Microlocal analysis and application to control of waves

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Recent advances on control theory of PDE systems

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Outline

- Motivation : Observability estimates
- Pseudo-differential operators and wave front set
- Propagation of singularities
- Microlocal defect measures
- Applications to observation of waves

Setting

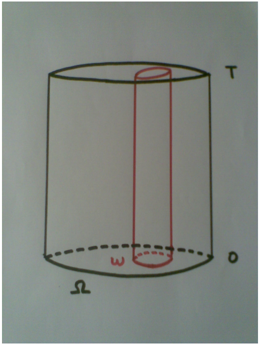
$$(W) \qquad \begin{cases} \partial_t^2 u - \Delta_x u = 0 & \text{in} \quad]0,+\infty[\times M \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in H^1 \times L^2 \end{cases}$$

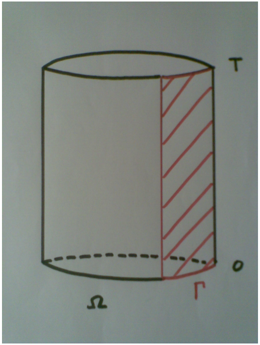
- M Riemannian manifold, connected, compact, without boundary, with dimension n.
- $M = \Omega$ open subset of $\mathbb{R}^n$, connected, bounded, with smooth boundary (homogeneous Dirichlet condition).

$$H = \mathcal{C}^0([0,+\infty[, H^1) \cap \mathcal{C}^1([0,+\infty[, L^2)$$

$$Eu(t) = ||\nabla_x u(t,.)||^2_{L^2(\Omega)} + ||\partial_t u(t,.)||^2_{L^2(\Omega)} = Eu(0)$$

$\omega \subset \Omega, \quad \Gamma \subset \partial\Omega, \quad$ and $T > 0$ (suitable)





The Goal : Observability estimate

Provide an observability estimate for the wave equation (W)

$$Eu(0) \leq c \int_0^T \int_\omega |\partial_t u(t,x)|^2 dxdt \qquad (IO)$$

$$Eu(0) \leq c \int_0^T \int_\Gamma |\partial_n u_{|\partial\Omega}(t,x)|^2 d\sigma dt \qquad (BO)$$

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Or at least

$$Eu(0) \leq c \int_0^T \int_\omega |\partial_t u(t,x)|^2 dxdt + c||(u_0,u_1)||^2_{L^2\times H^{-1}} \qquad (R-IO)$$

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Applications

→ **Exact controllability** (HUM)

Given (u_0, u_1), find a control vector f s.t the solution of

$$\begin{cases} \partial_t^2 u - \Delta_x u = \chi_\omega f \\ (u(0), \partial_t u(0)) = (u_0, u_1) \end{cases}$$

satisfies $u(T) = \partial_t u(T) = 0$.

→ **Stabilization**

$$Eu(t) \leq C\, exp^{-\gamma t} Eu(0)$$

for solutions of the damped equation

$$\partial_t^2 u - \Delta_x u + a(x)\partial_t u = 0$$

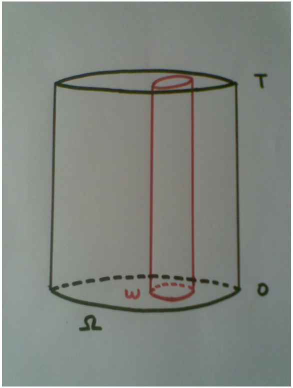
→ **Inverse problems**

Stability,....

State of the art

- 80' : Observability estimates under the Γ-condition of J.L. Lions.
  $\rightarrow$ Metric of class C^1, multiplier techniques.
- 90': Microlocal conditions and microlocal tools : Rauch -Taylor 74', Bardos, Lebeau and Rauch 92', Burq and Gérard 97'.
  The geometric control condition **(G.C.C)** : a microlocal condition, stated in the (compressed) cotangent bundle (Melrose-Sjöstrand 78').
  $\rightarrow$ Microlocal and pseudo-differential techniques : propagation of wave front sets and supports of microlocal defect measures.
  $\rightarrow$ This condition is optimal but....... a priori needs smooth metric and smooth boundary.
- 97' N. Burq : Boundary observability: C^2-metric and C^3-boundary.
- Fanelli-Zuazua 15' and D-Ervedoza 17'.
- 22' Burq-D-Le Rousseau : Observability: C^1-metric and C^2-boundary.

Back to internal observability



$$||(u_0, u_1)||^2_{H^1 \times L^2} \leq c \int_0^T \int_\omega |\partial_t u(t,x)|^2 dxdt + c||(u_0, u_1)||^2_{L^2 \times H^{-1}}$$

$\rightarrow$ Implies observability for high frequency data.

$$(\mathrm{u}_0, u_1) \in L^2 \times H^{-1} \quad \text{and} \quad u \in H^1((0,T) \times \omega)$$

$$\Downarrow$$

$$(\mathrm{u}_0, u_1) \in H^1 \times L^2$$

In other words

$$u \in L^2((0,T) \times \Omega) \text{ and } u \in H^1((0,T) \times \omega) \Rightarrow u \in H^1((0,T) \times \Omega)$$

$\rightarrow$ Propagation of the H^1- regularity

Remarks

a) This condition is necessary !

b) With $\omega_T = (0,T) \times \omega$,

$$u \in H^1_{loc}(\omega_T) \Longleftrightarrow \partial_t u \in L^2_{loc}(\omega_T) \Longleftrightarrow \nabla_x u \in L^2_{loc}(\omega_T)$$

since u is a wave.

Denote

$$E = H_0^1 \times L^2, \qquad E_{-1} = L^2 \times H^{-1},$$

and consider the following assumptions :

For every $(u_0, u_1) \in E_{-1} = L^2 \times H^{-1}$,

A 1. $\partial_t u \in L^2((0,T) \times \omega) \Longrightarrow (u_0, u_1) \in E$: propagation of the regularity .

A 2. $\partial_t u = 0$ in $(0,T) \times \omega \Longrightarrow (u_0, u_1) = 0$: unique continuation .

Theorem

a) A 1 $\Longrightarrow$ Relaxed observability

b) A 1 + A 2 $\Longrightarrow$ Observability

Proof 1 : Bardos-Lebeau-Rauch (1992)- Propagation of the WF set.

$$F = \left\{ (u_0, u_1) \in E_{-1},\ \partial_t u \in L^2((0,T) \times \omega) \right\}$$

$$\|(u_0,u_1)\|_F^2 = \int_0^T \int_\omega |\partial_t u|^2 + \|u\|^2_{L^2((0,T)\times\Omega)},\ \|(u_0,u_1)\|_G^2 = \|u_0\|^2_{H_0^1} + \|u_1\|^2_{L^2}$$

$\rightarrow$ $F = E$ + both are Banach spaces + Banach isomorphisms theorem
$\rightarrow$ Conclude by contradiction.

Proof 2 : Burq-Lebeau- ($\geq$ 1995) - Microlocal defect measures.

$\rightarrow$ Contradiction argument and propagation of mdm's.

To summarize:

We need a "tool" to propagate

a) The H^1 regularity from $(0,T)\times\omega$ to $(0,T)\times\Omega$ ....WF^1- set.

b) The H^1-compactness from $(0,T)\times\omega$ to $(0,T)\times\Omega$ ......microlocal defect measures.

Geometric Control Condition
(Rauch-Taylor 74' , Bardos-Lebeau-Rauch 92')

GCC at time T : The couple (ω, T) satisfies GCC if every geodesic issued from M at $\{t=0\}$ and travelling with speed 1, enters in ω before the time T.

The key problem

- How do the regularity/singularity of solutions of a wave equation travel ?
- How can we track singularities ? what is their path ?
- Same questions for compactness/lack of compactness.

1. Singular support of a distribution.

Let Ω be an open subset of $\mathbb{R}^n$, x_0 some point in Ω and $u \in D'(\Omega)$. The following statements are equivalent and define the singular support of the distribution u.

- $x_0 \notin \text{singsupp}u$.
- u is C^∞ in a neighborhood of x_0.
- There exists a neighborhood V_{x_0} of x_0 such that $\varphi u \in C_0^\infty(V_{x_0})$, for every $\varphi \in C_0^\infty(V_{x_0})$.
- There exists $\varphi \in C_0^\infty(\Omega)$, $\varphi \equiv 1$ near x_0 such that $\varphi u \in C_0^\infty(\Omega)$.

Remarks

1. Actually, $x_0 \notin$ singsupp$u$ iff
there exists V_{x_0} : $\forall \varphi \in C_0^\infty(V_{x_0})$, $\widehat{\varphi u}$ is rapidly decaying

$$\forall k \in \mathbb{N}, \exists C_k > 0, |\widehat{\varphi u}(\xi)| \leq C_k(1+|\xi|)^{-k}, \quad \forall \xi \in \mathbb{R}^n \tag{1.1}$$

or equivalently
there exists $\varphi \in C_0^\infty(\Omega)$, $\varphi \equiv 1$ near x_0 such that que $\widehat{\varphi u}$ is rapidly decaying.

