$

③ $K_G(X,0) = K_G(X)$

$\boxed{\text{K-theoretical critical convolution algebras}}$

X = smooth quasi projective

X_0 = affine variety

$\pi : X \longrightarrow X_0$ G-equivariant proper

ϕ regular G-invariant

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & \mathbb{C} \\ & X_0 \arrow[ur, "\phi"'] & \end{tikzcd}$$

$Z := X \times_{X_0} X$ $\qquad$ $L := X \times_{X_0} \{x_0\}$ $\qquad$ $x_0 \in \phi^{-1}(0)^G$

$\phi^{(2)} := \phi \oplus (-\phi) : X^2 \longrightarrow \mathbb{C}$ $\qquad$ $Z \subseteq (\phi^{(2)})^{-1}(0)$

Here exists a monoidal structure on $DCoh_G(X^2, f^{(2)})_Z$

$R_G = K_G(pt) =$ representation ring of G

$$\boxed{\begin{aligned} K_G(X^2, \phi^{(2)})_Z &= R_G\text{-algebra} \\ &= K\text{-theoretical critical convolution algebra} \\ &\circlearrowright K_G(X, \phi),\ K_G(X, \phi)_L \end{aligned}}$$

3) CRITICAL K-THEORY AND QUANTUM LOOP GROUPS

- $C = (c_{ij})_{i,j \in I}$ a Cartan matrix

$\mathcal{O} \subseteq I \times I$ an orientation : $(i,j) \in \mathcal{O} \Rightarrow (j,i) \notin \mathcal{O}$; (i,j) or $(j,i) \in \mathcal{O} \iff c_{ij} < 0$

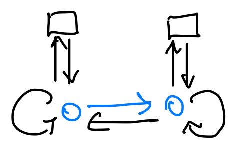
$\rightsquigarrow Q = (I, Q_1 = \{\alpha_{ij} : j \to i \mid (i,j) \in \mathcal{O}\})$

C of type $B_2 = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ $\qquad Q = \underset{1}{\circ} \xrightarrow{\alpha_{21}} \underset{2}{\circ}$

$(d_i)_{i \in I}$ symmetrizers of C $\qquad d_i c_{ij} = d_j c_{ji}$ $\qquad (d_1 = 1, d_2 = 2)$

- $\tilde{Q}$ = triple quiver of Q

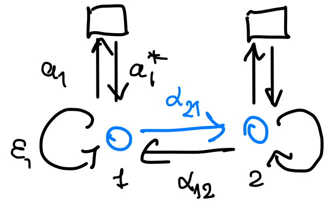
$\tilde{Q}_f$ = framed triple quiver of Q



$\tilde{\mathfrak{M}}(W) = \bigsqcup_v \tilde{\mathfrak{M}}(v,W)$ = moduli space of stable reps of $\mathbb{C}\tilde{Q}_f$

$\tilde{\mathfrak{M}}_0(W), \tilde{\mathcal{L}}(W), \tilde{\mathcal{Z}}(W) \quad \circlearrowleft G_W \times \mathbb{C}^\times$ as before

• grading on $\tilde{Q}_{f,1}$ $\begin{cases} \alpha_{ij} \mapsto d_i c_{ij} \\ \varepsilon_i \mapsto 2d_i \\ a_i, a_i^* \mapsto -d_i \end{cases}$



