What is Chaos?

The Importance of being Nonlinear

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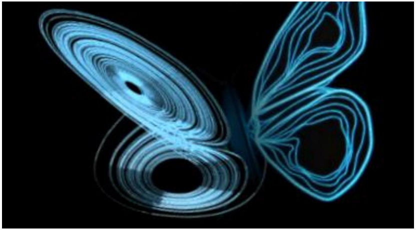


homepage: https://web.iisermohali.ac.in/dept/physics/Sudeshna_Sinha.html

- Unresonable Effectiveness of Classical Mechanics: Mathematically Rigorous Toolkit to understand phenomena over an extraordinary range of scales

- Concept of Chaos : a new frontier in Classical Mechanics

Chaos theory has caused a fundamental reassessment of the way in which we view the classical physical world

- Eternal quest : Forsee the Future!
- One of the beautiful things about classical physics is that it's deterministic.
- If you know all the properties of a system (where "system" can mean anything from a single particle in a box to weather patterns on the Earth) and you know the laws of physics, then you can perfectly predict the future.
- Concept of a Clockwork Universe

Surprise : Large range of seemingly simple natural processes can exhibit enormously complex behavior, often appearing as if they were evolving under random forces rather than deterministic laws

Some completely deterministic systems displayed exponential sensitivity to initial conditions and thus were essentially unpredictable

- Necessitated a paradigm shift in understanding classical systems
- Emergence of a "New Science": CHAOS Theory

Does God Play Dice?

First hints that nature can be both deterministic and unpredictable goes way back to the 1800s, when the king of Sweden offered a prize to anyone who could solve the so-called three-body problem

French mathematician Henri Poincare won the prize without actually solving the problem

Instead of solving it, he wrote about the problem, describing all the reasons why it couldn't be solved

- One of the most important reasons he highlighted was how small differences in the beginning of the system would lead to big differences at the end: i.e. the present determines the future, but approximate present does not approximately determine the future

Edward Lorenz was studying a simple model of the Earth's weather on an early computer

When he stopped and restarted his simulation, he ended up with wildly different results

What he found was a surprising sensitivity to the initial conditions: One tiny rounding error, no more than 1 part in a million, would lead to a completely different behavior of the weather in his model.

What Lorenz essentially discovered was chaos.

Dynamical Systems: deterministic mathematical prescription for evolving the state of a system forward in time

FLOWS : Continuous Time

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$$

MAPS : Discrete Time

$$\mathbf{X}_{n+1} = \mathbf{F}(\mathbf{X}_n)$$

State $\mathbf{X}$: a point in N-dimensional Phase Space

For any initial state of the system we can in principle solve the equations to obtain the future system of the system : ORBIT, TRAJECTORY , FLOW

N : dimension of the dynamical system ; degrees of freedom

Question : How large does N have to be in order for Chaos to be possible ?

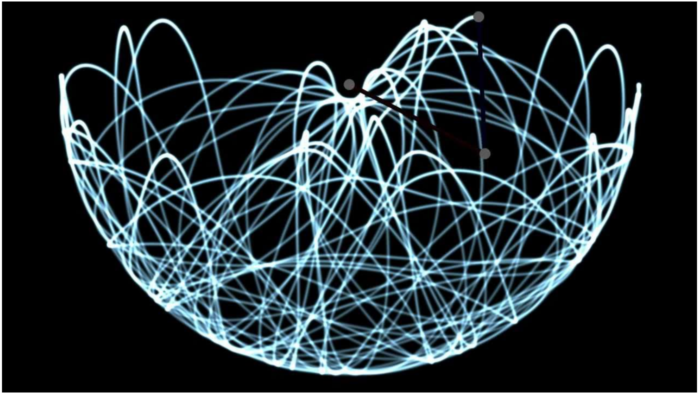
For a system whose dynamics is described by N first-order autonomous Ordinary Differential Equations, the smallest system capable of displaying chaos is $N = 3$.

$$N \geq 3$$

is the necessary condition for Chaos

Long exposure of Double Pendulum exhibiting chaotic motion (tracked with an LED)

(Image credit: Wikimedia Commons/Cristian V.)



For maps $\mathbf{X}_{n+1} = \mathbf{F}\mathbf{X}_n$

where $\mathbf{X} = \{X_1, X_2 \ldots X_N\}$

If invertible   i.e. if its inverse F^{-1} exists and is unique

Can get chaos for $N \geq 2$

If the Map F is non-invertible

i.e. if its inverse F^{-1} does not exist or is not unique

Can get chaos for $N \geq 1$

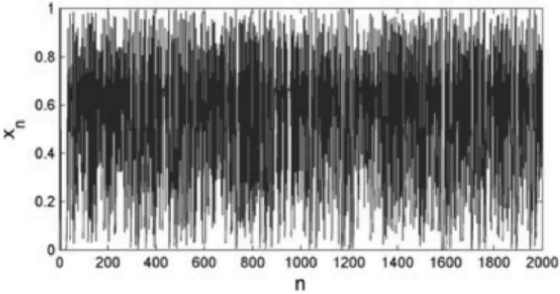
One dimensional non-invertible maps then are the simplest systems capable of chaotic motion

- Convenient starting point
- Surprisingly large proportion of the phenomena encountered in higher dimensional systems is already present in 1-d maps

Examples of simple one-dimensional nonlinear systems that can display chaos:

- Logistic Map: $x_{n+1} = 4\, x_n(1 - x_n)$
- Binary Shift Map: $x_{n+1} = 2x_n \quad (\bmod 1)$
- Tent Map : $x_{n+1} = 1 - 2|x_n - \frac{1}{2}|$

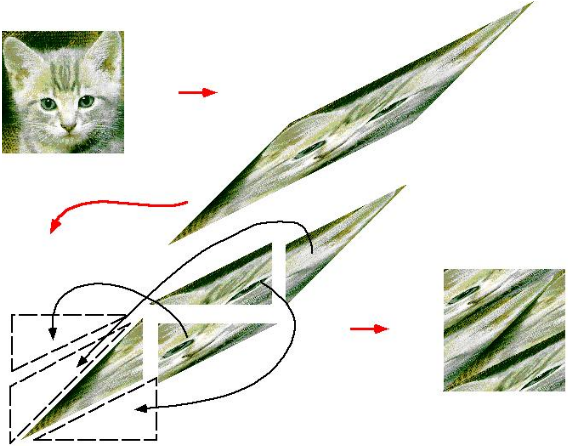
Action of the Maps: Stretch – Fold Operation



Arnold's Cat Map : invertible area-preserving two-dimensional map

$$x_{n+1} = 2x_n + y_n \mod 1$$
$$y_{n+1} = x_n + y_n \mod 1$$

Courtesy: University of Maryland



Dynamics of Iterated maps : produced by putting a number through a function then taking the result and putting it through the function again, then repeating

- Convenient starting point
- Surprisingly large proportion of the phenomena encountered in higher dimensional systems is already present in 1-d maps

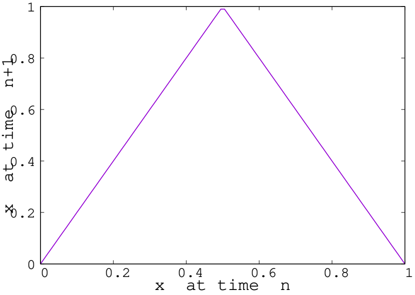
PIECEWISE LINEAR ONE-DIMENSIONAL MAPS

First example : the TENT map

$$x_{n+1} = F(x_n) = 1 - 2|x_n - 1/2|$$

For $x_n < 1/2 : x_{n+1} = 2x_n$

For $x_n > 1/2 : x_{n+1} = 2(1 - x_n)$



x-axis : x_n

y-axis : $x_{n+1} \; = \; F(x_n) \; = \; 1 - 2|x_n - 1/2|$

- Initial conditions that are negative move off to $-\infty$

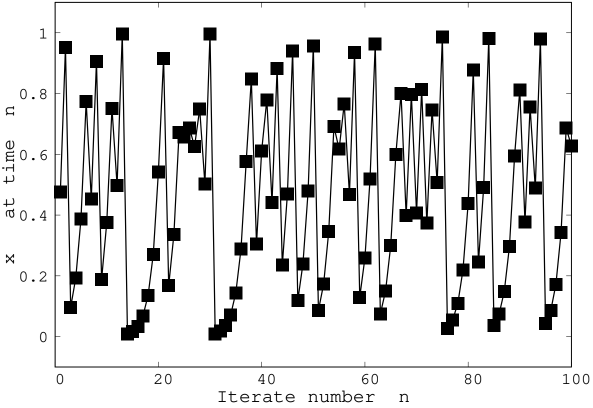
  Doubling their distance from the origin on each iterate