2. Consider $u \in D'(\mathbb{R}^2), u(x_1, x_2) = 0$ if $x_1 < 0$ and 1 otherwise.

$$singsupp(u) = \Delta = \{(0, x_2),\ x_2 \in \mathbb{R}\}$$

The singular support mixes the good and bad spectral directions.

Examples.

1. In $D'(\mathbb{R})$, singsupp $H =$ singsupp $\delta_0 = \{0\}$.
2. In $D'(\mathbb{R})$,singsupp $u' =$ singsupp u.
3. For $u \in D'(\mathbb{R}^n)$ and $P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$, $a_\alpha \in C^\infty(\mathbb{R}^n)$, we have

$$\text{singsupp}(Pu) \subset \text{singsupp}(u)$$

4. For every elliptic differential operator P, with constant coefficients in $\mathbb{R}^n$, and every distribution u in $\mathbb{R}^n$, we have the equality

$$\text{singsupp}(Pu) = \text{singsupp}(u)$$

We say that P is hypoelliptic.

2. The wave front set

Definition: Conical set
A subset Γ of $\Omega \times \mathbb{R}^n \backslash \{0\}$ is conical if $(x, \xi) \in \Gamma$ and $\lambda > 0 \Rightarrow (x, \lambda\xi) \in \Gamma$.

Definition: The C^∞ wave front
Let Ω be an open subset of $\mathbb{R}^n$ and $u \in D'(\Omega)$. We say that a point $\omega_0 = (x_0, \xi_0)$ of $\Omega \times \mathbb{R}^n \backslash \{0\} = T^*\Omega \backslash \{0\}$ is not in the wave front of u and we write $\omega_0 \notin WF(u)$ iff there exists a neighborhood V of x_0, contained in Ω, a conical neighborhood W of ξ_0 in $\mathbb{R}^n \backslash \{0\}$, s.t. for every $\varphi \in C_0^\infty(V)$, one has

$$\forall k \in \mathbb{N}, \exists C_k > 0, |\widehat{\varphi u}(\xi)| \leq C_k (1 + |\xi|)^{-k}, \qquad \forall \xi \in W \tag{2.1}$$

Remarks

1. For $u \in D'(\Omega)$, $WF(u)$ is a closed conical subset of $\Omega \times \mathbb{R}^n \backslash \{0\}$.
2. A point $\omega_0 = (x_0, \xi_0)$ of $\Omega \times \mathbb{R}^n \backslash \{0\}$ is not in $WF(u)$ if locally near x_0, the distribution u has the behavior of a "C^∞ function" near the spectral direction ξ_0.
3. To analyze $WF(u)$, we first localise the distribution u near x_0, then we study the behavior of $\widehat{\varphi u}$ in a conical neighborhood of the spectral direction ξ_0 : it is a **microlocal analysis.**

Examples.

1.In $\mathbb{R}$, we have $WF(\delta_0) = WF(H) = \{0\} \times \mathbb{R}^*$.

2.We come back to the distribution u on $\mathbb{R}^2$ given by the characteristic function of the half-plan $\{(x_1, x_2), x_1 \geq 0\}$.

$$WF(u) = \{(x_1, x_2; \xi_1, \xi_2), x_1 = 0, \xi_2 = 0\}.$$

3.

$$u(x) = \int_0^{+\infty} \frac{\exp(ixt)}{(1+t^2)^2} dt, \qquad x \in \mathbb{R}$$

$u \in C^\infty(\mathbb{R} \setminus 0)$ since $x^k u \in C^{k+2}(\mathbb{R})$, by integration by parts.

$$singsupp(u) = \{0\} \text{ and } WFu = \{(0, \xi), \ \xi > 0\}$$

.

4. Fix $\alpha \in]0,1[$ and $\xi_0 \in S^{n-1}$, and set

$$u(x) = \sum_{k \geq 1} k^{-2} \psi(k^{\alpha} x) exp(ikx.\xi_0)$$

with $\psi \in C_0^{\infty}(\mathbb{R}^n)$ such that $\int \psi(x)dx = 1$, and $\widehat{\psi} \geq 0$.

$$singsuppu = \{0\} \quad \text{and} \quad WF(u) = \{(0, \lambda\xi_0), \lambda > 0\}.$$

3. Properties of the C^∞ wave front

In this section, we describe the action of differential operators on the wave front and we give the relation between the wave front and the singular support of a distribution.

Proposition 3.1

1. If $x_0 \notin \text{singsupp} u$, then for every $\xi \in \mathbb{R}^n \backslash \{0\}$, $(x_0, \xi) \notin WF(u)$.
2. $WF(u + v) \subset WF(u) \cup WF(v)$.
3. If $\varphi \in C^\infty$, then $WF(\varphi u) \subset WF(u)$.
4. $WF(\partial u / \partial x_j) \subset WF(u)$.

Theorem 3.2
For every $u \in D'(\Omega)$ and every differential operator P with C^∞ coefficients in Ω, we have the inclusion

$$WF(Pu) \subset WF(u) \tag{3.1}$$

We say that differential operators satisfy the pseudolocal property.

Theorem 3.3: Denote by π the canonical projection

$$\begin{cases} \pi : T^*(\Omega) = \Omega \times \mathbb{R}^n \backslash \{0\} \to \Omega \\ \qquad (x, \xi) \to x \end{cases}$$

Then the following identity holds true

$$\pi(WF(u)) = \text{singsupp} u \tag{3.2}$$

4.Wave front H^s

Let $s \in \mathbb{R}$, Ω open set in $\mathbb{R}^n$, $u \in D'(\Omega)$ and $\omega_0 = (x_0, \xi_0)$ a point of $\Omega \times \mathbb{R}^n \backslash \{0\}$.

Definition: We say that $\omega_0 = (x_0, \xi_0)$ is not in the wave front H^s of u and we write $\omega_0 \notin WF^s(u)$ iff there exists a neighborhood V_{x_0} of x_0, contained in Ω, a conical neighborhood W of ξ_0 in $\mathbb{R}^n \backslash \{0\}$, such that for every function $\varphi \in C_0^\infty(V_{x_0})$, we have

$$(1 + |\xi|^2)^{s/2} \widehat{\varphi u}(\xi) \in L^2(W) \qquad (4.1)$$

Remark

If $\omega \notin WF(u)$ then $\omega \notin WF^s(u)$, for every $s \in \mathbb{R}$.

2-Pseudo-Differential Operators

- The goal is to study the behavior of the wave front set of a distribution u solution of a PDE $\quad P(x,D)u = f \in C^\infty$.

- For this purpose, we define an **algebra of operators** containing the differential operators (smooth coefficients) and the "inverses" (in some sense to be precised), of the elliptic operators.

Let $P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha$ a differential operator of order m, with coefficients in $C^\infty(\mathbb{R}^n)$. For every $u \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{cases} Pu(x) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha u(x) \\ = \sum_{|\alpha| \leq m} a_\alpha(x) (2\pi)^{-n} \int e^{ix\xi} \xi^\alpha \widehat{u}(\xi) d\xi \\ = (2\pi)^{-n} \int e^{ix\xi} p(x,\xi) \widehat{u}(\xi) d\xi \end{cases}$$

where

$$p(x,\xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$$

This representation suggests that $p(x,\xi)$ can be replaced by a more general function living in a suitable class of symbols.

1. Symbols

Definition: For $m \in \mathbb{R}$, we denote $S^m = S^m(\mathbb{R}^n \times \mathbb{R}^n)$ the set of functions $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all multi-idexes α and $\beta \in \mathbb{N}^n$, there exists a constant $C_{\alpha\beta} > 0$ s.t.

$$\left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}, \quad (x,\xi) \in \mathbb{R}^{2n} \tag{1.1}$$

A function of S^m is called a symbol of order m.

We denote $S^{-\infty} = \cap S^m$ and $S^{+\infty} = \cup S^m$.

Examples.

1. If $a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$, with $a_\alpha \in C^\infty(\mathbb{R}^n)$, bounded as well as all its derivatives, then $a(x,\xi) \in S^m$. We say that a is a differential symbol of order m.
2. If $a(\xi) \in S(\mathbb{R}^n)$, then $a \in S^{-\infty}$
3. $a(\xi) = (1+|\xi|^2)^{m/2} \in S^m$.
4. If $a(x,\xi) \in S^m$ then $\partial_x^\beta \partial_\xi^\alpha a(x,\xi) \in S^{m-|\alpha|}$.
5. If $a \in S^m$ and $b \in S^{m'}$ then $ab \in S^{m+m'}$.
6. If $a(x,\xi) \in S^m$ satisfies $|a(x,\xi)| \ge C(1+|\xi|)^m$ (we say that $a(x,\xi)$ is elliptic), then $1/a \in S^{-m}$.
7. Attention: $a(x,\xi) = e^{ix\xi}$ is not a symbol !
8. Denote $\xi = (\xi', \xi'')$ and let $a(x,\xi) \in S^m$ independent of ξ'', then $a(x,\xi)$ is a polynomial symbol (of order m) in ξ'.

