Improving

Continuous Case

Sphere

$$\widehat{d\sigma}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi}\, d\sigma(x)$$

Thm

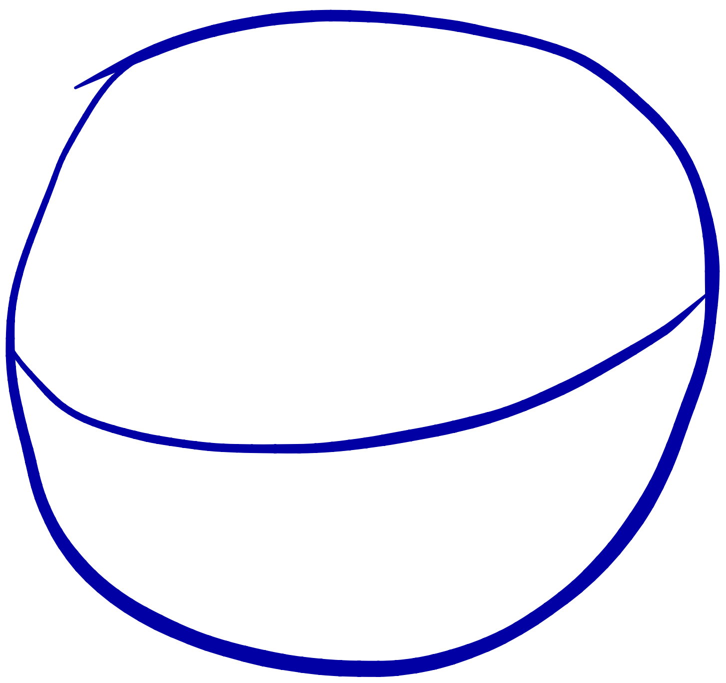
$$\widehat{d\sigma}(\xi) = c\,|\xi|^{-\frac{d-2}{2}} \underbrace{J_{\frac{d}{2}-1}(2\pi|\xi|)}_{\text{Bessel } f^{\underline{n}}}$$

$$\approx |\xi|^{-\frac{d-1}{2}} \cos(2\pi|\xi| + c_d)$$

$|\xi| \to \infty$

(we will see periodic Bessel $f^{\underline{ns}}$ later)

A heuristic approach to this estimate



Thm (Littman, Strichartz)

$$Af = f * d\sigma$$

maps $L^{\frac{d+1}{d}} \to L^{d+1}$, $d = 2, \dots$

Pf of this is delicate complex interpolation; which has extensions to e.g. hypercubes

High / Low approach to $L^{\frac{d+1}{d},1} \to L^{d+1,\infty}$.

i.e. $F, G \subset [0,1]^d$, show

$$\langle A 1_F, 1_G \rangle \lesssim \left(|F| \cdot |G|\right)^{\frac{d}{d+1}}$$

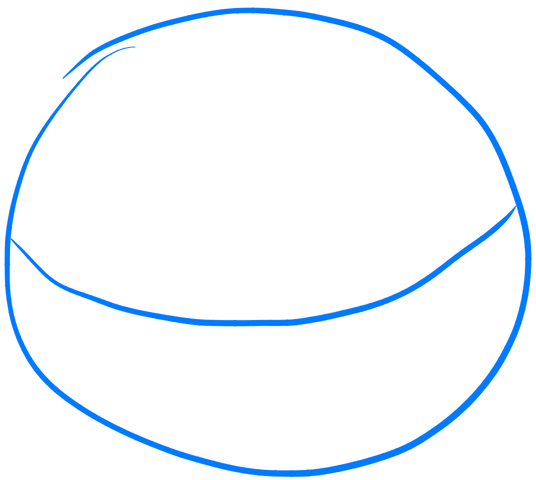
$N > 1$, determined by F, G

$$A 1_F = Hi + Lo$$

$$Lo = \varphi_{1/N} * A 1_F$$

Low pass filter of spatial scale 1/N

$$\langle Lo , 1_G \rangle \leq$$



$$\langle Hi , 1_G \rangle \leq$$

Square Root Avg

$$A f(x) = \int_0^1 f(x-y) \frac{dy}{\sqrt{y}}$$

$y = u^2$

$$= c \int_0^1 f(x-u^2)\, du$$

The Fourier transform of measure is

$$\int_0^1 e^{-i\xi y^2} dy$$

$$\left| \int_0^1 e^{i\lambda y^2}\, dy \right| \lesssim \frac{1}{\sqrt{|\lambda|}}$$