- If $x_0 > 1$, then $x_1 < 0$ and the subsequent orbit moves off to $-\infty$

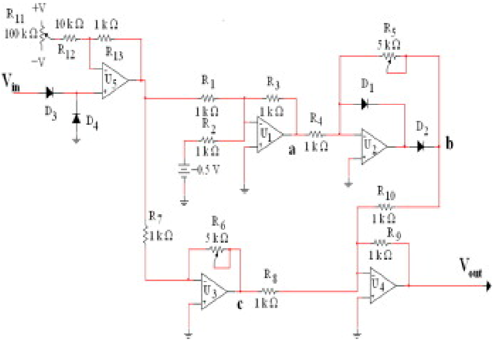
- For $x \in [0, 1]$, we have $0 \leq 1 - 2|x - 1/2| \leq 1$

  So all orbits are bounded in the interval

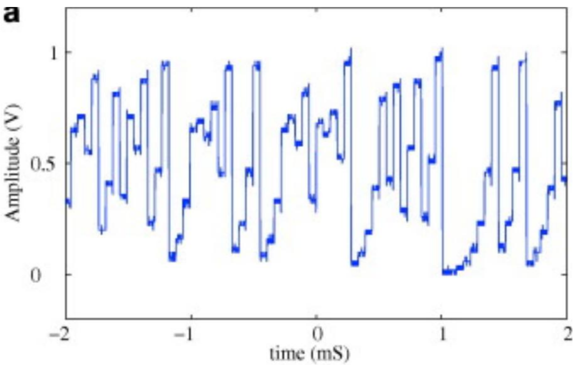
Time evolution of the state x



A simple electronic circuit realization of the tent map



Campos-Canton *et al*


a
Amplitude (V)
1
0.5
0
−2
−1
0
1
2
time (mS)

Map has 2 actions :

- Interval uniformly stretched to twice its original length
- Stretched interval is folded in half – so that the folded line segment is now contained in the original interval

STRETCHING and FOLDING :

Following a point under the map —

- Stretching leads to exponential divergence of nearby trajectories (by a factor of two on each iterate)

- Folding : keeps orbit bounded

  Folding makes it non-invertible : since it results in 2 different values of x_n mapping the same x_{n+1}

To demonstrate sensitive dependence on initial conditions:

Compose the map m times with itself to obtain F^m:

$$F^m(x) = F(F^{m-1}(x)) = F(F(F(\ldots(F(x)))\ldots))$$

with $F^1(x) = F$

Thus $x_{n+m}(x) = F^m(x_n)$

For instance, $x_{n+2} = F(F(x_n)) = F^2(x_n)$

So, in case of TENT MAP:

$$x_{n+2} = F(F(x_n)) = 1 - 2|\ (1 - 2|x_n - 1/2|)\ - 1/2|$$

- If $x_n = 0, \ \frac{1}{2}, \ 1$

  Then 2 applications of F yield $x_{n+2} = 0$

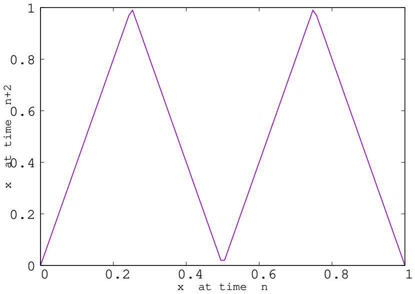
- If $x_n = \frac{1}{4}, \ \frac{3}{4}$

  Then 2 applications of F yield $x_{n+2} = 1$

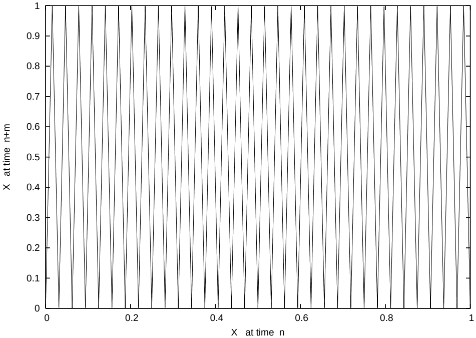
Linear in between

x-axis : x_n

y-axis : $x_{n+2} = F(F(x_n)) = F^2(x_n)$



x-axis – Initial state x_n
y-axis – m^{th} iterate starting from initial state x_n, i.e. state x at time $n + m$: $x_{n+m} = F^m(x_n)$



Sixth Iterate : i.e. $m = 6$

Initial condition lies within $\frac{1}{2^m}$ of some point, the x_m can lie anywhere in the interval $[0, 1]$

Absolutely no knowledge of the location of the future points after times $> m$ even though you know the initial point with reasonable precision

Exponential Sensitivity of chaotic orbits to small changes in initial conditions

Another simple map:

$$x_{n+1} = 2x_n \mod 1$$

Can be regarded as a map on a circle

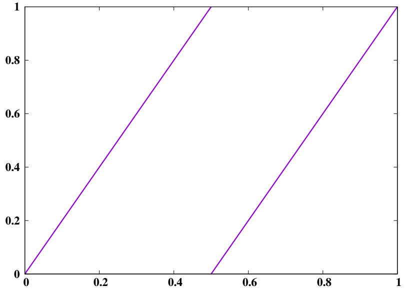
Modulo 1 : makes x like an angle variable : x increasing from 0 to 1 corresponds to one circuit around the circle

Action of the map: Stretch - Twist - Fold operation

Chaotic separation of nearby points (by a factor of 2) on each iterate of this map

x-axis – Initial state x_n

y-axis – Subsequent iterate starting from initial state x_n, i.e. state x at time $n+1$: $x_{n+1} = F(x_n) = 2x \mod 1$



Alternate way of viewing: Bernoulli Shift

Say we represent the initial condition x_0 as a binary decimal

$$x_0 = 0.a_1 a_2 a_3 \cdots \equiv \sum_{j=1}^{\infty} 2^{-j} a_j$$

where each digit a_j is either 0 or 1

Then next iterate is obtained by setting the first digit to 0 and moving the decimal point to the right :

$$x_1 = 0.a_2 a_3 a_4 \ldots$$

$$x_2 = 0.a_3 a_4 a_5 \ldots$$

and so on

So digits initially far right – having only very slight influence on the initial value of x eventually become first digit

Small change in initial condition

Say : a_{10} is changed from 0 to 1,

i.e. a change in x_0 by 2^{-10}

At time $n = 9$ makes huge change in x_n

Periodic orbit of a map

If the orbit successively cycles through p distinct points:

$x_0, x_1, \ldots x_{p-1}$

Such that $x_i \neq x_j$ if $i \neq j$

For instance if $x_0 = \frac{2}{3} = 0.10101010\ldots$

$$x_1 = 0.01010101\cdots = \frac{1}{3}$$
$$x_2 = 0.10101010\cdots = \frac{2}{3}$$

Period-2 cycle (x_0, x_1), namely $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3} \ldots$

General period of order p arises from initial conditions of the form:

$0.a_1a_2a_3\ldots a_pa_1a_2\ldots$

We ask : How many different initial conditions are there that return to themselves after p iterations ?

There are 2^p distinct sequences

2^p intersections with $x_{n+p} = x_n$

A point on the period p orbit is a fixed point of the composed map F^p as

$$y = F^p(y)$$

2 of these : fixed points 0 and 1

For $p = 2$: $\quad 2^p \ = \ 2^2 \ = \ 4$

i.e. there are 4 intersections of the diagonal line $x_{n+2} = x_n$ with $F^2(x_n)$

Intersections at $0, \ \ \frac{1}{3}, \ \ \frac{2}{3}, \ \ 1$

If p is a prime number then all these $2^p - 2$ points lie on orbits of period p

So number of distinct periodic orbits:

$$N_p = \frac{2^p - 2}{p}$$

This number gets large rapidly:

e.g.

For $p = 11$: $N_{11} = 186$

For $p = 17$: $N_{17} = 7710$

If p is not a prime number :

Has integer factors $p_1, p_2, \dots$ $\quad (p = p_1^{n_1} p_2^{n_2} \dots)$

Then some of these $2^p - 2$ points will lie on orbits of lower period $p_1, p_2 \dots$

e.g. For $p = 4$, $\quad 2^p - 2 = 14$

Out of which 2 points belong to $p = 2$

But here too for large p : $\quad N_p \sim \frac{2^p}{p}$

Question: What is the stability of these orbits?