Proposition 1.1: **Asymptotic expansion**
Let (m_j) be a decreasing sequence of real numbers, $m_j \to -\infty$, and $a_j(x,\xi) \in S^{m_j}$.Then there exists a symbol $a \in S^{m_0}$, unique modulo $S^{-\infty}$, s.t. supp$a \subset \cup$suppa_j and

$$a - \sum_{j=0}^{k-1} a_j \in S^{m_k}, \quad k \in \mathbb{N}^* \tag{1.2}$$

a is called the asymptotic sum of the symbols a_j and we denote $a \sim \sum a_j$.
In particular, a symbol a of order m is a classical symbol if $a \sim \sum a_j$, where the functions a_j are homogeneous of order $m - j$.

Example : $m_j = -j, \quad j \in \mathbb{N}$ (classic symbol).

2. Pseudo-differential Operators

For $a \in S^m$, we try to define an operator by the formula

$$a(x, D)u(x) = (2\pi)^{-n} \int e^{ix\xi} a(x, \xi)\widehat{u}(\xi)d\xi \tag{2.1}$$

Theorem 2.1: For $a \in S^m$, the formula above defines a function of $\mathcal{S}(\mathbb{R}^n)$ and the map

$$\begin{cases} \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \\ u \to a(x, D)u \end{cases}$$

is continuous.

Definition: The operator defined by the previous theorem is called pseudo-differential operator of symbol a. It's denoted by $op(a)$, $a(x, D)$ or A.

Remark: If $u \in C_0^\infty(\mathbb{R}^n)$, then $a(x, D)u \in \mathcal{S}(\mathbb{R}^n)$; it's **not anymore compactly supported** since the formula uses the Fourier transform $\widehat{u}$.

3. Symbolic calculus .

Theorem 3.1 (adjoint)
If $a(x, D) \in op(S^m)$, then its adjoint $a^*(x, D) \in op(S^m)$ and one has

$$a^*(x,\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{a}(x,\xi) \tag{3.1}$$

Consequently, $a(x, D)$ is bounded from S' to S'.

Attention: Here the duality is defined by : $(Au, v) = (u, A^*v)$ where $u \in \mathcal{S}'$, $v \in \mathcal{S}$ and $(u, v) = \langle u, \overline{v}\rangle_{\mathcal{S}',\mathcal{S}}$

Theorem 3.2 (composition)
If $a_1 \in S^{m_1}$ and $a_2 \in S^{m_2}$, then there exists $b \in S^{m_1+m_2}$ such that $b(x, D) = a_1(x, D)a_2(x, D)$. Moreover we get the asymptotic expansion

$$b(x,\xi) \sim \sum_{\alpha} \frac{i^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha a_1(x,\xi) \partial_x^\alpha a_2(x,\xi) \tag{3.2}$$

Remarks

1. The symbol b of formula (3.2) is denoted $b = a_1 \# a_2$.

2. If a_1 and a_2 are two differential symbols (polynomials), the asymptotic formulae are exact.

3.In practice, one rarely needs the whole asymptotic expansion; the most usefull terms are the first ones. This is summarized in the following corollary.

Corollary 3.3

If $a_1 \in S^{m_1}$ and $a_2 \in S^{m_2}$, then

1. $a_1(x, D)a_2(x, D) = (a_1 a_2)(x, D) + R(x, D)$ where $R(x, \xi) \in S^{m_1+m_2-1}$.

2. $[a_1(x, D), a_2(x, D)] = C(x, D) + R(x, D)$ where

$$C(x, \xi) = \frac{1}{i}\{a_1(x, \xi), a_2(x, \xi)\} \quad \text{and} \quad R(x, \xi) \in S^{m_1+m_2-2}$$

Here $\{a_1, a_2\} = \sum_j (\partial a_1/\partial \xi_j \partial a_2/\partial x_j - \partial a_1/\partial x_j \partial a_2/\partial \xi_j)$ is the Poisson bracket of a_1 and a_2.

4. Action of Pdo's on Sobolev spaces

Theorem 4.1: If $a \in S^0$, then $a(x, D)$ is bounded on $L^2(\mathbb{R}^n)$.

Hint : Consider the kernel + Symbolic calculus + Schur Lemma

Corollary 4.2: If $a \in S^m$, then $a(x, D)$ is bounded from $H^s(\mathbb{R}^n)$ to $H^{s-m}(\mathbb{R}^n)$.

Conisder the pseudo-differential operator $\Lambda^r = op((1 + |\xi|^2)^{r/2})$
$\rightarrow$ Isomorphism between H^r and L^2.

$$a(x, D) = \Lambda^{m-s}(\Lambda^{s-m} a(x, D)\Lambda^{-s})\Lambda^s$$

Remark: In this way, it's easy to see that a pseudo-differential operator A in the class $op(S^{-\infty})$ is bounded from H^s to H^t for all s and t.
We say that A is infinitely smoothing.

Theorem 4.2 : Gärding inequality (weak form)
Consider a symbol $a(x,\xi) \in S^{2m}$, and assume there exists $c > 0$ such that

$$Re\, a(x,\xi) \geq c(1+|\xi|^2)^m \qquad \text{for} \quad |\xi| \geq R.$$

Then for every $N \geq 0$, there exists $C_N > 0$ such that

$$Re \left(a(x, D_x)u, u \right)_{L^2} \geq \frac{c}{2}\|u\|^2_{H^m} - c_N\|u\|^2_{H^{-N}}.$$

Proof : Notice that

$$Re \left(a(x, D_x)u, u \right)_{L^2} = \left((a(x, D_x) + a^*(x, D_x))u, u \right)_{L^2}$$

$$= \left(Op(Re\, a(x,\xi))u, u \right)_{L^2} + \left(C_{2m-1}(x, D_x)u, u \right)_{L^2}$$

where $C_{2m-1}(x,\xi) \in S^{2m-1}$.

5. Inversion of PDO

Theorem 5.1: Let $a \in S^m$, satisfying $|a(x,\xi)| \geq C(1+|\xi|)^m$ (we say that a is elliptic). Then there exists b_1 and $b_2 \in S^{-m}$ such that

$$\begin{cases} b_1(x,D)a(x,D) = Id + R(x,D) \\ a(x,D)b_2(x,D) = Id + R'(x,D) \end{cases} \tag{5.1}$$

where $R, R' \in op(S^{-\infty})$.
$b_1(x,D)$ (resp. $b_2(x,D)$) is a left (resp. right) parametrix of $a(x,D)$.

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where $R, R' \in op(S^{-\infty})$.
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Proof: $c_1(x,\xi) = (a(x,\xi))^{-1}$ satisfies $c_1(x,D)a(x,D) = Id - r(x,D)$ with $r(x,\xi) \in S^{-1}$. And one easily checks that the symbol $q \backsim \sum_{k\geq 0} r^k$ is an inverse modulo $S^{-\infty}$ of $1-r$. Thus $b_1(x,D) = q(x,D) \circ c_1(x,D)$ provides a left parametrix of $a(x,D)$.

Remarks.

1.We have the same result under the relaxed condition $|a(x,\xi)| \geq C\,|\xi|^m$ for $|\xi| \geq R$.

2.Consider the case of elliptic differential operators.

Theorem 5.2: Let $(x_0,\xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \backslash 0$ and $V_{x_0} \times \Gamma_{\xi_0}$ a conical neighborhood of (x_0,ξ_0), and consider $a \in S^m$, satisfying $|a(x,\xi)| \geq C(1+|\xi|)^m$ for $(x,\xi) \in V_{x_0} \times \Gamma_{\xi_0}$, $|\xi| \geq R$ ($a(x,\xi)$ is microlocally elliptic at (x_0,ξ_0)). Then for all $\psi(x) \in C_0^\infty(V_{x_0})$, $\psi = 1$ near x_0, $\chi(\xi) \in S^0$, supp$(\chi) \subset \Gamma_{\xi_0}$, $\chi = 1$ in a conical neighborhood of $\xi_0 \cap (|\xi| \geq R)$, there exists $b_1, b_2 \in S^{-m}$ such that

$$\begin{cases} b_1(x,D)a(x,D) = \chi(D)\psi(x) + R(x,D) \\ a(x,D)b_2(x,D) = \chi(D)\psi(x) + R'(x,D) \end{cases} \tag{5.2}$$

with $R, R' \in S^{-\infty}$.

6. Wave front set and pseudo-differential operators

Proposition 6.1
If $a(x,\xi) \in S^{-\infty}$, then the pdo $a(x,D)$ continuously maps

$$\mathcal{S}'(\mathbb{R}^n) \to C^\infty(\cap\mathcal{S}')$$

and

$$\mathcal{E}'(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

We say that $a(x,D)$ is infinitely smoothing.

