Dynamic Polymers

Yuri Bakhtin

joint work with Hong-Bin Chen and Liying Li

Courant Institute, NYU

July 2022
ICTS Bengaluru

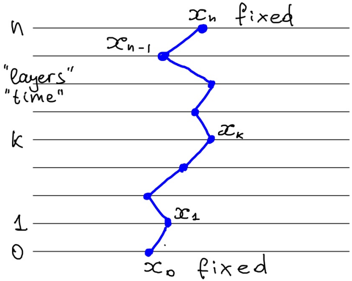
Keywords

- directed polymers
  - Gibbs distributions (LPP-type model if temperature = 0)
  - random potential
  - infinite volume/length (thermodynamic limit)
- dynamical viewpoint
  - ∞-dimensional stochastic gradient flows
  - invariant measures
  - monotonicity
  - one force – one solution principle (1F1S)

Initial motivation:

SPDEs, ergodic theory of stochastic Burgers / KPZ / heat equations

Point-to-point action minimizers (geodesics)



Fix endpoints x_0, x_n, minimize

$$E(x) = \frac{1}{2}\sum_{k=1}^{n} |x_k - x_{k-1}|^2 + \sum_{k=1}^{n} F_k(x_k)$$

$F_k(x) =$ i.i.d. in k, mixing in x.

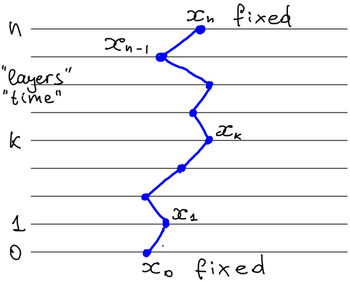
Bakhtin, Cator, Khanin (2014), Bakhtin (2016)

- $n \to \infty$, $\frac{x_n}{n} \to v$
- semi-infinite one-sided minimizers, mutual behavior
- applications to Burgers equation with random kick forcing

This LPP-type model is a "temperature $T = 0$" Gibbs measure
Directed polymers: $T > 0$ (for Burgers equation, viscosity> 0)

Finite volume/length directed polymers

point-to-point ("p2p") Gibbs polymer measure:



$$\frac{d\mu^{0,n}_{x_0,x_n,F}}{d\,\mathrm{Leb}}(x) = \frac{e^{-\beta E(x)}}{Z} \quad \text{on } \mathbb{R}^{n-1}$$

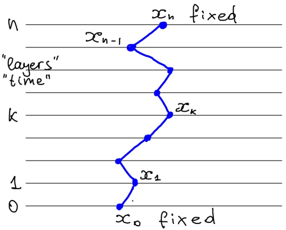
Z = normalizing constant
("partition function")

$$E(x) = \frac{1}{2}\sum_{k=1}^{n} |x_k - x_{k-1}|^2 + \sum_{k=1}^{n} F_k(x_k)$$

inverse temperature $\beta = 1/T = const$

as $\beta \to \infty$, $T \to 0$, $\quad \mu^{0,n}_{x_0,x_n,F}$ concentrates on minimizer

Infinite Volume Polymer Measures (IVPM)



$$\frac{d\mu^{0,n}_{x_0,x_n,F}}{d\,\text{Leb}}(x) = \frac{e^{-\beta E(x)}}{Z} \quad \text{on } \mathbb{R}^{n-1}$$

$$E(x) = \frac{1}{2}\sum_{k=1}^{n} |x_k - x_{k-1}|^2 + \sum_{k=1}^{n} F_k(x_k)$$

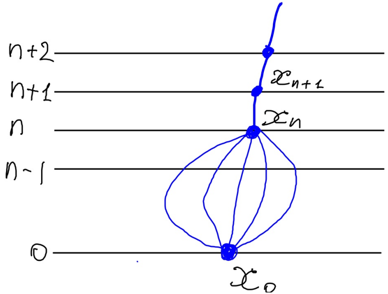
DLR (Dobrushin – Lanford – Ruelle) condition:

μ on $\mathbb{R}^{\mathbb{N}}$ is IVPM if:

for all n,

$$\mu\Big(\ \cdot\ \Big|\ x_n, x_{n+1}, \ldots\Big) = \mu^{0,n}_{x_0,x_n,F}$$

μ-a.s.