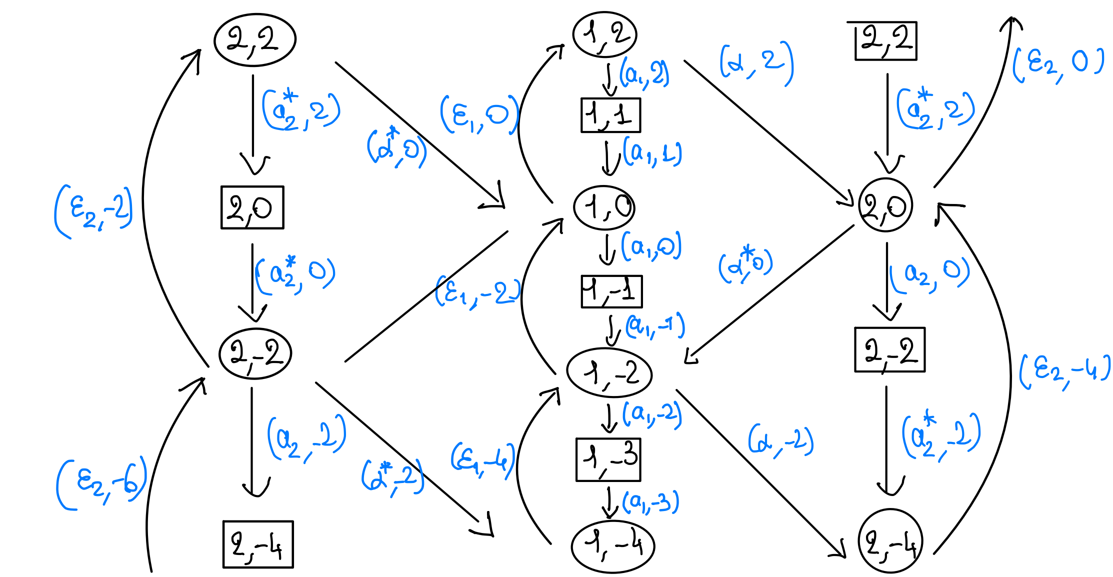
⇝ new quiver

$$\tilde{Q}^\bullet_f \begin{cases} I^\bullet = I \times \mathbb{Z} \\ \tilde{Q}_{f,1} \times \mathbb{Z} \supseteq \tilde{Q}^\bullet_{f,1} = \{ (h,k) \ : (s(h),k) \longrightarrow (t(h), k+\deg(h)) \} \end{cases}$$

EX B_2 $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ $\alpha = \alpha_{21}$, $\alpha^* = \alpha_{12}$

$\deg \alpha, \alpha^*, a_2, a_2^* = -2$ $\deg a_1, a_1^* = -1$ $\deg \varepsilon_1 = 2$ $\deg \varepsilon_2 = 4$

One of two connected components of $\tilde{Q}^\bullet_f$



$\widetilde{\mathcal{M}}^\bullet, \widetilde{\mathcal{M}}^\bullet_0, \widetilde{\mathcal{L}}^\bullet, \widetilde{\mathcal{Z}}^\bullet$ graded versions of $\widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}_0, \widetilde{\mathcal{L}}, \widetilde{\mathcal{Z}}$

RMK $\widetilde{\mathcal{M}}^\bullet = \widetilde{\mathcal{M}}^A$ for a 1-parameter subgp of $G_W \times \mathbb{C}^\times$

- **DEF** potential of Q = linear combination of cycles in Q

$\tilde{p}, \tilde{p}^\bullet$ = homogenous degree 0 potentials of $\tilde{Q}_f, \tilde{Q}^\bullet_f$

$\tilde{\phi} = \mathrm{tr}\,\tilde{p} : \widetilde{\mathcal{M}}(W) \longrightarrow \mathbb{C}$, $\quad \tilde{\phi}^\bullet = \mathrm{tr}\,\tilde{p}^\bullet : \widetilde{\mathcal{M}}^\bullet(W) \longrightarrow \mathbb{C}$

The Geiss–Leclerc–Schröer potentials

$$\boxed{\tilde{p} := \sum_{i,j \in I} o_{ij}\, \varepsilon_i^{-c_{ij}}\, \alpha_{ij}\alpha_{ji}}$$

$$\tilde{p}^\bullet := \sum_{i,j \in I} o_{ij}\, \varepsilon_{i,k-2d_i}\, \varepsilon_{i,k-4d_i} \cdots \varepsilon_{i,k+2b_{ij}}\, \alpha_{ij,k+b_{ji}}\, \alpha_{ji,k}$$

$$o_{ij} = \begin{cases} 1 & (i,j) \in \Omega \\ -1 & (j,i) \in \Omega \\ 0 & \text{else} \end{cases} \qquad b_{ij} = d_{ij} c_{ij}$$

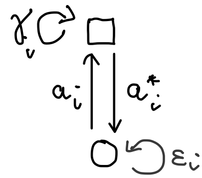
EX B_2 $\quad \tilde{p} = \varepsilon_2 \alpha_{21}\alpha_{12} - \varepsilon_1^2 \alpha_{12}\alpha_{21}$