Say we have a periodic orbit of period p :

$x_0, x_1, x_2 \dots x_{p-1}, x_p \dots$ where $x_0 = x_p$

So for each point $x_j = F^p(x_j)$ for $j = 0, 1, \dots, p-1$

Say we take a slightly different initial condition

$x_0 \to x_0 + \delta_0$

As a result : $x_0 + \delta_p = F^p(x_0 + \delta_0)$

Since δ is small — Taylor expand to first order to obtain

$$\delta_p = \lambda_p \delta_0$$

where

$$\lambda_p = \frac{dF^p(x)}{dx}|_{x=x_0} = \frac{dx_{n+p}}{dx}|_{x=x_0}$$

$$\frac{dx_{n+p}}{dx} = \frac{dx_{n+1}}{dx} \quad \frac{dx_{n+2}}{dx_{n+1}} \dots \frac{dx_{n+p}}{dx_{n+p-1}} = F'(x_n)F'(x_{n+1}) \dots$$

where $F' = \frac{dF}{dx}$

So λ_p is same for all points on the orbit (just a cyclic permutation of the terms in the product)

Thus the deviation from the periodic orbit grows (if $|\lambda_p| > 1$) or shrinks (if $|\lambda_p| < 1$) by a factor $|\lambda_p|$ on each circuit around the periodic circuit

$|\lambda_p|$: STABILITY COEFFICIENT

If $|\lambda_p| > 1$: UNSTABLE

If $|\lambda_p| < 1$: STABLE

If $|\lambda_p| = 0$: SUPERSTABLE

For Tent map, $2x$ modulo 1 map : all orbits unstable

As $|F'| = 2$ always

So $|\lambda_p| = 2^p > 1$

All orbits unstable : Dense Set of Unstable Periodic Orbits

Given x and ϵ however small : around any point x, there is at least one periodic point in the interval $[x - \epsilon, x + \epsilon]$

One point of period p in each interval $[2^{-p}(m-1), 2^{-p}m]$, $m = 1, 2, \ldots 2^p$

If $p > \ln(1/\epsilon)/\ln 2$, i.e. $2^{-p} < \epsilon$ at least one point of the period p orbit in the small interval around x

Periodic Points are Countably infinite

Measure zero set

Implies : if one chooses an initial condition x_0 at random in $[0, 1]$ then the probability of x_0 being a periodic point is zero

Chaotic orbits are TYPICAL

- The set of rational numbers is infinite, their measure is 0
- In contrast, the irrational numbers from zero to one have a measure equal to 1
- Hence, the measure of the irrational numbers is equal to the measure of the real numbers – in other words, “almost all” real numbers are irrational numbers.

Consider a histogram of the fraction of time a finite length orbit originating from a typical initial point falls in N bins of equal size along the x axis

As N and orbit length $\to \infty$, this probabilty $\to \frac{1}{N}$

Can define a Natural Invariant Density $\rho(x)$

Fraction of time typical orbits spend in an interval $[a, b]$ is given by $\int_b^a \rho(x)dx$

For the tent map this is Uniform : $\rho(x) = 1$ in $[0, 1]$

$$x_{n+1} = F(x_n) = r\, x_n(1 - x_n)$$

Parameter : r $\quad x \geq 0$

Simple idealised model of seasonal variations of a population

Relevant to seasonally breeding population in which generations do not overlap

Temperate zone insects, including many economically important crop and orchard pests, are of this kind

Simple mathematical models with very complicated dynamics, Robert May, *Nature* 1976

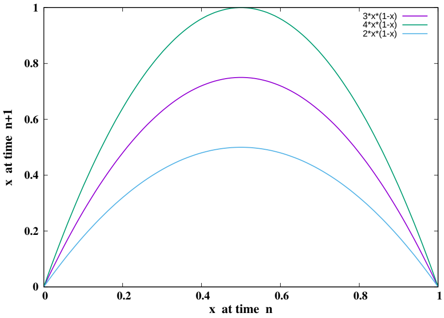
Assuming constant conditions : the population of a year n determines the population of the next year $n+1$

i.e. $x_{n+1} = r\, x_n$: namely $x_n = r^n\, x_0$

- If $r < 1$: Extinction as $x_n \to 0$ as $n \to \infty$
- If $r > 1$: Exponential Growth as $x_n \to \infty$ as $n \to \infty$

But competition for food and death due to overcrowding must modify the growth term

Logistic Map $x_{n+1} = r\ x_n(1 - x_n)$



Nonlinearity Parameter $r = 2, \ 3, \ 4$

Unimodal Map : One-humped Map – with one maximum, in the interval

The map function has a maximum at $x = \frac{1}{2}$ and has value $F(\frac{1}{2}) = \frac{r}{4}$

Thus for $0 \leq r \leq 4$ if x_n is in $[0, 1]$ then so is x_{n+1}

So orbit bound in the interval $[0, 1]$

- Outside the interval $[0 : 1]$:
- If $x < 0$, then $F(x) < 0$: $\quad x$ remains negative
- If $x > 1$, next iterate is negative (as $F(x) < 0$ for $x > 1$), and from then on all iterates are negative

Consider $r \leq 4$ and $x \in [0, 1]$

Existence of Fixed Points

Fixed Point implies $x_{n+1} = F(x_n) = r\ x_n(1 - x_n) = x_n$

So there are 2 Fixed Points which are solutions of $x = r\ x\ (1 - x)$

$$x^* = 0 \quad \text{and} \quad x^* = 1 - \frac{1}{r}$$

For $r \geq 1$: both fixed points lie in the interval [0:1]

Stability of the Fixed Ponts

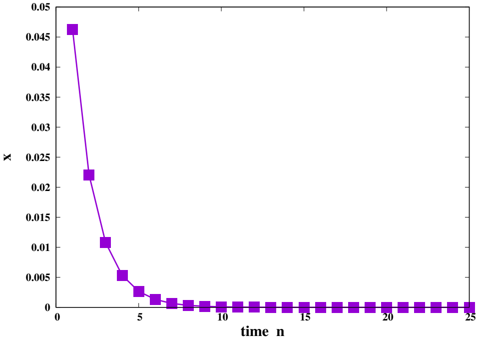
For stability, examine $F'(x^*) \;=\; r - 2r\; x^*$

- For $x^* = 0$:

$$F'(0) \;=\; r$$

Fixed point is STABLE if $|F'| < 1$ i.e. when $r < 1$

Time series when $r < 1$



$r = 0.5$ After transcience $x^* = 0$

- For $x^* = 1 - 1/r$:

$$F'(1 - \frac{1}{r}) = r - 2r\ (x^*)\ = 2 - r$$

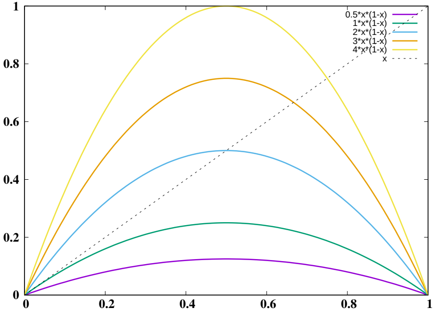
- Fixed point is STABLE if $|F'| < 1$ i.e. $|2 - r| < 1$

  So stability condition is satisfied for $1 < r < 3$

- Superstable when $|F'| = 0$, i.e. $r = 2$

- F' is positive for $r < 2$ and negative for $r > 2$

Graphical Solution of the Fixed Point



Solid lines: x_{n+1} for parameter $r = 0.5, 1, 2, 3$ and 4
Dashed: 45^0 line

Intersections: Fixed Point solutions
Slope at intersections: Stability of the Fixed Points

Fixed point $1 - \frac{1}{r}$ is the attractor in the parameter range $[1:3]$

Any initial x_0 satisfying $0 < x_0 < 1$ approaches the fixed point attractor

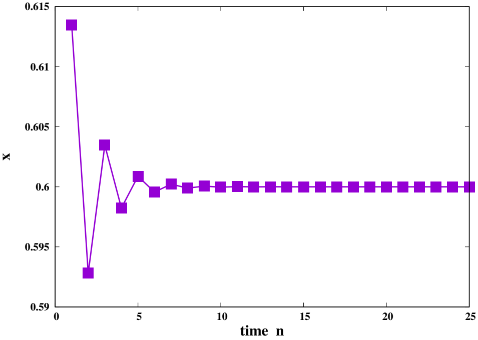
No other periodic point

$[0, 1]$ is the Basin of Attraction of the fixed point

Attracts all points in $[0, 1]$ except for a set of measure zero

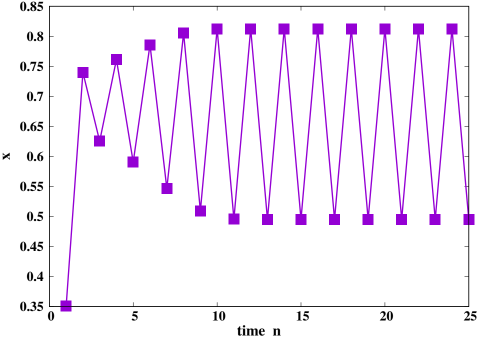
Roughly the non-attracted set has zero length : essentially the unstable fixed point 0, and its pre-image 1