Theorem 6.2: If $a(x, \xi) \in S^m$, then for all $u \in \mathcal{S}'(\mathbb{R}^n)$

$$\left\{ \begin{array}{c} \text{singsupp}(a(x, D)u) \subset \text{singsupp}u \\ WF(a(x, D)u) \subset WFu \\ WF_{s-m}(a(x, D)u) \subset WF_s u \end{array} \right. \tag{6.1}$$

We say that the pdo $a(x, D)$ is pseudo-local.
Finally, we give the elliptic microlocal regularity theorem.

Theorem 6.3: Microlocal elliptic regularity
Let $u \in \mathcal{S}'(\mathbb{R}^n)$, $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \backslash 0$ and $a \in S^m$ elliptic at (x_0, ξ_0), i.e. verifying $|a(x, \xi)| \geq C(1 + |\xi|)^m$ for x close to x_0 , and ξ in a conical neighborhood of $\xi_0, |\xi| \geq R$.

- If $(x_0 , \xi_0) \notin WF(a(x, D)u)$ then $(x_0 , \xi_0) \notin WFu$.
- If $(x_0 , \xi_0) \notin WF_s(a(x, D)u) \Rightarrow (x_0 , \xi_0) \notin WF_{s+m}u$.

Corollary 6.4: Consider a differential operator $P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ with coefficients in $C^\infty(\mathbb{R}^n)$, and $u \in \mathcal{S}'(\mathbb{R}^n)$.
Denote

$$CharP = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \backslash 0, \quad p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha = 0\}$$

the characteristic set of P.
We get the inclusions

$$WF(Pu) \subset WFu \subset CharP \cup WF(Pu)$$

$$WF_{s-m}(Pu) \subset WF_s u \subset CharP \cup WF_{s-m}(Pu)$$

Example

Consider $u \in L^2((0,T) \times \Omega)$, $\quad \omega \subset \Omega$ and

$$\begin{cases} \Box u = \partial_t^2 u - \Delta_x u = 0 \quad (0,T) \times \Omega \\ \\ \partial_t u \in L^2((0,T) \times \omega) \end{cases}$$

Then $u \in H^1_{loc}((0,T) \times \omega)$.

Indeed, $Char(\partial_t^2 - \Delta_x) \cap Char(\partial_t) = \{0\}$.

$$\tau^2 - |\xi|^2 = 0 \qquad \text{and} \quad \tau = 0 \quad \Longrightarrow \quad \tau = \xi = 0.$$

The action of a pseudo-differential operator

- does not "increase" the wave front set
- satisfies the microlocal elliptic regularity property

$$WF(Pu) \subset WFu \subset CharP \cup WF(Pu)$$

with

$$CharP = \{(x,\xi) \in \Omega \times \mathbb{R}^n \backslash 0,\ p(x,\xi) = 0\}$$

Here p is the principal symbol of P (characteristic manifold of P).

Goal: Localize more precisely the singularities of solutions of a pseudo-differential equation of type $Pu = f$.

$\rightarrow$These singularities live in $Char(P)$ and are essentially carried by the integral curves of the hamiltonian field H_p of p.

Consider p a real valued C^∞ function on $\mathbb{R}^n \times \mathbb{R}^n$.

Definition: The hamiltonian field or bicharacteristic field H_p of p, is the vector field on $\mathbb{R}^n \times \mathbb{R}^n$ given by

$$H_p(x,\xi) = \sum_{j=1}^{n} \left(\frac{\partial p}{\partial \xi_j}(x,\xi)\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}(x,\xi)\frac{\partial}{\partial \xi_j}\right) \tag{1.1}$$

A hamiltonian curve or bicharacteristic curve of p is an integral curve of H_p, i.e a maximal solution $\mathbb{R} \supset I \ni s \to (x(s), \xi(s))$ of the differential system

$$\dot{x}_j = \frac{\partial p}{\partial \xi_j}(x,\xi), \quad \dot{\xi}_j = -\frac{\partial p}{\partial x_j}, \quad x(0) = x^0, \xi(0) = \xi^0 \tag{1.2}$$

Remarks

1. The hamiltonian field H_p has an intrinsic definition.It is the only field in $\mathbb{R}^{2n}$ that satisfies

$$\sigma(V, H_p(x,\xi)) = dp(x,\xi)V$$

for every $V \in \mathbb{R}^{2n}$, dp beeing the differential of p and σ the symplectic form on $T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$, i.e the exterior differential of the Liouville form.

2. Since $H_p p = 0$, p is constant along its bicharacteristic curve. In particular, $p = 0$ on each curve issued from a point (x_0, ξ_0) s.t $p(x_0, \xi_0) = 0$.

Examples

1.Consider $p(t,x;\tau,\xi)=\tau^2-\xi^2$.

$$\dot{t}=2\tau, \quad \dot{x}=-2\xi, \quad \dot{\tau}=\dot{\xi}=0$$

The null bicharacteristic issued from (0,0;1,1) is given by $\gamma(s)=(2s,-2s;1,1)$.

2.For $p(t,x;\tau,\xi)=\tau^4-\xi^4$, we get $\gamma(s)=(4s,-4s;1,1)$.

3. If p and q are two hamiltonians on $\mathbb{R}^n\times\mathbb{R}^n$, with q elliptic, then the null bicharacteristic curves of p and (pq) issued from the same point are identical.

Theorem 2.1 (Hörmander '71)
Let P be a pseudo-differential operator of order m in $\mathbb{R}^n$; assume that P is classic and with real principal symbol. Consider $u \in D'(\mathbb{R}^n)$ s.t $Pu \in C^\infty(\mathbb{R}^n)$ and Γ a bicharacteristic curve of P. Then we have

$$\Gamma \subset WFu \quad \text{or} \quad \Gamma \cap WFu = \varnothing$$

In other words, WFu is invariant under the hamiltonian flow of P.

Corollary 2.2: Under assumptions of Theorem 2.1 , WFu is a union of null bicharacteristics of P.

Valid on a domain Ω far from the boundary !

Remark: The conclusion of Theorem 2.1 can be stated as follows: let Γ be a bicharacteristic of P and ω a point of Γ. Then one has:
-If $\omega \notin WFu$ then $\Gamma \cap WFu = \varnothing$: propagation of the regularity.
-If $\omega \in WFu$ then $\Gamma \subset WFu$: propagation of the singularity.

Theorem 2.3: Sobolev wave front
Under assumptions of Theorem 2.1, for $s \in \mathbb{R}$, we have

$$\Gamma \subset WF^s u \quad \text{or} \quad \Gamma \cap WF^s u = \varnothing$$

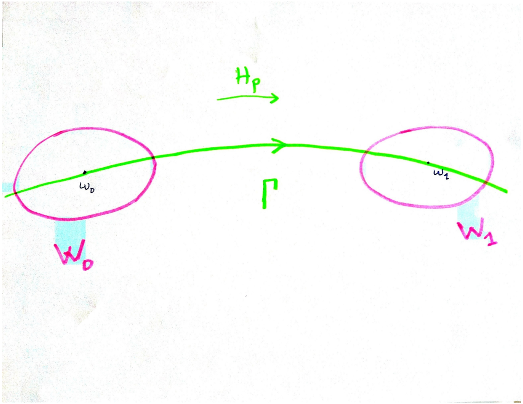
Proof of the theorem: a microlocal ODE

Assume P of order 1 and consider ω_0, ω_1 two points of the bicharacteristic curve Γ, sufficiently close.

Lemma 2.4

Take $a_0(x,\xi) \in S^0(\mathbb{R}^{2n})$. Then there exist a neighborhood W_0 of ω_0 , and W_1 of ω_1 such that for every symbol $c_s(x,\xi) \in S^s(\mathbb{R}^{2n})$, supp $c_s(x,\xi) \subset W_1$, there exists a symbol $q_{2s} \in S^{2s}(\mathbb{R}^{2n})$, supported near Γ, and $r_{2s}(x,\xi) \in S^{2s}(\mathbb{R}^{2n})$ supported in W_0, such that:

$$H_{p_1} q_{2s} + a_0 q_{2s} = |c_s(x,\xi)|^2 + r_{2s}(x,\xi)$$



Choose W_1 sufficiently small and $c_s(x,\xi)$ elliptic at ω_1. By assumption, the quantity

$$I = ((P^* Q_{2s} - Q_{2s} P)u, u)_{L^2} = (Q_{2s} u, Pu)_{L^2} - (Pu, Q_{2s}^* u)_{L^2}$$

is bounded.

$$I = ((PQ_{2s} - Q_{2s} P)u + (P^* - P)Q_{2s} u, u)_{L^2}$$

$$P^* - P = a_0(x,D) + a_{-1}(x,D), \qquad a_{-j} \in S^{-j}$$

Therefore

$$\begin{aligned} I &= ((PQ_{2s} - Q_{2s} P)u + a_0(x,D) Q_{2s} u, u)_{L^2} + (a_{-1}(x,D) Q_{2s} u, u)_{L^2} \\ &= \|c_s(x,D)u\|_{L^2}^2 + (r_{2s}(x,D)u, u)_{L^2} + (a_{-1}(x,D) Q_{2s} u, u)_{L^2} \end{aligned}$$

is bounded if we assume u is $H^{s-1/2}$ microlocally near Γ..