The potentials $F_k(x)$ we will work with

$F_k(x) =$ random potential, $k \in \mathbb{Z}$, $x \in \mathbb{R}$

- stationarity
- i.i.d. in k
- mixing in x (e.g., finite dependence range)
- some exponential moments
- some smoothness in x
- more (to be discussed)

Archetypal examples

- smooth Gaussian process with finite dependence range
- convolution of Poisson point process with a compact smooth kernel ("shot noise")

Infinite Volume Polymer Measures (IVPM)

Existence / uniqueness / description of all DLR measures is a basic problem of statistical mechanics

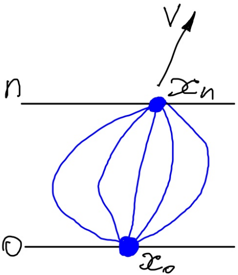
Theorem (with Liying Li, CPAM 2018)

Fix $v \in \mathbb{R}$. For a.e. F, if $x_n/n \to v$, then for all x_0,

$$\mu^{0,n}_{x_0,x_n,F} \Rightarrow \mu_{v,F}$$

$\mu_{v,F}$ is a unique IVPM concentrated on

$$S(v) = \{(x_k)_{k\in\mathbb{N}} : x_k/k \to v\}$$



- These measures, along with this and other convergence results are used to study Burgers equation with random kick forcing and prove unique ergodicity and, moreover, 1F1S on each ergodic component $\left\{\text{velocities } u : \mathbb{R} \to \mathbb{R} \text{ with average } v\right\}$

IVPMs for other polymer models: mostly an open problem

More general

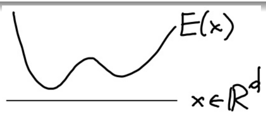
- energies? (not just $|x_i - x_{i-1}|^2$)
- dimensions? $d \geq 2$? (existence OK. Uniqueness?)
- potentials? (not i.i.d.)
- lattice models?

The latter is open except a couple of exactly solvable models:

- Log-gamma polymers
  [Georgiou, Rassoul-Agha, Seppäläinen, Yilmaz, 2016 – 2017]
- O'Connell–Yor polymers [Alberts, Rassoul-Agha, Simper, 2020]
  Also zero temperature.

PDE approach to Burgers and its generalizations (no polymers) in papers by Drivas, Dunlap, Graham, La, Ryzhik (2020 – 2022+)

Gibbs measure are invariant for stochastic gradient flow



Let $d < \infty$

$$E : \mathbb{R}^d \to \mathbb{R}$$

$$\frac{d\mu}{d\text{Leb}}(x) = \frac{e^{-\beta E(x)}}{Z}, \quad Z = \int_{\mathbb{R}^d} e^{-\beta E(x)} dx$$

Theorem (Kolmogorov, 1937, was he really first? I am not sure)

Gibbs $\iff$ invariant under

$$dX_t = -\nabla E(X_t)dt + \sigma dW,$$

where $\sigma = \sqrt{2\beta^{-1}} = \sqrt{2T}$

(can be seen from Fokker–Planck equation)

Archetypal example: $E(x) = \frac{x^2}{2}, \quad x \in \mathbb{R}.$

The only invariant density for $dX = -Xdt + \sqrt{2}dW$ is $\frac{e^{-x^2/2}}{\sqrt{2\pi}}$.

Study IVPMs as invariant for ∞-dim diffusion?

Define

$$\text{“}\, E(x) = \frac{1}{2}\sum_{k=1}^{\infty} |x_k - x_{k-1}|^2 + \sum_{k=1}^{\infty} F_k(x_k)\,\text{”}$$

and try to make sense of $\quad dX = -\nabla E(X)dt + \sigma dW$

$$\begin{aligned} dX_k &= (\Delta_k X + f_k(X_k))dt + \sigma dW_k, \quad k \in \mathbb{N}, \\ X_0 &\equiv 0. \end{aligned}$$

Discrete Laplacian:

$$\Delta_k x = x_{k-1} - 2x_k + x_{k+1} = 2\Big(\frac{x_{k-1} + x_{k+1}}{2} - x_k\Big)$$

$$f_k(x) = -\partial_x F_k(x)$$

$$
\begin{aligned}
dX_k &= (\Delta_k X + f_k(X_k))dt + \sigma dW_k, \quad k \in \mathbb{N} \\
X_0 &\equiv 0
\end{aligned}
$$

Theorem (with Hong-Bin Chen, PTRF, 2021)

The solution map $(\Phi^t_{F,W})$

- is well-defined on $\mathbb{L} = \left\{ x \in \mathbb{R}^{\mathbb{N}} : \ \|x\|_{\mathbb{L}} = \sup_{k \in \mathbb{N}} \frac{|x_k|}{k} < \infty \right\}$
- defines an order-preserving continuous RDS

Fix $v \in \mathbb{R}$. For a.e. realization of (F_k) or (f_k):

- $S(v) = \{x : \ x_k/k \to v\}$ is preserved
- $\exists!$ invariant measure on $S(v)$; mixing
- it coincides with unique IVPM on $S(v)$
- one force — one solution Principle (1F1S, synch) on $S(v)$.

Also, *ordering by noise* holds

Existing work on Gibbs and invariance in ∞-dim

In general, in dim=∞

- "Gibbs $\Longrightarrow$ invariant" typically holds and often not hard
- "invariant $\Longrightarrow$ Gibbs" is harder, still holds often (not always).