Quantum loop gps

Fix W $\dot{I}$-graded

Fix $\gamma = (\gamma_i)_{i\in I}$, $\gamma_i \in \mathfrak{g}^{\circ}_{W_i} := \bigoplus_{k,l} \mathrm{Hom}_{\mathbb{C}}(W_{i,k}, W_{i,l})$

γ_i nilpotent homogeneous of deg $2d_i$



$$\boxed{\tilde{P}_\gamma := \tilde{P} + \sum_{i\in I} \varepsilon_i a_i^* a_i - \sum_{i\in I} \gamma_i a_i a_i^*}$$

$\tilde{P}_\gamma$ = a γ-deformed potential

THM (V-V) $q \in \mathbb{C}^\times$, $\mathfrak{g} = \mathfrak{g}_Q$

We have an algebra homomorphism

$$U_q(L\mathfrak{g}) \longrightarrow K\left(\tilde{\mathfrak{M}}^{\bullet 2}(W), \tilde{\phi}^{\bullet(2)}_\gamma\right)_{\mathcal{Z}^\bullet(W)}$$

RMK Let $\gamma = 0$

① C symmetric

$$\tilde{p}_0 = \varepsilon[\alpha, \alpha^*] + \varepsilon a^* a$$

$$\operatorname{crit}(\tilde{\Phi}_0) = \{ x \in \tilde{\mathfrak{M}}(w) \mid [x_\alpha, x_{\alpha^*}] + x_{a^*} x_a = 0 = [x_\varepsilon, x_\alpha] = x_\varepsilon x_{a^*} = x_a x_\varepsilon \}$$

stability $\Rightarrow x_\varepsilon = 0$

$$\Rightarrow \operatorname{crit}(\tilde{\Phi}_0) = \mathfrak{M}(W) = \text{stable reps of } \pi_{\overline{Q}_f}$$

$$K_{G_W \times \mathbb{C}^*}\left(\tilde{\mathfrak{M}}(W)^2, \tilde{\Phi}_0^{(2)}\right)_{\tilde{Z}(W)} \simeq K^{G_W \times \mathbb{C}^*}(Z(W)) \quad \text{as algebras}$$

(dimensional dimension)

② C non symmetric

$$\operatorname{crit}(\tilde{\Phi}_0) = \text{stable reps of } \pi_{\tilde{\overline{Q}}_f}$$

no more smooth

a framed version of the GLS generalized preprojective algebra $= \operatorname{Jac}(\tilde{Q}_f, \tilde{\Phi}_0)$

Shifted quantum loop gps

Finkelberg–Tsymbaliuk ('19) :

shifted q-loop group $\mathcal{U}_F^{\lambda^+,\lambda^-}(Lg)$ $\quad \lambda^+, \lambda^- \in \mathbb{Z}I$

gens : $\quad x^{\pm}_{i,m}$, $\psi^{\pm}_{i,\pm n_i}$, $h_{i,r}$ $\quad r, m, n_i \in \mathbb{Z}$ $\quad n_i \geqslant -\lambda_i^{\pm}$ $\quad r \neq 0$

rels : as in usual q-loop gp except that

$$\psi_i^{\pm}(u) = \sum_{n \geqslant -\lambda_i^{\pm}} \psi^{\pm}_{i,\pm n} u^{\mp n} = \psi_{i,\mp\lambda_i^{\pm}} u^{\pm\lambda_i^{\pm}} \exp\Big(\pm(q_i - q_i^{-1}) \sum_{r>0} h_{i,\pm r} z^{\mp r}\Big)$$

$$\psi_i^{\pm}(u) = \sum_{n \geqslant -\lambda_i^{\pm}} \psi^{\pm}_{i,\pm n} u^{\mp n} = \psi_{i,\mp\lambda_i^{\pm}} u^{\pm\lambda_i^{\pm}} \exp\Big(\pm(q_i - q_i^{-1}) \sum_{r>0} h_{i,\pm r} z^{\mp r}\Big)$$