$\rightarrow$ Iterate the process.

Application 1: Relaxed internal observation

$\Box u = 0 \quad , \quad u_{|\partial\Omega} = 0, \quad u \in L^2(]0,T[\times\Omega) \quad \text{and} \quad \partial_t u \in L^2(]0,T[\times\omega)$

→ Prove global regularity $u \in H^1(]0,T[\times\Omega)$ under G.C.C.

→ Use Hörmander's theorem (propagation up to the boundary).

$$WF^1 u \subset \{(t,x;\tau,\xi) \in T^*(]0,T[\times\Omega)\backslash 0, \tau^2 = |\xi|^2\}.$$

$$WF^1 u \cap T^*((0,T)\times\omega) \subset \{(t,x;\tau,\xi),\ \tau = 0\}.$$

This yields $WF^1 u \cap T^*((0,T)\times\omega) = \emptyset$, i.e $u \in H^1_{loc}(]0,T[\times\omega)$. Now, take $\rho_0 = (t_0,x_0;\tau_0,\xi_0) \in]0,\varepsilon[\times\Omega\times\mathbb{R}^{1+n}\backslash 0$; the bicharacteristic Γ_0 issued from this point necessarely enters in the region $]0,T[\times\omega$, i.e in the region where u is H^1. Therefore, by propagation, we obtain that $\rho_0 \notin WF^1 u$ and $u \in H^1(]0,T[\times\Omega)$.

We consider the Klein-Gordon equation

$$(K-G) \quad \begin{cases} \partial_t^2 u - \Delta_x u + u = 0 & \text{in }]0,+\infty[\times M \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in H^1 \times L^2 \end{cases}$$

- M Riemannian manifold, compact, connected, without boundary. (Torus, sphere ...)
- ω open subset of M
- Assume (ω, T) doesn't satisfy **GCC**.

There exists $m_0 = (x_0, v_0) \in TM$ such the geodesic γ_{m_0} satisfies

$$\left\{\gamma_{m_0}(s),\ s \in [0, T]\right\} \cap \omega = \emptyset.$$

Thus, there exists $\xi_0 \in T^*_{x_0}M$ such that

$\tilde{\rho}_0 = (0, x_0, \tau_0 = |\xi_0|_{x_0}, \xi_0) \in T^*(\mathbb{R} \times M)$ satisfies

$$\left\{\Gamma_{\tilde{\rho}_0}(s),\ s \in [0, T]\right\} \cap T^*(\mathbb{R} \times \omega) = \emptyset.$$

Consider the family of functions (in local coordinates) :

$$v_{0\varepsilon}(x) = \varepsilon^{1-n/4} exp\Big(\frac{i}{\varepsilon}(x.\xi_0)\Big) exp\Big(- \frac{|x - x_0|^2}{\varepsilon}\Big), \qquad \varepsilon > 0$$

- The sequence $(v_{0\varepsilon})_\varepsilon$ weakly converges to 0 in $H^1(\mathbb{R}^n)$ and satisfies

$$\|v_{0\varepsilon}\|_{H^1} \sim 1, \qquad \text{for} \quad \varepsilon \to 0^+.$$

- For $b = b(x;\xi) \in S^0(T^*M)$ pseudo-differential symbol of order 0 such that $(x_0, \xi_0) \notin supp(b)$, we have for every $s \geq 1$,

$$\|b(x; D_x)v_{0\varepsilon}\|_{H^s} = o(1) \quad \text{for} \quad \varepsilon \to 0^+.$$

Theorem : The point $\rho = (0, x; \tau = |\xi|_x, \xi) \in T^*(\mathbb{R} \times M) \setminus 0$ satisfies $\rho \neq \tilde{\rho}_0 = (0, x_0; \tau_0 = |\xi_0|_x, \xi_0)$, we have for any $s > 1$, and we have

$$WF^s u_\varepsilon \cap \Gamma_\rho = \emptyset.$$

$$\begin{cases} P_A u_\varepsilon = \partial_t^2 u_\varepsilon - \Delta_A u_\varepsilon + u_\varepsilon = 0 \\ u_\varepsilon(0,.) = v_{0\varepsilon}, \qquad \partial_t u_\varepsilon(0,.) = i\lambda(D) v_{0\varepsilon} \end{cases}$$

where

$$\lambda(D) = \sqrt{-\Delta_A + 1}$$

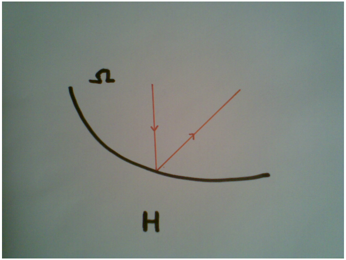
is a pseudo-differential operator classic, of order 1 , on M with principal symbol $\sigma_1(\lambda)(x,\xi) = |\xi|_x$.

$$\|u_{0\varepsilon}(0)\|_{H^1}^2 + \|\partial_t u_{0\varepsilon}(0)\|_{L^2}^2 \approx 1 \qquad \text{and} \qquad \lim_{\varepsilon \to 0} \|\partial_t u_\varepsilon\|_{L^2(0,T)\times\omega)} = 0$$

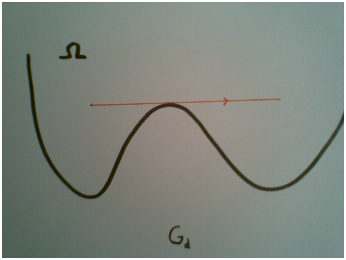
$$\partial_t u_\varepsilon - i\lambda(D) u_\varepsilon = 0 \qquad u_\varepsilon(0,.) = v_{0\varepsilon}$$

→ Follow backward any bicharacteristic curve issued from $(0, T) \times \omega$.

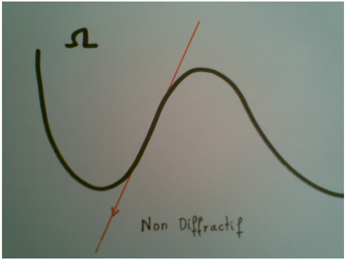
Geometry at the boundary and generalized bicharacteristics



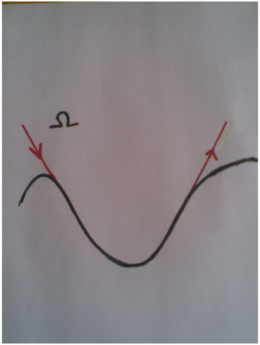
Hyperbolic



Glancing Diffractive



Non diffractive



Gliding ray

Geodesic coordinates

Ω a bounded domain of $\mathbb{R}^n$ with smooth boundary, $m_0 \in \partial\Omega$.

$$\Box = -\partial_t^2 + \sum_{1 \le i,j \le n} \partial_{x_j} b_{ij}(x) \partial_{x_i}$$

Near m_0 one can find a system of geodesic local coordinates

$$x = (x_1, x_2, ..., x_n) \longrightarrow y = (y_1, y_2, ..., y_n)$$

such that

$$\Omega = \{(y_1, y_2, ..., y_n),\ y_n > 0\}, \quad \partial\Omega = \{(y_1, y_2, ..., y_{n-1}, 0)\} = \{(y', 0)\}$$

$$P = \Box = -\partial_t^2 + \left(\partial_{y_n}^2 + \sum_{1 \le i,j \le n-1} \partial_{y_j} b_{ij}(y) \partial_{y_i}\right) + M_0(y)\partial_{y_n} + M_1(y, \partial_{y'})$$

Come back to initial notation : (t, x) coordinates.

$$p = \sigma(\Box) = -\xi_n^2 + \left(\tau^2 - \sum_{1 \le i,j \le n-1} a_{ij}(x)\xi_i\xi_j\right) = -\xi_n^2 + r(x, \tau, \xi')$$

$$\sum_{1 \le i,j \le n-1} a_{ij}(x)\xi_i\xi_j \ge c_0|\xi|^2$$

We shall write

$$r_0(x', \tau, \xi') = r(x, \tau, \xi')_{|x_n=0} = r(x', 0, \tau, \xi')$$

$$p_{|x_n=0} = \sigma(\Box)_{|x_n=0} = -\xi_n^2 + r_0(x', \tau, \xi')$$

$$\mathcal{L} = \mathbb{R} \times \Omega, \qquad \partial\mathcal{L} = \mathbb{R} \times \partial\Omega$$