(stationary phase arg.)

$$\int_0^1 e^{i\lambda y^2}\, dy = c \int_0^1 e^{iu} \frac{du}{\sqrt{\lambda u}} \le \frac{1}{\sqrt{\lambda}}$$

$u = \lambda y^2$

$du = 2\lambda y\, dy$

Thm $A : L^{4/3} \longrightarrow L^4$

Proof

$\langle A 1_F , 1_G \rangle$

A much more interesting Improving Estimate

local spherical maximal $f^{\underline{n}}$

$$Mf = \sup_{1 \le t \le 2} |A_t f|$$

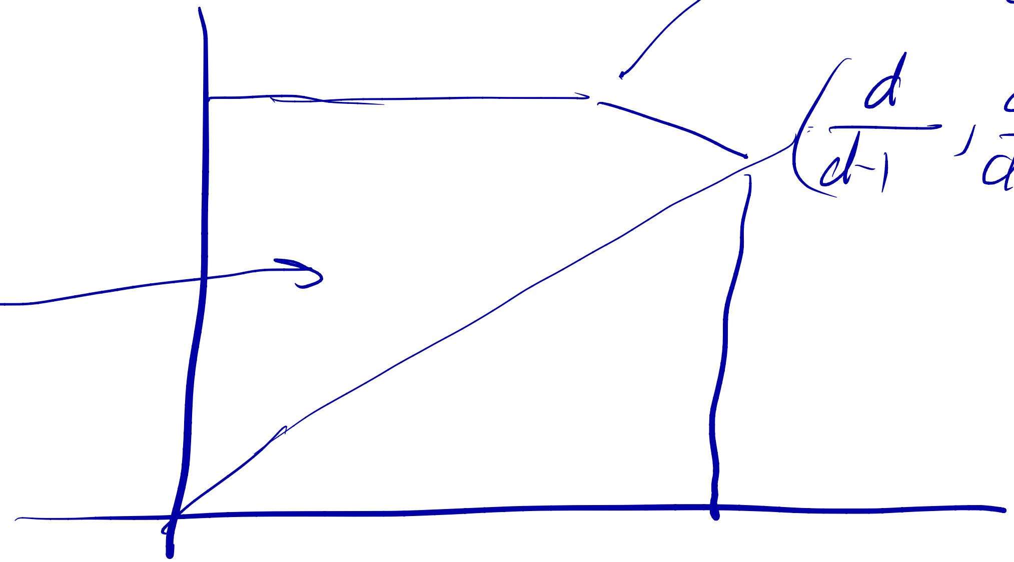
avg over sphere of radius t

Thm (Schlag Sogge)

$M : L^p \to L^q$

Strichartz corner

$\left(\frac{1}{p}, \frac{1}{q}\right)$



$\left(\frac{d}{d-1}, \frac{d}{d-1}\right)$

$\frac{d}{d-1}$

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 131, Number 5, Pages 1433–1442
S 0002-9939(02)06781-3
Article electronically published on September 19, 2002

ENDPOINT ESTIMATES FOR THE CIRCULAR MAXIMAL FUNCTION

SANGHYUK LEE

(Communicated by Andreas Seeger)

Improving Inequalities

$$\Longleftrightarrow \quad \text{Smoothing inequalities}$$

with spheres

$$A : L^2(\mathbb{R}^d) \longrightarrow W^{\frac{d-1}{2}}(\mathbb{R}^d)$$

discrete versions : 'Arithmetic' Sobolev space.

Improving Inequalities imply 'sparse bounds'.