$+ \quad \psi^{\pm}_{i,\mp\lambda_i^{\pm}}$ invertible

it depends only on $\lambda_+ + \lambda_-$

$\lambda_+ + \lambda_- = 0 \iff$ usual $q\ell$ gp

$$\tilde{p} = \sum_{i,j \in I} \sigma_{ij} \varepsilon_i^{-c_{ij}} \alpha_{ij} \alpha_{ji}$$ the GLS potential

$$\tilde{\phi}^{\circ} = \mathrm{tr}(\tilde{p}^{\bullet})$$

THM (V-V) W $\dot{I}$-graded, $w = \dim W \in \mathbb{N}I$

We have an algebra homomorphism

$$\mathcal{U}_q^{0,-w}(L\mathfrak{g}) \longrightarrow K\left(\tilde{\mathfrak{M}}^{\bullet}(W)^2, \tilde{\phi}^{\bullet}\right)_{\tilde{\mathcal{Z}}(W)}$$

4) CRITICAL K-THEORY and REPS of QLGs

$q \neq \sqrt{1} \qquad w = \sum_i w_i \delta_i \in \mathbb{N}I$

Hernandez : $\exists$ subcategory $\mathcal{O}^{0,-w}$ of $\mathcal{U}_q^{0,-w}(L\mathfrak{g})$-mod

$$\{\text{simples of } \mathcal{O}^{0,-w}\}$$

$$\updownarrow$$

$$\left\{ L(\underline{\Psi}) \,\middle|\, \underline{\Psi} = (\Psi_i(u))_{i\in I},\ \Psi_i(u) \text{ rational fct regular at } 0 \text{ of degree } -w_i \right\}$$

$\underline{\Psi} = (-w)$- dominant loop highest weight

NB: if $w=0$ (ie $U_q(L\mathfrak{g})$) then there exists f.d. simples

$$L(\underline{\Psi}) \text{ is f.d.} \iff \Psi_i(u) = q^{\deg P_i} \frac{P_i(1/qu)}{P_i(q/u)}$$

$(P_i(u))_{i\in I} =$ Drinfeld polynomials (Chari - Pressley)

THM (V-V)

1) Let $KR^{\ell}_{i,k}$ be the simple $U_q(Lg)$-module with Drinfeld polynomials $(P_j)_{j\in I}$

$$\begin{cases} P_j(u) = 1 \quad \forall j \neq i \\ P_i(u) = (1-q_i^{k-\ell+1}u)(1-q_i^{k-\ell+3}u)\cdots(1-q_i^{k+\ell-1}u) \end{cases}$$

$$\boxed{KR^{\ell}_{i,k} \simeq K(\tilde{\mathfrak{m}}^{\bullet}(W_\ell), \tilde{\phi}^{\bullet}_{\gamma})_{\tilde{\mathcal{L}}^{\bullet}(W_\ell)} \simeq K(\tilde{\mathfrak{m}}^{\circ}(W_\ell), \tilde{\phi}^{\bullet}_{\gamma})}$$

$W_\ell = I^{\bullet}$-graded of dimension

$$\delta_{i,k-(\ell-1)d_i} + \delta_{i,k-(\ell-3)d_i} + \cdots + \delta_{i,k+(\ell-1)d_i}$$

$\gamma \in g^{\bullet}_{W_\ell}$ = regular rep

2) $$\boxed{L^{-}(\delta_{i,k}) \simeq K(\tilde{\mathfrak{m}}^{\bullet}(\delta_{i,k}), \tilde{\phi}^{\bullet})_{\tilde{\mathcal{L}}(\delta_{i,k})} \simeq K(\tilde{\mathfrak{m}}^{\circ}(\delta_{i,k}), \tilde{\phi}^{\circ})}$$

as $\mathcal{U}_q^{0,-T}(Lg)$ modules

$L^{-}(\delta_{i,k})$ = prefundamental module $= L\left(\underline{Y}_j(u) = 1 + \delta_{ij}\dfrac{q^k}{u-q^k}\right)$

RMKS

1) Nakajima - Okounkov (unpublished) :

a similar result of 1) in the SYMMETRIC case

Liu (Columbia PhD's thesis '21) :

a similar result of 2) (sl_2, quasi maps)

2)

CONJ $\forall w \in \mathbb{N}\dot{I}$, the $U_q(Lg)$-modules

$$K(\tilde{\mathfrak{M}}^\bullet(W), \tilde{\Phi}^\bullet_0)_{\tilde{\mathcal{L}}^\bullet(W)} \quad \text{and} \quad K(\tilde{\mathfrak{M}}^\bullet(W), \tilde{\Phi}^\bullet_0)$$

are isomorphic to the standard and costandard modules

(symmetric case : Nakajima)