The compressed cotangent bundle of Melrose-Sjöstrand is given by

$$T_b^*\mathcal{L} = T^*\mathcal{L} \cup T^*\partial\mathcal{L}$$

We recall the natural projection

$$\pi \ : \ T^*\mathbb{R}^{n+1} \,|_{\overline{\Omega}} \to T_b^*\mathcal{L} \tag{1}$$

and we equip $T_b^*\mathcal{L}$ with the induced topology.
We have a partition of $T^*(\partial\mathcal{L})$ into elliptic, hyperbolic and glancing sets:

$$\#\left\{\pi^{-1}(\rho) \cap Char(P)\right\} = \begin{cases} 0 & if \quad \rho \in \mathcal{E} \\ 1 & if \quad \rho \in \mathcal{G} \\ 2 & if \quad \rho \in \mathcal{H} \end{cases} \tag{2}$$

In geodesic coordinates we have locally

$$\mathcal{L} = \{(t,x) \in \mathbb{R}^{n+1},\, x_n > 0\} \quad \text{and} \quad \partial\mathcal{L} = \{(t,x) \in \mathbb{R}^{n+1},\, x_n = 0\}.$$

$\longrightarrow$

$$p_{|x_n=0} = -\xi_n^2 + r_0(x', \tau, \xi')$$

Take $\rho = (t, x', \xi') \in T^*(\partial\mathcal{L})$

Thus one defines the elliptic, hyperbolic and glancing sets

$$\mathcal{E} = \{r_0 < 0\}, \qquad \mathcal{H} = \{r_0 > 0\}, \qquad \mathcal{G} = \{r_0 = 0\}.$$

Definition

1. A point $\rho \in T^*\partial\mathcal{L}\backslash 0$ is nondiffractive if $\rho \in \mathcal{H}$ or if $\rho \in \mathcal{G}$ and the free bicharacteristic $(\exp sH_p)\widetilde{\rho}$ passes over the complement of $\overline{\mathcal{L}}$ for arbitrarily small values of s, where $\widetilde{\rho}$ is the unique point in $\pi^{-1}(\rho) \cap Char(P)$. $(\mathcal{G}_{nd})$.

2. $\rho \in T^*\partial\mathcal{L}\backslash 0$ is strictly gliding if $\rho \in \mathcal{H}$ or if $\rho \in \mathcal{G}$ and $H_p^2(x_n)(\rho) < 0$.$(\mathcal{G}_{sg})$.
   In this case, the projection on the $(t, x)-$space of the free *bicharacteristic* ray γ issued from ρ leaves the boundary $\partial\mathcal{L}$ and enters in $T^*(\mathbb{R}^{n+1} \setminus \overline{\mathcal{L}})$ at $\widetilde{\rho} = \pi^{-1}(\rho)$.

3. $\rho \in T^*\partial\mathcal{L}\backslash 0$ is strictly diffractive if $\rho \in \mathcal{G}$ and $H_p^2(x_n)(\rho) > 0$. $(\mathcal{G}_d)$.

   This means that there exists $\varepsilon > 0$ such that $(\exp sH_p)\widetilde{\rho} \in T^*\mathcal{L}$ for $0 < |s| < \varepsilon$.

Generalized bicharacteristics

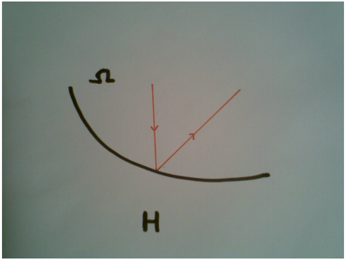
A generalized bicharacteristic ray is a continuous map

$$\mathbb{R} \supset I \setminus B \ni s \mapsto \gamma(s) \in T^*\mathcal{L} \cup \mathcal{G} \subset T^*\mathbb{R}^{n+1}$$

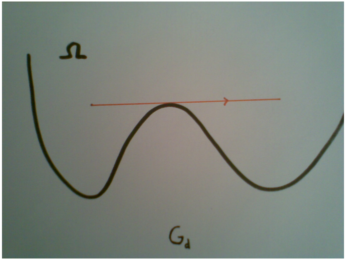
where I is an interval of $\mathbb{R}$, B is a set of isolated points.
For every $s \in I \setminus B$, $\gamma(s) \in \pi(Char(P))$ and γ is differentiable as a map with values in $T^*\mathbb{R}^{n+1}$, and

1. If $\gamma(s_0) \in T^*\mathcal{L} \cup \mathcal{G}_d$ then $\dot{\gamma}(s_0) = H_p(\gamma)(s_0)$.
2. If $\gamma(s_0) \in \mathcal{G} \setminus \mathcal{G}_d$ then $\dot{\gamma}(s_0) = H_p^G(\gamma(s_0))$, where
$$H_p^G = H_p + (H_p^2 x_n / H_{x_n}^2 p) H_{x_n}$$
3. For every $s_0 \in B$, the two limits $\gamma(s_0 \pm 0)$ exist and are the two different points of the same hyperbolic fiber of the projection π.

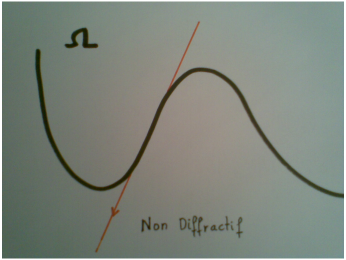
Remark : If H_p has only finite order contact with $\partial T^*\mathcal{L}$, through every $\rho \in T_b^*\mathcal{L}$ passes a unique maximal generalized bicharacteristic γ.



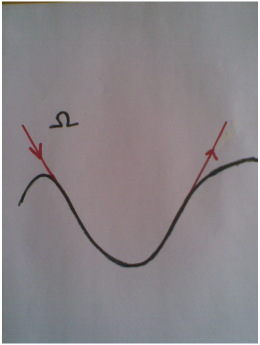
Ω
H



Ω
G_d



Ω
Non Diffractif



Ω

Wave front up to the boundary

We work in geodesic coordinates.

$$\mathcal{L} = \{(t, x', x_n),\, x_n > 0\} \quad \text{and} \quad \partial\mathcal{L} = \{(t, x', 0)\}.$$

Take $\rho = (t, x', \tau, \xi') \in T^*(\partial\mathcal{L})$ and assume that $Pu = 0$ in $\mathcal{L}$.

We say that $\rho \notin W_b^s(u)$ iff for some $\varepsilon > 0$ small, there exists a tangential pseudo-differential operator $A = A(t, x, D_t, D_{x'})$, of order 0, elliptic at ρ, such that

$$Au \in H^s\Big(B_\varepsilon(\pi(\rho)) \cap \{x_n > 0\}\Big).$$

Remark : For $\rho = (t, x, \tau, \xi) \in T^*\mathcal{L}$ (ie $x \in \Omega$), use the classical definition of the wave front.

Theorem (Melrose -Sjöstrand '78)
Consider $u \in \mathcal{D}'(\mathbb{R} \times \Omega)$ solution to

$$\Box u = 0, \quad \text{in} \quad \mathbb{R} \times \Omega, \qquad u_{|\partial\Omega} = 0$$

Then $W_b(u)$ is invariant under the hamiltonian flow of the wave operator .

If Γ is a generalized bicharacteristic curve of P, we have

$$\Gamma \subset WF_b(u) \quad \text{or} \quad \Gamma \cap WF_b(u) = \varnothing$$

Remark

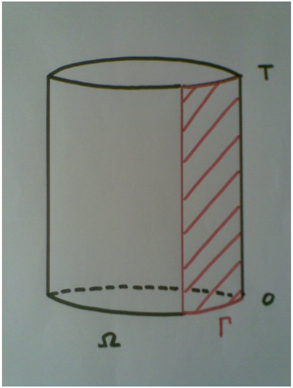
→ Other boundary conditions.

→ Similar result for $WF_b^s(u)$.

Geometric Control Condition (Boundary control)

Consider $\Gamma \subset \partial\Omega$ and $T > 0$.

The couple (Γ, T) satisfies the geometric control condition (G.C.C), if every generalized bicharacteristic of the wave symbol, issued at $t = 0$, intersects the boundary subset Γ at a nondiffractive point, before the time T.



The Lifting Lemma of Bardos-Lebeau-Rauch

Consider a nondiffractive point $\rho \in T^*(\partial\mathcal{L})$ and $u \in \mathcal{D}'(\mathbb{R} \times \Omega)$ solution to

$$\Box u = 0, \quad \text{in} \quad \mathbb{R} \times \Omega$$

Assume that

$$\rho \notin WF^s(u_{|\partial\Omega}) \quad \text{and} \quad \rho \notin WF^{s-1}(\partial_n u_{|\partial\Omega})$$

Then $\rho \notin WF_b^s u$.

$\rightarrow$ Key point in B-L-R paper.

4-Microlocal Defect Measures

- A tool to analyze the **lack of compactness** of sequences weakly converging to 0 in $L^2(\mathbb{R}^n)$.
- The first one is the old notion of "defect measure"
- The second was introduced independently by L.Tartar and P. Gérard and called " microlocal defect measure" (mdm's).