Time series when $1 < r < 3$



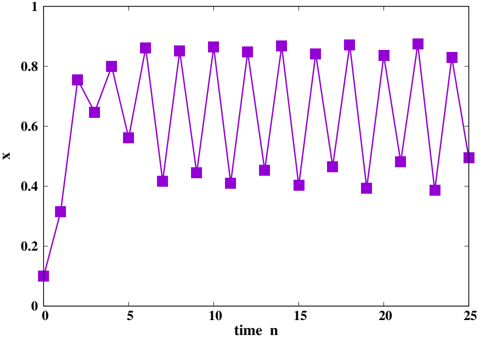
$r = 2.5$ After transcience $x^* = 3/5$

Time series when $r > 3$



$r = 3.25$ After transience Period 2-cycle: $x_0 \rightarrow x_1 \rightarrow x_0 \ldots$

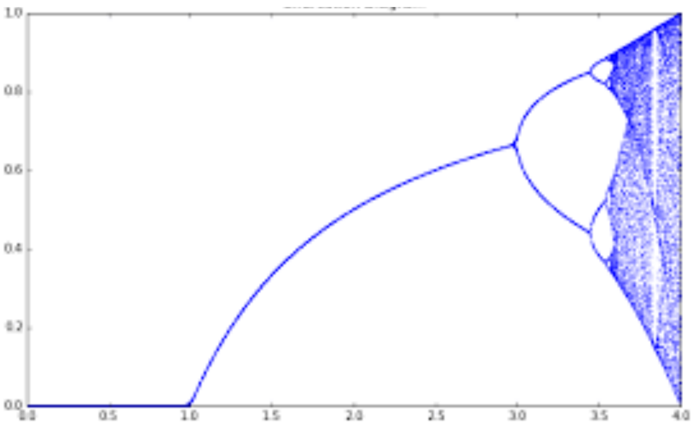
Time series when $3.449\cdots < r < 3.544\ldots$



$r = 3.5$

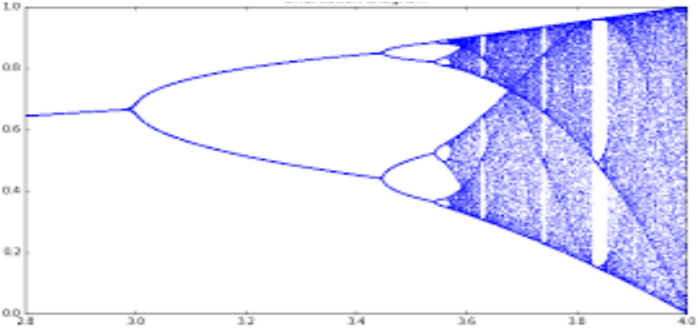
After transience Period 4-cycle: $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_0 \ldots$

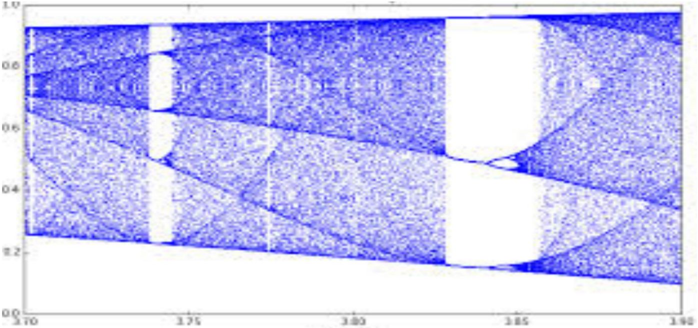
BIFURCATION DIAGRAM



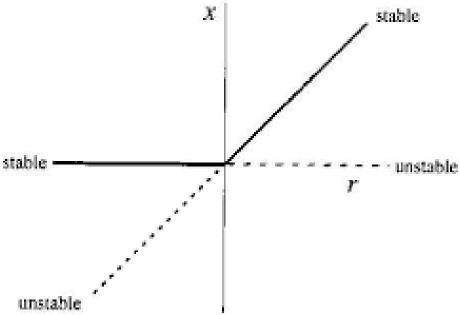
BIFURCATIONS : Qualitative change in the dynamics which occurs as a system parameter varies

The bifurcation diagram is obtained by plotting, as a function of parameter r, a series of values for x_n obtained by starting with a random value x_0, iterating many times, and discarding the first points corresponding to values before the iterates converge to the attractor (so-called "transience").


1.0
0.8
0.6
0.4
0.2
0.0
2.8
3.0
3.2
3.4
3.6
3.8
4.0



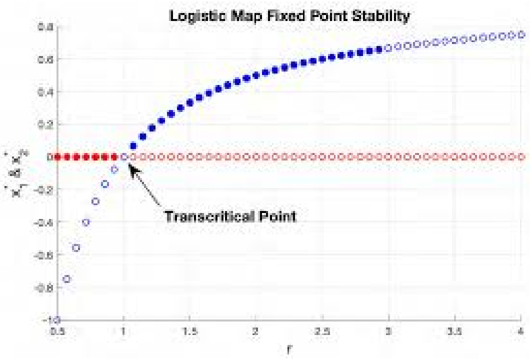
Transcritical Bifurcation



Two solutions exchange stability

Unstable solution : dashed line ; Stable solution : Solid line

Transcritical Bifurcation in Logistic Map at $r = 1$



Two solutions $x^* = 0$ and $x^* = 1 - \frac{1}{r}$ exchange stability at $r = 1$

How does one get the period 2 orbit from period 1?

Creation of Two New Fixed Points at the Bifurcation Point

- Precisely at the point when Period 1 becomes unstable Period 2 is created
- Period 2 is stable when created

For $1 < r < 3$ the stability coeffcent $-1 < \lambda_1 = F'(x^*) < 1$
$\lambda_1 = -1$ at $r = r_0 = 3$ (i.e. at the birth of period 2)

What is the fate of the period 2 orbit as nonlinearity parameter r increases?

Period $2 \ \rightarrow$ Period 4

r_1 : Value of parameter r when Period 2 crosses to Period 4

In general:

r_m : Value of parameter r when Period 2^m crosses to Period 2^{m+1}

- $r_0 = 3$
  Period $1 = 2^0$ orbit loses stability
  Birth of a stable Period $2 = 2^1$ orbit
- $r_1 = 3.449\ldots$
  Period $2 = 2^1$ orbit loses stability
  Birth of a stable Period $4 = 2^2$ orbit
- $r_2 = 3.54409\ldots$
  Period $4 = 2^2$ orbit loses stability
  Birth of a stable Period $8 = 2^3$ orbit

Process of period doubling continues

Infinite Cascade of of Period Doublings – with stable ranges

$$r_{m-1} < r \leq r_m$$

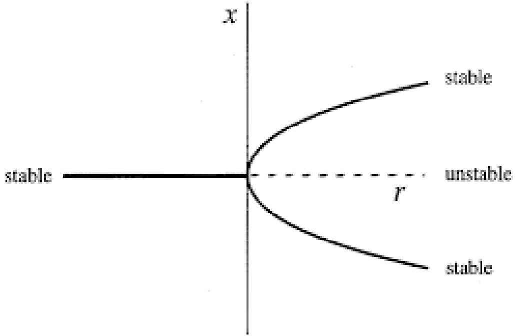
for orbit of period 2^m

As $m \to \infty$

Accumulation point of an infinite number of period doublings :

$$r_\infty = 3.57\ldots$$

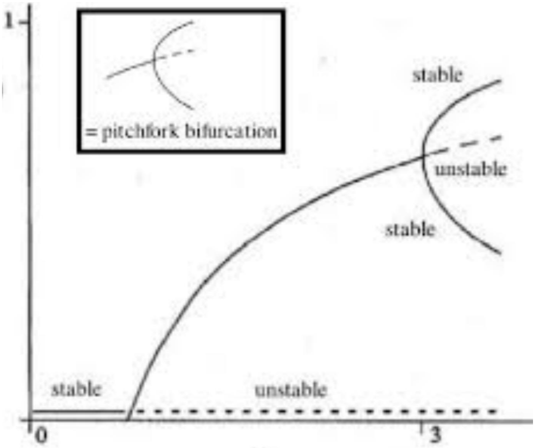
Pitchfork Bifurcation



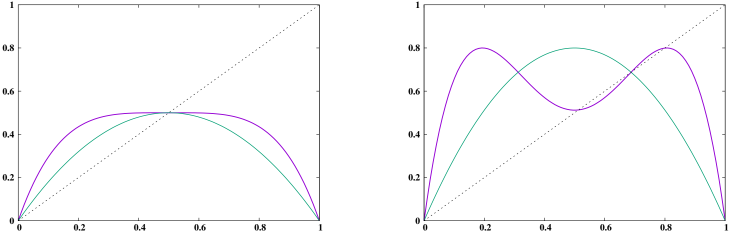
Two new solutions are created : 1 solution gives rise to 3 solutions as the parameter increases