[Fritz,1982]
[V. Bogachev, Röckner, Wang 2004]
[Albeverio, Kondratiev, Röckner, Tsikalenko 2001]
[Funaki, Spohn 1997]
[Liggett's book on interacting particles 1985]
[Guionnet, Zegarlinski lectures on log-Sobolev inequalities 2003]
[many more]

If $F \equiv 0$ then synchronization or 1F1S

Let $X, Y \in S(v)$ (same slope) be two solutions

$$\begin{aligned} dX &= \Delta X\, dt + dW \\ dY &= \Delta Y\, dt + dW \\ d(X - Y) &= \Delta(X - Y)\, dt \end{aligned}$$

so $X(t) - Y(t) \to 0$ (slowly, no spectral gap), mixing follows.

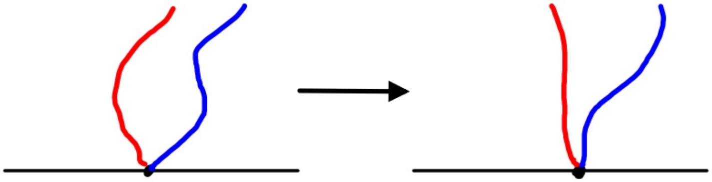
On each $S(v)$, a unique invariant distribution is the Gaussian RW with drift v (can be interpreted as GFF.)

If different slopes, convergence to nonzero fixed point of ΔX (a ray).

True in any dimension.

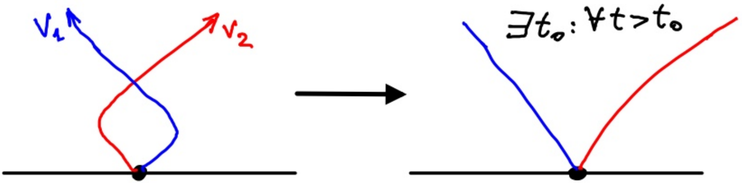
Order-preservation, aka monotonicity, comparison principle

$$dX_k = (\Delta_k X + f_k(X_k))dt + \sigma dW_k, \quad k \in \mathbb{N}$$

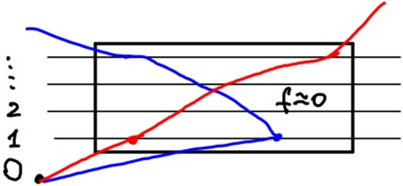


Results on synchronization for monotone systems:
Chueshov, Scheutzow, Flandoli, Gess, Butkovsky …
Our system does not seem to fit.

Ordering by noise

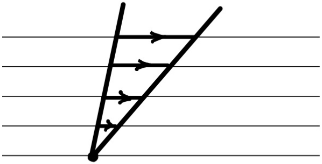


The mechanism (and an extra condition we need):



Synchronization and uniqueness of invariant measure

- IVPMs are invariant (take limit of finite-dim result).
- Ordering by noise $\Longrightarrow$ in a stationary regime, if subject to the same noise $X_{v_1} \ll X_{v_2}$ (coordinate-wise) for $v_1 < v_2$. So for a fixed v and each $k \in \mathbb{N}$ possible values of $X_{v,k}$ occupy a finite interval or one point.
- Dynamics in $S(v_1)$ is conjugated to dynamics in $S(v_2)$ by a shear:



"shear-invariance"

- There are uncountable many $v \in \mathbb{R}$, so probability that possible values of $X_{v,k}$ occupy a whole interval must be zero.

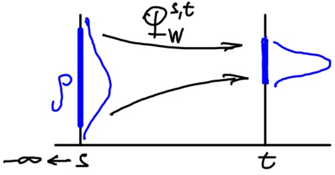
The rigorous version of this argument is based on pullback sample measures (Le Jan 1985, Ledrappier, Young 1988)

Pullback sample measures

If ρ is invariant for the Markov process generated by RDS, then the pullback limits are well-defined:

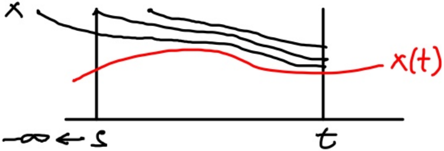
$$\rho_{W,t} = \lim_{s \to -\infty} \rho(\Phi_W^{s,t})^{-1}$$

limit of the pushforward of ρ under the solution map $\Phi_W^{s,t}$.



- weak 1F1S: $\rho_{W,t} = \delta_{x(t)}$ for some $x(t)$
- true 1F1S: for all $x \in S(v)$, $\lim_{s \to -\infty} \Phi_W^{s,t} x = x(t) \in S(v)$
  $x(t)$ is a global (defined for all all $t \in \mathbb{R}$) stationary solution:

$$\Phi_W^{s,t} x(s) = x(t)$$



Corollary:

- Inv. distr on $S(v)$ is unique (has to be IVPM)
- mixing

Conjecture

In broad generality (non-quadratic interactions, higher dim, non-i.i.d. potentials):

For all $v \in \mathbb{R}$, $\exists!$ IVPM $\in S(v)$, coincides with unique invariant measure, depends on v continuously.

Are available methods good enough?
In $d \geq 2$ with quadratic potential, existence is similar to $d = 1$.
Usual strategy for uniqueness: irreducibility + regularity.
Not elliptic enough?