3) Motivation 1

CONJ There exists a sheaf theoretical construction of simple modules in the non symmetric case generalizing Nakajima's construction

④ $\Pi_{\tilde{Q}^\bullet}$ = GLS's generalized preprojective algebra of

= the Jacobi algebra of $(\tilde{Q}^\bullet, \tilde{P}^\bullet)$

Previous thm is based on

THM (VV) $\mathrm{Crit}(\tilde{\phi}^\bullet_\gamma) \cap \tilde{\mathcal{L}}^\bullet(\omega) = \mathrm{Gr}_{\Pi_{\tilde{Q}^\bullet}}(I_\gamma)$

where $I_\gamma \in \Pi_{\tilde{Q}^\bullet}$-mod , generic kernel, and

$\mathrm{Gr}_{\Pi_{\tilde{Q}^\bullet}}$ is the quiver grassmanian

Motivation 2: cluster algebras

Hernandez-Leclerc

$\mathfrak{g}$ simple $/\mathbb{C}$

$\mathcal{C}^- \underset{\otimes}{\subseteq} U_q(L\mathfrak{g})$-fd mod (all f.d. simple are in $\mathcal{C}^-$ up to spectral shift)

$K_0(\mathcal{C}^-)$ = cluster algebra s.t. $KR^{e}_{i,k}$ are cluster variables

$\mathcal{R}$ = {cluster variables} $\subseteq$ {cluster monomials } $\subseteq$ { simple modules }

Kashiwara-Kim-Oh-Park

Cluster theory $\Rightarrow \forall\, L \in \mathcal{R}$ of loop h.w. $w \in \mathbb{N}I^{\bullet}$

$\exists\, I_L \in \Pi_{\tilde{Q}^{\bullet}}$ -mod s.t.

$$q\text{-}ch(L) = \sum_{v \in \mathbb{N}I^{\bullet}} \chi\left(Gr_{\Pi_{\tilde{Q}^{\bullet}}}(v, I_L)\right) e^{w - cv}$$

(Derksen-Weyman-Zelevinski, Caldero-Chapoton)

$\boxed{Q:}$ $\forall\, L \in \mathcal{R}$: $\exists\, W$, $\exists\, \tilde{\Phi}^{\bullet}_L$ such that

$$L \simeq K(\tilde{\mathfrak{M}}^{\bullet}(W), \tilde{\Phi}^{\bullet}_L) \text{ as } U_q(L\mathfrak{g})\text{-modules}$$

(True for Kirillov-Reshetikhin modules)

Motivation 3: K-theoretical Hall algebras

- Q quiver

p potential

$deg : Q_1 \to \mathbb{Z}$ a grading s.t. p homogeneous of deg 0

$X(v) = \{\text{reps of dim } v \text{ of } Q\} \curvearrowleft G_v \times \mathbb{C}^\times$

induced by the grading

$$\mathcal{X}(v) = \left[X(v) / G_v \right]$$

$$\mathcal{X} = \bigsqcup_{v \in \mathbb{N}I} \mathcal{X}(v) = \text{stack of reps of } Q$$

$$\phi = \mathrm{tr}(p) : \mathcal{X} \longrightarrow \mathbb{C}$$

- Padurariu : $\exists$ monoidal structure on $D\mathrm{Coh}_{\mathbb{C}^*}(\mathcal{X}, \phi)$

$$\mathcal{X}^{(2)} = \{(x,y) \in \mathcal{X}^2 \mid x \subset y\}$$

$$\begin{tikzcd} & \mathcal{X}^{(2)} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{X} \times \mathcal{X} & & \mathcal{X} \end{tikzcd} \qquad \begin{tikzcd} & x \subset y \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ x, y/x & & y \end{tikzcd}$$

q smooth, p proper

$$\mathcal{E} * \mathcal{F} = (Rp)_* (Lq)^* (\mathcal{E} \boxtimes \mathcal{F})$$

DEF $\mathrm{KHA}(Q, p) = K_{\mathbb{C}^*}(\mathcal{X}, \phi)$

$= \boxed{\text{K-theoretical Hall algebra}}$

- $\tilde{Q}$ = triple quiver associated to Q

(1)