Also exchange of stability: existing solution loses stability; new solutions are stable

Bifurcations



Now consider the two times iterated logistic map F^2



$r = 2$ $\qquad\qquad$ $r = 3.2$

- The fixed point of F is also fixed point of F^2
- Two new solutions (intersections) for $r > 3$

Existence of Period 2 Orbit

To find period 2 points, we need to solve for the points where the graph intersects the diagonal, i.e., we need to solve the fourth-degree equation

$$F^2(x) - x \ = \ r^2 \ x \ (1-x)[1 - rx(1-x)] - x \ = \ 0$$

Luckily, the fixed points $x^* = 0$ and $x^* = 1 - \frac{1}{r}$ are trivial solutions of this equation, as $F(x^*) = x^*$ implies that $F^2(x^*) = x^*$ also is satisfied.

After factoring out the fixed points, the problem reduces to solving a quadratic equation.

$$
\begin{aligned}
x\{r^2[1 - x(1+r) + 2rx^2 - rx^3] - 1\} &= 0 \\
x[-r^3x^3 + 2r^3x^2 - r^2(1+r)x + (r^2 - 1)] &= 0 \\
-r^3x\ [x - (1 - r^{-1})]\ [x^2 - (1 + r^{-1})x + r^{-1}(1 + r^{-1})] &= 0
\end{aligned}
$$

Factoring out x and $x - (1 - r^{-1})$, and solving the resulting quadratic equation, we obtain a pair of roots

$$
\frac{r + 1 \pm \sqrt{(r-3)(r+1)}}{2r}
$$

- These are real for $r > 3$: So a 2-cycle exists for all $r > 3$
- For $r < 3$ the roots are not real, which imples that a 2-cycle doesn't exist
- At $r = 3$, the roots coincide and equal $x^* = 1 - \frac{1}{r} = \frac{2}{3}$

- Slope of $F^2(x^*)$ at $x^* = 0$ is

$$[F'(0)]^2 \ = \ r^2$$

So the stability coefficient is greater than 1 for all $r > 1$

- Slope of $F^2(x^*)$ at $x^* = 1 - \frac{1}{r}$ is

$$[F'(1 - \frac{1}{r})]^2 \ = \ (2 - r)^2$$

As r increases through $r = 3$: $F'(x^*)$ decreases through -1, and the slope of F^2 at $x^* = 1 - \frac{1}{r}$ increases to above 1

Now for stability of period 2 orbit, consider the stability coefficient:

$$\lambda_2 = F'(x_1)F'(x_2)$$

with $x_1,\ x_2$ being the points on the 2 - cycle

where

$$x_{1,2} = \frac{r + 1 \pm \sqrt{(r-3)(r+1)}}{2r}$$

Calculating r_1

- Strategy: To analyze the stability of a cycle, reduce the problem to a question about the stability of a fixed point
- Both x_1 and x_2 are solutions of $F^2(x) = x$: $F(x_1) = x_2$ and $F(x_2) = x_1$; so $F(F(x_1)) = x_1$ and $F(F(x_2)) = x_2$
- So x_1 and x_2 are fixed points of the second-iterate map $F^2(x)$
- The 2-cycle is stable precisely if x_1 and x_2 are stable fixed points for $F^2(x)$

Computing the Stability Coefficient:

$$\lambda_2 = [\frac{dF(F(x))}{dx}]_{x=x_1} = F'(F(x_1))F'(x_1) = F'(x_2)F'(x_1)$$

Same λ_2 is obtained at $x = x_2$

Simultaneous splitting of the two branches of period 2 in the bifurcation diagram

Differentiating and substituting for x_1 and x_2 gives:

$$\begin{aligned}
\lambda_2 &= r\ (1 - 2x_2)\ r\ (1 - 2x_1) \\
&= r^2[1 - 2(x_1 + x_2) + 4x_1x_2] \\
&= r^2[1 - 2(r+1)/r + 4(r+1)/r^2] \\
&= 4 + 2r - r^2
\end{aligned}$$

So the 2-cycle is stable for $|4 + 2r - r^2| < 1$

That is, for $3 < r < 1 + \sqrt{6}$.

Picture essentially same:

$$F \equiv F^2 \quad \text{and} \quad \mathrm{F}^2 \equiv \mathrm{F}^4$$

Process of period doubling continues

Infinite Cascade of of Period Doublings – with stable ranges

$$r_{m-1} < r \leq r_m$$

for orbit of period 2^m

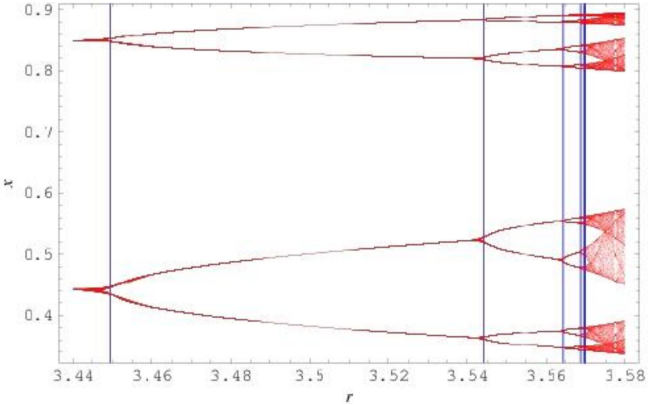
- $r_0 = 3$
  Period 1 orbit loses stability; Birth of a stable Period 2 orbit
- $r_1 = 3.449\ldots$
  Period 2 orbit loses stability; Birth of a stable Period 4 orbit
- $r_2 = 3.54409\ldots$
  Period 4 orbit loses stability; Birth of a stable Period 8 orbit
- $r_3 = 3.5644\ldots$
  Period 8 orbit loses stability; Birth of a stable Period 16 orbit
- $r_4 = 3.568759\ldots$
  Period 16 orbit loses stability; Birth of a stable Period 32 orbit

  .

  .

  .
- $r_\infty = 3.569946\ldots$
  Accumulation Point of the Period Doubling Sequence

Period Doubling Cascade



Decreases geometrically with common ratio $1/\delta < 1$:

$$\frac{r_m - r_{m-1}}{r_{m+1} - r_m} \to 4.669201 \cdots \equiv \delta$$

As $m \to \infty$

Accumulation point of an infinite number of period doublings :

$$r_\infty = 3.57\ldots$$

δ : Feigenbaum Constant

Reference:
M J Feigenbaum, *J Stat. Phys.*, **19** (1978) 25;
J Stat. Phys., **21** (1979) 669

m	Bifurcation parameter r_m	Ratio $\frac{r_m - r_{m-1}}{r_{m+1} - r_m}$
0	3.0	–
1	3.4494897	4.7514
2	3.5440903	4.6562
3	3.5644073	4.6683
4	3.5687594	4.6686
5	3.5696916	4.6692
6	3.5698913	4.6694
7	3.5699340	–

The ratio in the last column converges to the first Feigenbaum constant.