After, we give some examples that illustrate the precision of the des m.d.m's with respect to defect measures. And finally, for bounded sequences (in H^1 or L^2...) of solutions of partial differential equations, we present two theorems.

- Localization of the support of the measure
- Propagation ...along the bicharacteristic flow.

Let (u_k) be a bounded sequence of $L^2(\mathbb{R}^n)$ weakly converging to 0.

Definition: The defect measure of (u_k) is the limit, in the sense of the measures, of the sequence $\alpha_k = |u_k(x)|^2 \, dx$, where dx is the Lebesgue measure on $\mathbb{R}^n$.
This measure α is given by

$$\langle \alpha, \varphi \rangle = \lim_{k \to +\infty} (\varphi u_k, u_k), \quad \varphi \in C_0^\infty(\mathbb{R}^n)$$

The microlocal defect measures generalize the notion of defect measures in the sense that the test functions used in the limit above are not any more functions of $C_0^\infty(\mathbb{R}^n)$ but **0-order pseudodifferential symbols** (essentially in the class $S^0(\mathbb{R}^n \times \mathbb{R}^n)$).

- Denote by $\mathcal{A}^m$ the set of all classical pseudodifferential operators, of order m and compact support in $\mathbb{R}^n$.
- If $A \in \mathcal{A}^m$, its symbol $\sigma(A)$ can be taken in the form: $\sigma(A) = \varphi a \psi \in,$ where $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$ and $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$.
- For a given $A \in \mathcal{A}^0$ and (u_k) weakly converging to 0 in $L^2(\mathbb{R}^n)$, the sequence (Au_k, u_k) is bounded and thus admits a converging subsequence: $\lim_{k_n}(Au_{k_n}, u_{k_n})$ exists in $\mathbb{C}$.
- Moreover, writing $A = A_0 + A_{-1}$, where $A_{-1} \in \mathcal{A}^{-1}$ and $\sigma(A_0)$ is homogeneous of order 0, we see that

$$\lim_{k_n}((A_0 + A_{-1})u_{k_n}, u_{k_n}) = \lim_{k_n}(A_0 u_{k_n}, u_{k_n})$$

This limit only depends on $\sigma(A_0) = \sigma_0(A)$, since $\lim_k \|A_{-1}u_k\|_{L^2} = 0$, thanks to Riellich Lemma.

By a diagonal extraction process, one can prove the existence of a subsequence (u_{k_n}) such that (Au_{k_n}, u_{k_n}) converges for every $A \in \mathcal{A}^0$.

And consequently, the map

$$\begin{array}{rcl} L : C_0^\infty(\mathbb{R}^n \times S^{n-1}) & \longrightarrow & \mathbb{C} \\ \sigma_0(A) & \longmapsto & \lim_{k_n} (Au_{k_n}, u_{k_n}) \end{array}$$

is well defined.

The following proposition shows that L is positive and continuous for the uniform topology.

Lemma : **Gärding inequality**

Take $A \in \mathcal{A}^0$ and assume that its principal symbol is real and satisfies $\sigma_0(A) \geq 0$. Then there exists $C > 0$ and for evrey $\delta > 0$, there exists $C_\delta > 0$, such that, for every $v \in L^2_{com}(\mathbb{R}^n)$,

$$\begin{cases} \mathrm{Re}\,(Av, v)_{L^2} \geq -\delta \left\| v \right\|^2_{L^2} - C_\delta \left\| v \right\|^2_{H^{-1/2}} \\ \left| \mathrm{Im}\,(Av, v)_{L^2} \right| \leq C \left\| v \right\|^2_{H^{-1/2}} \end{cases}$$

Proposition: For every $A \in \mathcal{A}^0$, we have :

1-

$$\overline{lim}_{k \to +\infty} |(Au_k, u_k)| \leq \overline{lim}_{k \to +\infty} \|u_k\|_{L^2}^2 \sup_{\mathbb{R}^n \times S^{n-1}} |\sigma_0(A)|$$

2- If moreover $\sigma(A) \geq 0$, we obtain

$$lim_{k \to +\infty} \mathrm{Im}(Au_k, u_k) = 0 \quad \text{and} \quad \underline{lim}_{k \to +\infty} \mathrm{Re}(Au_k, u_k) \geq 0.$$

Consequence: There exists a positive Radon measure μ on $S^*\mathbb{R}^n = \mathbb{R}^n \times S^{n-1}$ satisfying :

$$\lim_{k_n \to +\infty} (Au_{k_n}, u_{k_n}) = \int_{\mathbb{R}^n \times S^{n-1}} \sigma_0(A)(x,\xi)\, d\mu, \qquad A \in \mathcal{A}^0$$

Definition: The measure μ is called the microlocal defect measure attached to the sequence (u_{k_n}).

1. Sequences with concentration effect.
Let $u_k(x) = k^\beta \psi(kx)$ with $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\beta \in \mathbb{R}$. We have

$\|u_k\|_{L^2}^2 = k^{2\beta-n} \|\psi\|_{L^2}^2$, hence

* If $\beta < n/2$, $u_k \longrightarrow 0$ in $L^2(\mathbb{R}^n)$.
* If $\beta > n/2$, $\|u_k\|_{L^2} \longrightarrow +\infty$.
* If $\beta = n/2$, the sequence (u_k) is bounded in L^2.
In this case, we can prove that for $A \in \mathcal{A}^0$

$$(Au_k, u_k) \longrightarrow (2\pi)^{-n} \int_{\mathbb{R}^n} \left|\widehat{\psi}(\xi)\right|^2 a(0,\xi)\, d\xi \ , \qquad a = \sigma_0(A)$$

Therefore, the defect measure α of (u_k) is given by

$$\alpha = \|\psi\|_{L^2}^2 \, \delta_{x=0}$$

Moreover, the set $M(u_k)$ of all microlocal defect measures associated to this sequence is reduced to one single measure: (u_k) is pure.

$$\mu = \delta_{x=0} \otimes h(\xi) d\sigma(\xi)$$

$$h(\xi) = (2\pi)^{-n} \int_0^{+\infty} \left| \widehat{\psi}\,(r\xi) \right|^2 r^{n-1} dr$$

2. Sequences with oscillation effect

a) $u_k(x) = \psi(x)\, e^{ikx\xi_0}$ where $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\xi_0 \neq 0$.

For $A \in \mathcal{A}^0$,

$$(Au_k, u_k) \longrightarrow \int_{\mathbb{R}^n} a(x, \frac{\xi_0}{|\xi_0|})\, |\psi(x)|^2\, dx$$

$$\alpha(u) = |\psi(x)|^2\, dx \quad \text{and} \quad \mu = |\psi(x)|^2\, dx \otimes \delta_{\frac{\xi_0}{|\xi_0|}}$$

Remark: In this example, the oscillation of (u_k) leaded to the term $\delta_{\xi_0/|\xi_0|}$ in the expression of the mdm μ, while this term is completely hidden in the defect measure $\alpha(u)$.

Other question

Assume $u_k \rightharpoonup 0$ in $L^2(\mathbb{R}^n)$ and $supp(u_k) \subset B(0,R)$.
Then for all fixed $M > 0$,

$$\int_{|\xi|\leq M} |\hat{u}_k(\xi)|^2 d\xi = 0 \qquad \text{for} \quad k \to \infty$$

This says that the lack of compactness occurs in high frequencies.

Question : At which scale ???

→ Semiclassical measures

Properties of the mdm's

- P is a differential operator of order m, with principal symbol p_m.
- (u_k) a bounded sequence in $L^2(\mathbb{R}^n)$, weakly converging to 0, **pure**.
- μ is the mdm attached to (u_k).

Proposition 1: The following conditions are equivalent:
a) $Pu_k \to 0$ in $H^{-m}(\mathbb{R}^n)$ (strong convergence),
b) $Supp(\mu) \subset Char(P) = p_m^{-1}(0)$.

Consequence: If $\omega_0 \in S^*(\mathbb{R}^n \times \mathbb{R}^n)$ is such that $p_m(\omega_0) \neq 0$, then $u_k \to 0$ in L^2 microlocally near ω_0 (strong convergence).

Proof: Condition a) leads to

$$\left\| Q_{m/2} P u_k \right\|_{L^2} = (Q_m P u_k, P u_k)_{L^2} \to 0$$

where Q_m is a pseudodifferential operator in the class $\mathcal{A}^{-2m}$, with principal symbol

$$\sigma_{-2m}(Q_m) = |\xi|^{-2m} \quad \text{for } |\xi| \geq 1.$$

But, by definition of the measure μ, this limit also satisfies

$$\lim (Q_m P u_k, P u_k) = \lim (P^* Q_m P u_k, u_k)$$

$$= \int_{\mathbb{R}^n \times S^{n-1}} \sigma_{-2m}(Q_m) \left| p_m(x,\xi) \right|^2 d\mu = \int_{\mathbb{R}^n \times S^{n-1}} \left| p_m(x,\xi) \right|^2 d\mu$$

and this gives the desired result, i.e

$$\operatorname{supp}(\mu) \subset \{(x,\xi),\ p_m(x,\xi) = 0\}.$$

Proposition 2

Assume that

- P is self adjoint, i.e $P = P^*$.
- $Pu_k \to 0$ in $H^{1-m}(\mathbb{R}^n)$,

then one obtains $H_{p_m}\mu = 0$.