$\boxed{\text{THM (V-V)} \quad Q = \text{Dynkin type}, \; \mathfrak{g} = \mathfrak{g}_Q \quad U_R(L\mathfrak{g})^+ \simeq \mathrm{KHA}(\tilde{Q}, \tilde{p}) \text{ where } \tilde{p} = \varepsilon[\alpha, \alpha^*].}$

(2) there exists an explicit algebra homomorphism

$$\mathrm{KHA}(\tilde{Q}, \tilde{p}) \longrightarrow K_{G_W \times \mathbb{C}^*}\left(\tilde{\mathcal{M}}(W)^2, \tilde{\phi}^{(2)}\right)_{\tilde{Z}(W)}$$

where $\tilde{\phi} = \mathrm{tr}(\tilde{p})$ or $\mathrm{tr}(\tilde{p}_\gamma)$

(1)+(2) $\Longrightarrow$ For $\left(\tilde{Q}, \tilde{p} = \varepsilon[\alpha, \alpha^*], \tilde{\phi} = \mathrm{tr}(\tilde{p}_\gamma \text{ or } \tilde{p})\right)$ we have:

$$U = \mathrm{KHA} \otimes H \otimes \mathrm{KHA}^{op} \longrightarrow K_{G_W \times \mathbb{C}^*}\left(\tilde{\mathcal{M}}(W)^2, \tilde{\phi}^{(2)}\right)_{\tilde{Z}(W)} \otimes_R F$$

$$U = \begin{cases} U_F(L\mathfrak{g}) & \text{if } \tilde{\phi} = \mathrm{tr}(\tilde{p}_\gamma) \\ U_F^{0,-w}(L\mathfrak{g}) & \text{if } \tilde{\phi} = \mathrm{tr}(\tilde{p}) \end{cases}$$

ie. KCA = a "double" of KHA

General expectation: $\forall Q, \forall \tilde{p}$ there exists an algebra structure on $KHA \otimes H \otimes KHA^{op}$ for some H such that

$$KHA \otimes H \otimes KHA^{op} \longrightarrow K_{G_w \times \mathbb{C}^*}\left(\tilde{\mathcal{M}}(w)^2, \tilde{\phi}^{(2)}\right)_{\tilde{Z}(w)}$$

is an algebra map

idea of the proof

Want: $\boxed{K(\tilde{\mathcal{M}}^\circ(\delta_{i,k}), \tilde{\phi}^\circ) \simeq L^-(\delta_{i,k}) \text{ as } \mathcal{U}_q^{0,-1}(Lg)\text{-mod}}$

- both are loop h.w. modules of same ℓ.h w. ⇒ enough to prove

$$\forall v \in \mathbb{N}I^\circ: \quad q\text{-ch}\left(K(\tilde{\mathcal{M}}^\circ(v; \delta_{i,k}), \tilde{\phi}^\circ)\right) = q\text{-ch}(L^-(\delta_{i,k}))$$

- Hernandez-Jimbo:

$$q\text{-ch}(L^-(\delta_{i,k})) = \lim_{\ell\to\infty} q\text{-ch}(KR^{\ell}_{i,1+k-\ell})$$

- [V-V] ⇒ enough to prove: $\forall v \in \mathbb{N}I^\circ$

$$K(\tilde{\mathcal{M}}^\circ(v; \delta_{i,k}), \tilde{\phi}^\circ) \underset{\text{v.sp.}}{\simeq} \lim_{\ell\to\infty} K(\mathcal{M}^\circ(v; w^\circ(\ell)), \phi^\circ_\gamma)$$

$$w^\circ(\ell) = \delta_{i,2+k-2\ell} + \delta_{i,4+k-2\ell} + \cdots + \delta_{i,k}$$

- different spaces, different potentials

Key tool: Hirano deformed dimensional reduction

Hirano deformed dimensional reduction

X smooth $\qquad E, E^* \in \mathrm{Vect}(X)$

$\pi : E^* \longrightarrow X \qquad s \in \Gamma(X, E)$

$\sigma = (s, -) \quad : \mathrm{Tot}(E^*) \longrightarrow \mathbb{C}$

$\phi : X \longrightarrow \mathbb{C}$

$$\boxed{\sigma + \pi^*\phi \ , \ \phi|_{s^{-1}(0)} \text{ regular} \Rightarrow K(E^*, \sigma + \pi^*\phi) = K(s^{-1}(0), \phi|_{s^{-1}(0)})}$$