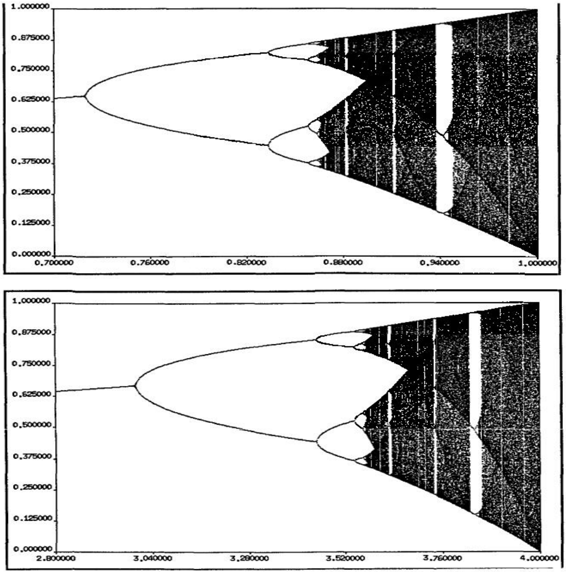
Universal Scaling Relations

Holds for all Unimodal One-dimensional Maps of an interval onto itself with a Quadratic Maximum

For instance it holds for $x_{n+1} = r \sin(\pi x_n)$

in the interval $[0:1]$, with $r \leq 1$

Typical of the Period Doubling Route to Chaos



Top: $x_{n+1} = r \sin(\pi x_n)$ Bottom: $x_{n+1} = r\, x_n(1 - x_n)$

$F(x) = r - x^2$

m	Bifurcation parameter r_m	Ratio $\frac{r_{m-1}-r_{m-2}}{r_m-r_{m-1}}$
0	0.75	–
1	1.25	4.2337
2	1.3680989	4.5515
3	1.3940462	4.6458
4	1.3996312	4.6639
5	1.4008286	4.6682
6	1.4010853	4.6689
7	1.4011402	–

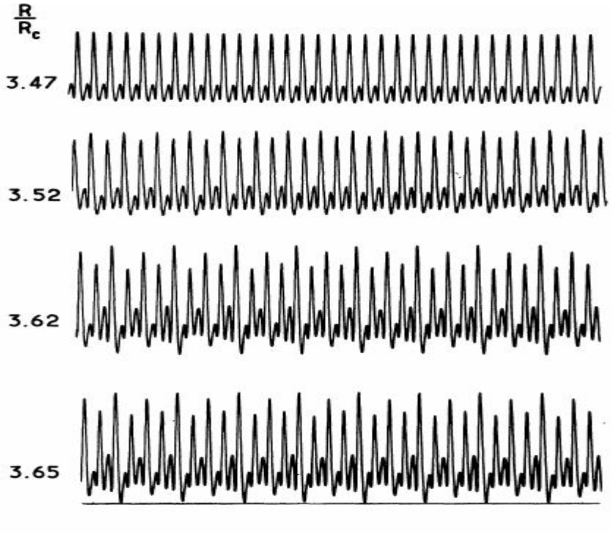
The ratio in the last column converges to the same Feigenbaum constant.

Period-doubling seen qualitatively and quantitatively in experiments in a variety of physical systems

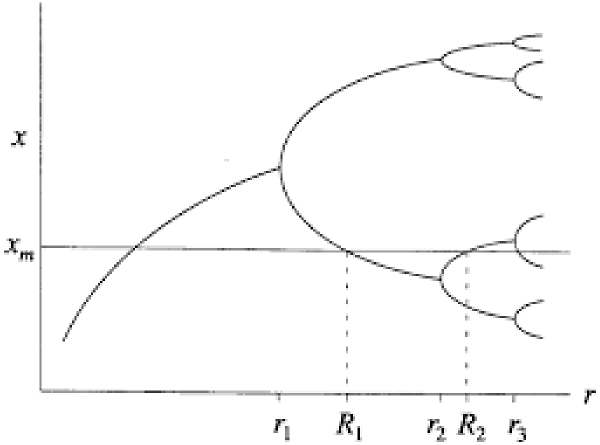
Including, strictly speaking, infinite-dimensional systems — such as fluid flows!

Results from one-dimensional maps also apply to higher dimensional systems

Experimental Results from a Rayleigh-Benard Cell



Libchaber *et al*, 1982



A superstable cycle of a unimodal map always contains the maximum x_m as one of its points

Define a Super Stable orbit of period 2^m as occuring at value R_m where stability coefficient is zero

In some sense, its in the "middle" of stability range

Since λ is zero, the critical point, i.e. the point with slope zero, namely $x = \frac{1}{2}$ must be a point on the superstable orbit

Let Δ_m be the distance between $1/2$ and the nearest of the other $2^m - 1$ points in the cycle - which happens to be the point which is one half period displaced from the critical point

$$\Delta = F^{2^{m-1}}(\frac{1}{2}) - \frac{1}{2}$$

Another Scaling Ratio...

Δ_m also decreases geometrically :

$$\frac{\Delta_m}{\Delta_{m+1}} \to -2.50280 \cdots \equiv \alpha$$

Minus sign: indicates that the nearest orbit point to $x = 1/2$ switches between above $1/2$ an below $1/2$ as m increases by 1

What happens beyond r_∞ ??

That is between : $r_\infty < r < 4$

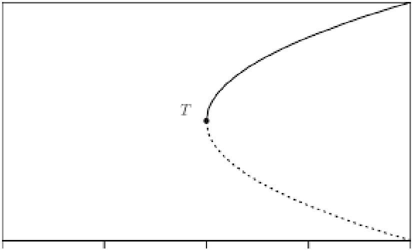
Chaotic Bands

Windows of Odd periods : for instance period 3 window

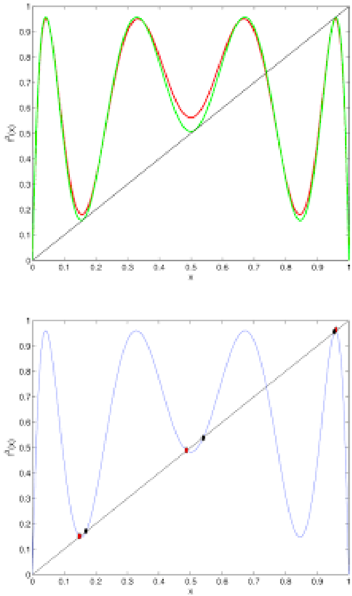
Birth of Odd-Period Orbits

Tangent Bifurcation   Also known as Saddle-Node Bifurcation

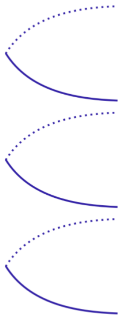
No Solutions $\rightarrow$ 2 new Solutions (One Stable and one Unstable)



Birth of Period 3 through the Tangent Bifurcation

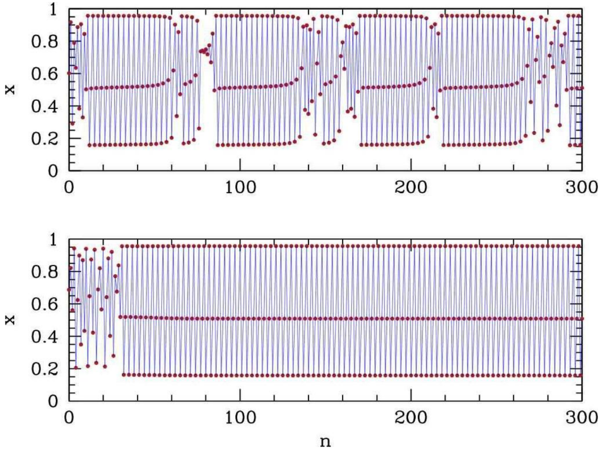


3 Cycle created through Tangent Bifurcations

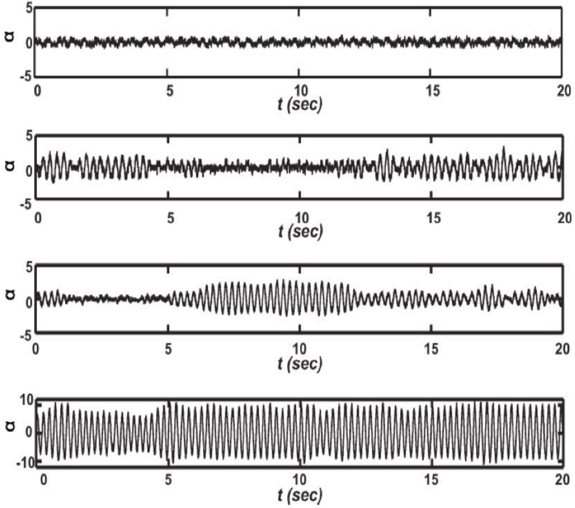


For r just below the period-3 window, the system exhibits an interesting kind of behaviour, called intermittency

(Pomeau and Manneville, 1980)

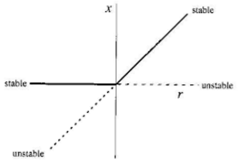


Intermittency in Experiments

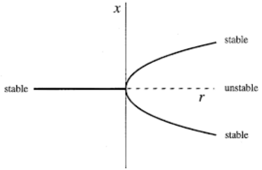


Wind Tunnel Experiments in IIT Madras

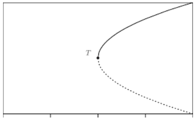
- Transcritical Bifurcation: Stability Exchange