Proof: For evry $Q \in \mathcal{A}^{1-m}$ with principal symbol $\sigma(Q) = q$. We have

$$\lim\left((QP - PQ)u_k, u_k\right)_{L^2} = \lim\left(Pu_k, Q^*u_k\right)_{L^2} - \lim(Qu_k, Pu_k)_{L^2} = 0$$

And this gives

$$\int_{\mathbb{R}^n \times S^{n-1}} \{q, p_m\}(x, \xi) d\mu = 0$$

which can be written in the form

$$\langle \mu, H_{p_m} q \rangle = 0$$

i.e

$$H_{p_m}\mu = 0$$

Consequence

Denote by $\phi_s(x,\xi) = \Big(x(s), \xi(s)\Big)$ the flow of H_p, and set

$$\tilde{\phi}_s(x,\xi) = \Big(x(s), \xi(s)/|\xi(s)|\Big)$$

For a symbol $a(x,\xi) \in S^0$ homogeneous, we have

$$a \circ \tilde{\phi}_s(x,\xi) = a \circ \phi_s(x,\xi)$$

$$\frac{d}{ds}\langle \mu, a \circ \tilde{\phi}_s \rangle = \frac{d}{ds}\langle \mu, a \circ \phi_s \rangle = \langle \mu, \frac{d}{ds}(a \circ \phi_s) \rangle = \langle \mu, H_p a \rangle = 0$$

Thus $\langle \mu, a \circ \tilde{\phi}_s \rangle = cte$, and for $\omega \in CharP \cap S^*(\mathbb{R}^{2n})$,

$$\omega \notin supp(\mu) \Leftrightarrow \tilde{\phi}_s(\omega) \notin supp(\mu), \qquad \forall s$$

$$\mu(\tilde{\phi}_s(B)) = \mu(B), \qquad \forall B \text{ borelian set.}$$

Application 3 : Observability (2^{nd} proof)

Here we use the mdm's properties to establish the following observability estimate, under **GCC** :

There exists a constant $C > 0$, such that inequality

$$\|u_0\|^2_{H^1_0} + \|u_1\|^2_{L^2} \leq C \int_0^T \int_\omega |\partial_t u(t,x)|^2 \, dxdt$$

holds for every $(u_0, u_1) \in E = H^1_0(\Omega) \times L^2(\Omega)$ and u solution of system

$$\begin{cases} \Box_{\mathcal{A}} u = \partial_t^2 u - \Delta_{\mathcal{A}} u = 0 & \text{in } \mathbb{R} \times \Omega \\ u(0) = u_0, \quad \partial_t u(0) = u_1 & \text{in } \Omega \end{cases}$$

Here we can take $\mathcal{A} = (g_{ij})$ of class $\mathcal{C}^2$.

The proof relies on a contradiction argument. Assuming that the estimate is false, one can find a sequence of data (u_0^k, u_1^k) in $H_0^1(\Omega) \times L^2(\Omega)$ such that

$$\left\| u_0^k \right\|_{H_0^1}^2 + \left\| u_1^k \right\| \quad \text{and} \quad \int_0^T \int_\omega \left| \partial_t u^k(t,x) \right|^2 dxdt \to 0 \qquad \text{as } k \to \infty$$

The sequence of solutions (u^k) is then bounded in $H^1(]0, T[\times\Omega)$ and we may assume that it is weakly convergent to 0 (unique continuation). Therefore, if μ is a microlocal defect measure attached to (u^k), we have $\mu = 0$ over $(0, T) \times \omega$, and by GCC and propagation for measures, $\mu = 0$ everywhere. And this contradicts our condition on the initial data.

Application 4 : Behavior of the HUM control operator

$$(W) \quad \begin{cases} \partial_t^2 u - \Delta_{\mathcal{A}} u = \chi_\omega^2(x) f & \text{in} \quad]0,T[\times\Omega \\ u = 0 & \text{on} \quad]0,T[\times\partial\Omega \\ (u(0), \partial_t u(0)) = (u_0, u_1) \in H_0^1 \times L^2 \end{cases}$$

We look for $f \in L^2(]0,T[\times\Omega)$, s.t

$$(u(T), \partial_t u(T)) = (0,0)$$

By HUM and under (G.C.C), we can take f solution of

$$(W') \quad \begin{cases} \partial_t^2 f - \Delta_{\mathcal{A}} f = 0 & \text{in} \quad]0,T[\times\Omega \\ f = 0 & \text{on} \quad]0,T[\times\partial\Omega \\ (f(0), \partial_t f(0)) = (f_0, f_1) \in L^2 \times H^{-1} \end{cases}$$

The map

$$\begin{cases} \Lambda : H_0^1 \times L^2 \to L^2 \times H^{-1} \\ \\ \quad (u_0, u_1) \to (f_0, f_1) \end{cases}$$

is an isomorphism.

This is HUM optimal control operator.

Two problems

a) Control of smooth data

$$U_0 = (u_0, u_1) \in E_k = H^{k+1} \times H^k, \quad k \geq 0$$

$\rightarrow$ Does the control ΛU_0 identify the data regularity ?

Remark: *Bardos-Lebeau-Rauch* : Observation estimates in each $H^s(M)$.

b) Treatment of the frequencies

$\rightarrow$ Does the control ΛU_0 load the frequencies carrying the data ?
$\rightarrow$ If U_0 has only low frequencies, how are the high frequencies of ΛU_0 ?
$\rightarrow$ Does it handle individually the frequencies of the data U_0 ?

Theorem (D-Lebeau, 2009)

For $\mathcal{A} = (g_{ij})$ of class $\mathcal{C}^\infty$ and under (G.C.C),
a) For all $s \geq 0$,

$$\Lambda : H^{s+1} \times H^s \to H^s \times H^{s-1}$$

is an isomorphism.

b)

$$\left\| \Lambda\psi(2^{-k}D) - \psi(2^{-k}D)\Lambda \right\| \leq C2^{-k/2}$$

c) If M is a Riemannian manifold without boundary, Λ is a pseudo differential operator.

Here

$$\sum_{k\geq 0} \psi(2^{-k}D) = Id$$

is the Littlewood-Paley decomposition.

Behavior of the HUM control process

Take $\mathcal{A} = (g_{ij})$ in $\mathcal{C}^\infty$, such that (ω, T) satisfies (GCC).

Theorem

For any $\mathcal{C}^\infty$- neighborhood $\mathcal{U}$ of $\mathcal{A}$, there exist $\mathcal{A}' \in \mathcal{U}$ and an initial data (u_0, u_1) , $||(u_0, u_1)||_{H^1 \times L^2} = 1$, s.t the respective solutions u and v of

$$\begin{cases} \partial_t^2 u - \Delta_{\mathcal{A}} u = \chi_\omega^2(x) f_{\mathcal{A}} \\ \partial_t^2 v - \Delta_{\mathcal{A}'} v = \chi_\omega^2(x) f_{\mathcal{A}} \\ (u(0), \partial_t u(0)) = (v(0), \partial_t v(0)) = (u_0, u_1) \in H_0^1 \times L^2 \end{cases}$$

satisfy

$$E_A(u - v)(T) = E_A(v)(T) \geq 1/2$$

Moreover, for some $C_T > 0$,

$$||f_{\mathcal{A}} - f_{\mathcal{A}'}||_{L^2((0,T)\times\omega)} \geq C_T$$

Proof

→ Choose $\mathcal{A}' = (1+\varepsilon)\mathcal{A}, \quad \varepsilon \neq 0,$ small .

→ (GCC) also satisfied by (ω, T) for the metric $\mathcal{A}'$.

→ Take a sequence (u_0^k, u_1^k) such that $||(\nabla_{\mathcal{A}} u_0^k, u_1^k)||_{L^2\times L^2} = 1$ and

$$u_0^k, u_1^k) \rightharpoonup (0,0) \qquad \text{in} \qquad H_0^1 \times L^2$$

→ $f_{\mathcal{A}}^k \rightharpoonup 0$ in $L^2((0,T)\times\Omega)$. Hence $f_{\mathcal{A}}^k \to 0$ in $H^{-1}((0,T)\times\Omega)$ and

$$\partial_t^2 f_{\mathcal{A}}^k - \Delta_{\mathcal{A}} f_{\mathcal{A}}^k = 0$$

$$E_{\mathcal{A}'}(v^k)(T) - E_{\mathcal{A}'}(v^k)(0) = 2\int_0^T \int_\Omega \chi_\omega^2(x) f_g^k \partial_t v^k \, dxdt \longrightarrow 0.$$

→ **Tool** : Localize the support of microlocal defect measures.

Remark : High frequency phenomena.