Additional facts needed are

- L^2 - continuity

$$\| A f(x-t) - A f(x) \|_{L^2} \lesssim |t|^{\delta} \|f\|_2$$

- A bit of Calderon-Zygmund theory

[Submitted on 28 Feb 2017 (v1), last revised 3 Dec 2018 (this version, v6)]

Sparse Bounds for Spherical Maximal Functions

Michael T. Lacey

We consider the averages of a function f on $\mathbb{R}^n$ over spheres of radius $0 < r < \infty$ given by $A_r f(x) = \int_{\mathbb{S}^{n-1}} f(x - ry)\, d\sigma(y)$, where σ is the normalized rotation invariant measure on $\mathbb{S}^{n-1}$. We prove a sharp range of sparse bounds for two maximal functions, the first the lacunary spherical maximal function, and the second the full maximal function.

$$M_{lac} f = \sup_{j \in \mathbb{Z}} A_{2^j} f, \qquad M_{full} f = \sup_{r>0} A_r f.$$

The sparse bounds are very precise variants of the known L^p bounds for these maximal functions. They are derived from known L^p-improving estimates for the localized versions of these maximal functions, and the indices in our sparse bound are sharp. We derive novel weighted inequalities for weights in the intersection of certain Muckenhoupt and reverse Hölder classes.

<u>Def</u> A family $\mathcal{Q}$ of cubes are <u>sparse</u> if for all $Q \in \mathcal{Q}$ there is a set $E_Q \subset Q$ s.t.

a) $\{E_Q\}$ are disjoint

b) $|E_Q| > \frac{1}{100} |Q|$

Q

Def A (p,q) sparse bound for operator T holds if for all cptly supported, bounded f, g there is sparse $\mathcal{Q} = \mathcal{Q}_{f,g}$

$$\langle Tf, g\rangle \leq \sum_{Q \in \mathcal{Q}} \langle f\rangle_{Q,p} \langle g\rangle_{Q,q} |Q|$$

$$\langle \phi\rangle_{s,Q} = \left[\frac{1}{|Q|}\int_Q |f|^s\right]^{1/s}$$

- Sparse bounds immediately imply wtd bounds $\omega \in A_p$

- They can be interpolated

- Frequently enoug to take f, g indicator

- Proofs reduce to a simple recursion

- Given Q_0 there are sparse children $Q_1, Q_2, \ldots \subset Q_0$ s.t.

$$\langle T(f 1_{Q_0}), g 1_{Q_0} \rangle \cdot K \langle f \rangle_{Q_0,p} \langle g \rangle_{Q_0,q} |Q_0|$$

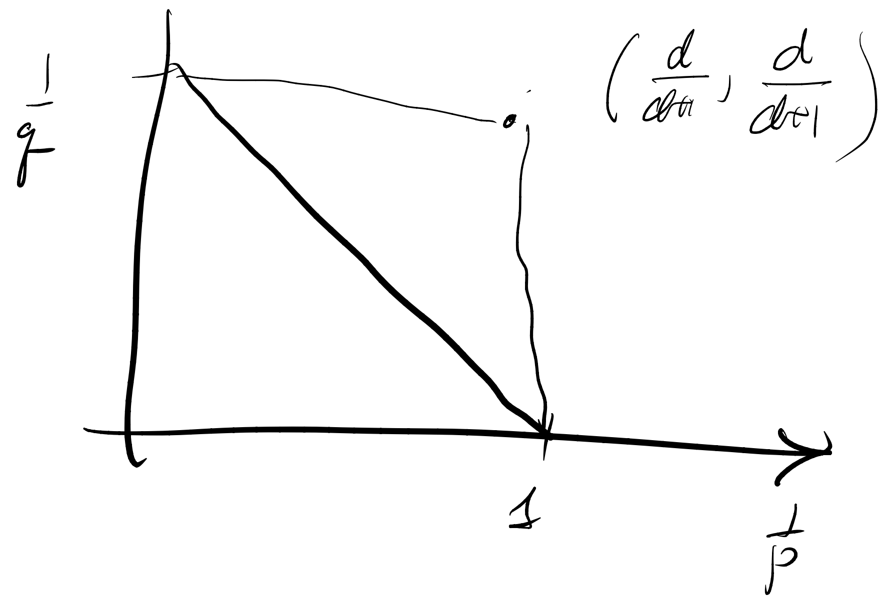
$$+ 1 \sum_j \langle T(f 1_{Q_j}), g 1_{Q_j} \rangle$$