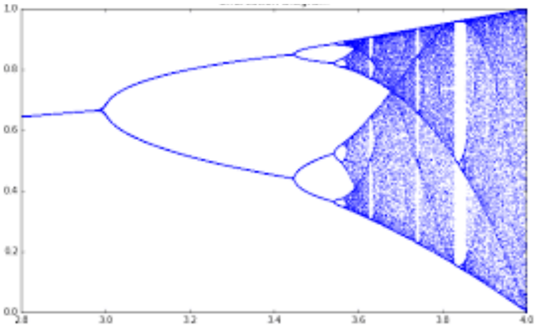


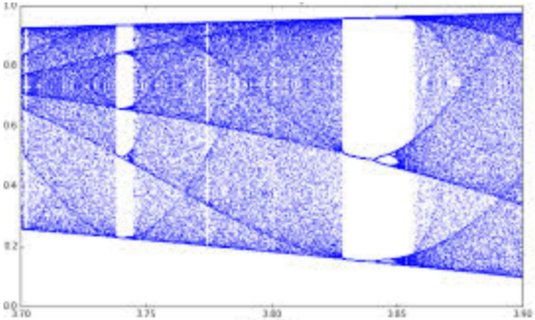
- Pitchfork Bifurcation: Gives rise to orbits wth periodicity 2^m



- Tangent Bifurcation: Gives rise to Odd Cycles




1.0
0.8
0.6
0.4
0.2
0.0
2.8
3.0
3.2
3.4
3.6
3.8
4.0


1.0
0.8
0.6
0.4
0.2
0.0
3.70
3.75
3.80
3.85
3.90

ORGANIZATION AND APPEARANCE OF PERIODIC ORBITS

Ordering of positive integers :

$3, 5, 7, \ldots 2 \times 3, 2 \times 5, 2 \times 7, \ldots,$
$2^2 \times 3, 2^2 \times 5, 2^2 \times 7, \ldots 2^3 \times 3, 2^3 \times 5, 2^3 \times 7, \ldots$
$2^5, 2^4, 2^3, 2^2, 2, 1$

All odds first (except 1), then 2 times odds, then 2^2 times odd, then list powers of 2 in decreasing order

Sarkovskii's Theorem: If period p occurs before another period l in the list then map F also has a periodic orbit of period l : "Period 3 implies Chaos"

Consider $r = 4$:

$$x_{n+1} = 4x_n(1 - x_n)$$

Change of variables :

$$x = \sin^2 \frac{\pi y}{2} = \frac{1}{2}[1 - \cos(\pi y)]$$

Substituting :

$$\sin^2(\pi y_{n+1}/2) = 1 - \cos^2(\pi y_n) = \sin^2(\pi y_n)$$

This yields:

$$\frac{\pi y_{n+1}}{2} = \pm \pi y_n + s\pi$$

where s is an integer

Now if y is defined to lie in $[0,1]$: Determines choice of s and sign of (πy_n)

Thus: $y_{n+1} = 2y_n$ for $0 \leq y_n \leq \frac{1}{2}$ (+sign and $s = 0$)

$y_{n+1} = 2 - 2y_n$ for $\frac{1}{2} \leq y_n \leq 1$ (-sign and $s = 1$)

Namely its the TENT MAP !

Since the tent map is chaotic, the logistic map at $r = 4$ must be chaotic too

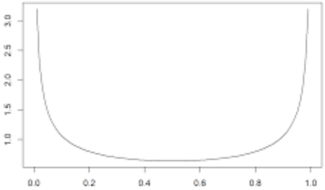
Must have the same dense set of unstable periodic orbits too

Obtain the invariant density $\rho(x)$ using the fact that the frequency of visiting an interval $[y, y+dy]$ is the same as visiting the corresponding transformed interval in x : $[x, x+dx]$

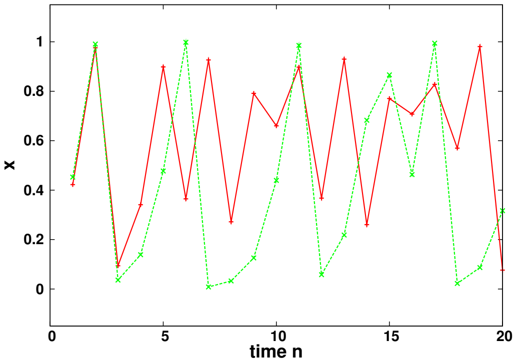
$$\rho(x) = |\frac{dy(x)}{dx}|\rho(y(x))$$

Since $\rho(y) = 1$ for the tent map :

$$\rho(x) = \frac{1}{\pi[x(1-x)]^{\frac{1}{2}}}$$



Sensitive dependence on initial conditions



Red : initial $x_0 = 0.12$ Green : initial $x_0 = 0.13$

Lyapunov Exponent

To be called chaotic a system should also show sensitive dependence on initial conditions, in the sense that neighboring orbits separate exponentially fast, on average.

Given some initial condition x_0, consider a nearby point $x_0 + \delta_0$ where the initial separation $\delta_0 \to 0$

If the separation after n iterates, δ_n, is:

$$|\delta_n| \approx |\delta_0| \ e^{\lambda n}$$

then λ is called the Lyapunov exponent

A positive Lyapunov exponent is a signature of chaos

A more precise and computationally useful formula for λ

Noting that $\delta_n = F^n(x_0 + \delta_0) - F^n(x_0)$, and taking logarithms and the limit $\delta_0 \to 0$:

$$\lambda \approx \frac{1}{n} \ln |\frac{\delta_n}{\delta_0}| = \frac{1}{n} \ln |\frac{F^n(x_0 + \delta_0) - F^n(x_0)}{\delta_0}| = \frac{1}{n} \ln |(F^n)'(x_0)|$$

Using Chain rule:

$$(F^n)'(x_0) = \prod_{i=0}^{n-1} F'(x_i)$$

So

$$\lambda \approx \frac{1}{n} \ln |\prod_{i=0}^{n-1} F'(x_i)| = \frac{1}{n} \sum_{i=0}^{n-1} \ln |F'(x_i)|$$

If this expression has a limit as $n \to \infty$, we define that limit to be the Lyapunov exponent for the orbit starting at x_0:

$$\lambda = \lim_{n\to\infty} \{ \frac{1}{n} \sum_{i=0}^{n-1} \ln |F'(x_i)| \}$$

- λ depends on x_0
- However, it is the same for all x_0 in the basin of attraction of a given attractor
- For stable fixed points and cycles: λ is negative
- For chaotic attractors : λ is positive

- Superstable Orbit: Lyapunov exponent is $-\infty$

  As $\lambda \approx \frac{1}{n} \ln | \prod_{i=0}^{n-1} F'(x_i)| = \frac{1}{n} \ln 0$

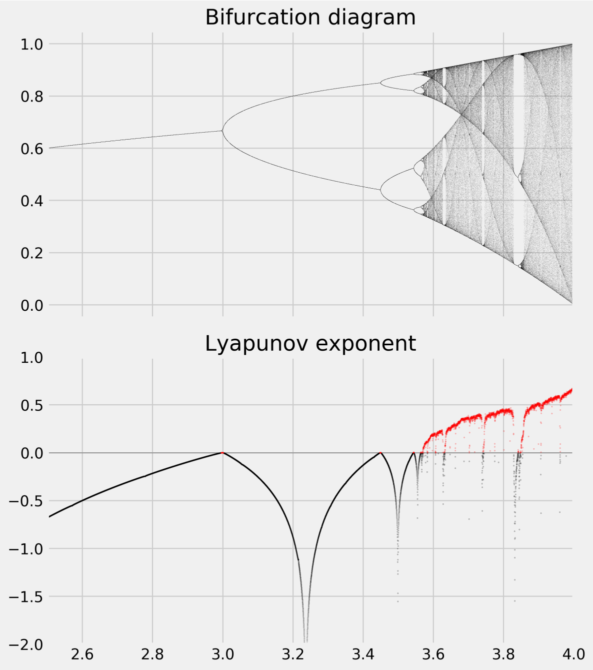
- Lyapunov Exponent for the Tent Map:

$$\lambda = \lim_{n\to\infty} \{ \frac{1}{n} \sum_{i=0}^{n-1} \ln |F'(x_i)| \} = \ln 2$$

  Independent of initial state x_0

Lyapunov Exponent for the Logistic Map

$$
\begin{aligned}
\lambda &= \lim_{n\to\infty} \{ \frac{1}{n} \sum_{i=0}^{n-1} \ln |F'(x_i)| \} \\
&= \lim_{n\to\infty} \{ \frac{1}{n} \sum_{i=0}^{n-1} \ln |r - 2rx_i| \}
\end{aligned}
$$



Bifurcation diagram
1.0
0.8
0.6
0.4
0.2
0.0
Lyapunov exponent
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
2.6
2.8
3.0
3.2
3.4
3.6
3.8
4.0

Renormalization

Notation:

- $f(x, r)$ denotes a unimodal map that undergoes a period-doubling route to chaos as r increases
- x_m is the maximum of $f(x, r)$
- r_n denotes the value of r at which a 2^n-cycle is born
- R_n denotes the value of r at which the 2^n-cycle is superstable

Following Feigenbaum, we will analyze the superstable cycles...