K abosolute

Lacunary Spherical Maximal F^n

$$A^* f = \sup_{k \in \mathbb{Z}} A_{2^k} f \quad \text{satisfies} \quad (p, q)$$

sparse bound for $\left(\frac{1}{p}, \frac{1}{q}\right) \in T$



Argument for the $\left(\frac{d}{d+1}, \frac{d}{d+1}\right)$ corner

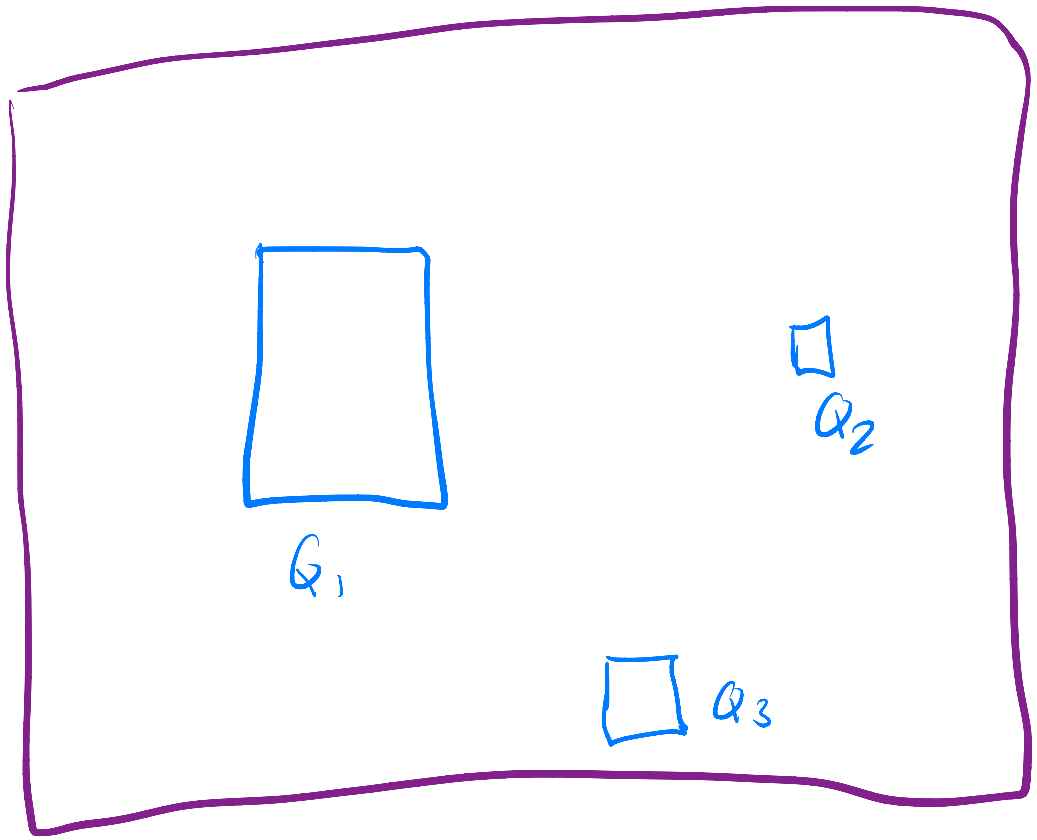
Recursive step for $f = 1_F$, $g = 1_G$, $F, G \subset Q_0$

Set $Q_1, Q_2, \ldots$ to be maximal

dyadic cubes in Q_0 s.t.

$$\langle f \rangle_{3Q_j} > K \langle f \rangle_{Q_0}$$

Let $\tau : Q_0 \setminus \bigcup_{j}^{\infty} Q_j$ be msbb map into $\{2^k : k \in \mathbb{Z}\}$



It suffices to show

$$\langle A_{\tau} f, g \rangle \lesssim \langle f \rangle_{Q_0}^{3/4} \langle g \rangle_{Q_0}^{3/4} |Q_0|$$

$$N = \left(|G|/|F|\right)^{\frac{d}{d+1}}$$

$$L_0 = A_{\tau/N} f \lesssim N\, Mf(x)$$

Then $\langle L_0, g\rangle \lesssim N \langle f\rangle_{Q_0} \langle g\rangle_{Q_0} |Q_0|$

High term