Consider the map $f(x,r) = r - x^2$

To find R_0 and R_1:

- Fixed point condition : $x^* = R_0 - (x*)^2$
- Superstability condition : $\frac{df}{dx}|_{x=x^*} = 0$
- Now $\frac{df}{dx} = -2x$, so $x^* = 0$ at superstable fixed point, which ocurs at the maximum of f
- Substituting $x^* = 0$ into the fixed point condition yields $R_0 = 0$

- At R_1 the map has a superstable 2-cycle
- Let x_1 and x_2 denote the points of the cycle
- Superstability requires that the stability coefficient $(-2x_1)(-2x_2) = 0$
- So the point $x = 0$ must be one of the points in the 2-cycle
- Then the period-2 condition $f^2(0, r) = 0$ implies $r - r^2 = 0$, with roots 0 and 1
- Since 0 corresponds to R_0, the other root, equal to 1, must be R_1

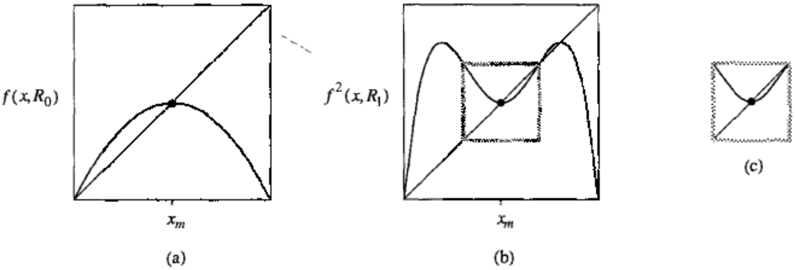
Illustrates a general rule: A superstable cycle of a unimodal map always contains x_m as one of its points

Renormalization theory is based on the self-similarity of the period-doubling cascade

Branches look like the preceding one – except they are scaled down in both the x and r directions

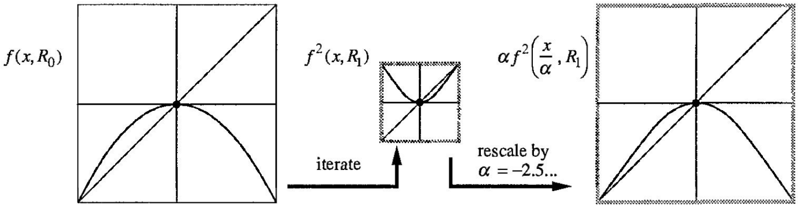
This structure reflects the endless repetition of the same dynamical processes: a 2^n-cycle is born, then becomes superstable, and then loses stability in a pitchfork bifurcation

To express the self-similarity mathematically, we compare $f(x, R_0)$ with its second iterate $f^2(x, R_1)$ at corresponding values of r, and then renormalize one map into the other


$f(x, R_0)$
x_m
(a)
$f^2(x, R_1)$
x_m
(b)
(c)

- Both maps have the same stability properties: x_m is a superstable fixed point for both of them
- r-shifting is a basic part of the renormalization procedure: we consider the second iterate of f and increased r from R_0 to R_1
- Key point is that small box looks practically identical to the figure for $f(x, R_0)$, except for a change of scale and a reversal of both axes.

- First step: translate the origin of x to x_m, by redefining x as $x - x_m$
- Also subtract x_m from f, since $f(x_n, r) = x_{n+1}$
- Next, to make (b) look like (a), we blow it up by a factor $\alpha > 1$ in both directions, and also invert it by replacing (x, y) by $(-x, -y)$.
- Both operations can be accomplished in one step if we define the scale factor α to be negative



$f(x, R_0)$
iterate
$f^2(x, R_1)$
rescale by
$\alpha = -2.5...$
$\alpha f^2\left(\frac{x}{\alpha}, R_1\right)$

Rescaling by α is equivalent to replacing $f^2(x, R_1)$ by $\alpha f^2(x/\alpha, R_1)$

The resemblance between (a) and (c) implies that

$$f(x, R_0) \;\approx\; \alpha\; f^2(\frac{x}{\alpha}, R_1)$$

In summary, f has been renormalized by:

- Taking its second iterate
- Rescaling $x \to x/\alpha$
- Shifting r to the next superstable value

We can also renormalize f^2 to generate f^4:

It too has a superstable fixed point if we shift r to R_2

The same reasoning yields

$$f^2(\frac{x}{\alpha}, R_1) \approx \alpha f^4(\frac{x}{\alpha^2}, R_2)$$

When expressed in terms of the original map $f(x, R_0)$, this equation becomes

$$f(x, R_0) \approx \alpha^2 f^4(\frac{x}{\alpha^2}, R_2)$$

After n times we get:

$$f(x, R_0) \approx \alpha^n f^{2^n}(\frac{x}{\alpha^n}, R_n)$$

Feigenbaum found numerically:

$$\lim_{n \to \infty} \alpha^n \ f^{2^n}(\frac{x}{\alpha^n}, R_n) = g_0(x)$$

where $g_0(x)$ is a universal function with a superstable fixed point

The limiting function exists only if α is chosen correctly, specifically, $\alpha = -2.5029\ldots$

Universal means that the limiting function $g_0(x)$ is (almost) independent of the original f

- The form of $g_0(x)$ suggests the explanation: $g_0(x)$ depends on f only through its behavior near $x = 0$, since that's all that survives in the argument x/α^n as $n \to \infty$
- With each renormalization, we're blowing up a smaller and smaller neighborhood of the maximum of f, so practically all information about the global shape of f is lost.
- One caveat: The order of the maximum is never forgotten!
- Hence a more precise statement is that $g_0(x)$ is universal for all f with a quadratic maximum
- A different $g_0(x)$ is found when the order of the maximum is different, e.g. for f's with a fourth-degree maximum, etc...

To obtain other universal functions $g_i(x)$, start with $f(x, R_i)$ instead of $f(x, R_0)$:

$$g_i(x) \ = \ \lim_{n\to\infty} \ \alpha^n \ f^{2^n}(\frac{x}{\alpha^n}, R_{n+i})$$

Here $g_i(x)$ is a universal function with superstable 2^i-cycle
The case where we start with $R_i = R_\infty$ (at the onset of chaos) is the most interesting and important, since then

$$f(x, R_\infty) \ \approx \ \alpha \ f^2(\frac{x}{\alpha}, R_\infty)$$

Now we don't have to shift r when we renormalize.
The limiting function $g_\infty(x)$, usually called $g(x)$, satisfies

$$g(x) \ = \ \alpha \ g^2(\frac{x}{\alpha})$$

- $g(x) = \alpha\ g^2(\frac{x}{\alpha})$ is a functional equation for $g(x)$ and the universal scale factor α
- It is self-referential: $g(x)$ is defined in terms of itself

- The functional equation is not complete until we specify boundary conditions on $g(x)$

- After the shift of origin, all our unimodal maps f have a maximum at $x = 0$: So we require $g'(0) = 0$

- Also, we can set $g(0) = 1$ without loss of generality: just defines the scale for x
  - If $g(x)$ is a solution of $g(x) = \alpha g^2(\frac{x}{\alpha})$, so is $\mu g(x/\mu)$, with the same α

Now we solve for $g(x)$ and α

- At $x = 0$ the functional equation gives $g(0) = \alpha\ g(g(0))$
- But $g(0) = 1$, so $1 = \alpha g(1)$
- Hence,

$$\alpha = 1/g(1)$$

- So α is determined by $g(x)$

No closed form solution for $g(x)$, so we resort to a power series solution

$$g(x) = 1 + c_2 x^2 + c_4 x^4 + \dots$$

which assumes that the maximum is quadratic

The coefficients are determined by substituting the power series into the functional equation for $g(x)$ and matching powers of x

Feigenbaum (1979) used a seven-term expansion, and found $\alpha \sim -2.5029$

Thus the renormalization theory has succeeded in explaining the value of a observed numerically.