Flows : Continuous Time

Consider the general system:

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x_1, \ldots, x_n) \\ &\;. \\ &\;. \\ &\;. \\ \frac{dx_n}{dt} &= f_n(x_1, \ldots, x_n) \end{aligned} \tag{1}$$

Solutions can be visualized as trajectories or orbits flowing through an n-dimensional phase space with coordinates $(x_1, x_2, \ldots, x_n)$.

We start with the simple case of $n = 1$, i.e. One-dimensional or First-Order systems, with a single equation of the form

$$\frac{dx}{dt} = f(x)$$

Here $x(t)$ is a real-valued function of time t, and $f(x)$ is a smooth real-valued function of x.

Flow is on a Line

An important techniques of dynamics: interpret a differential equation as a Vector Field

Illustrate this point by a simple nonlinear differential equation:

$$\frac{dx}{dt} = f(x) = \sin x$$

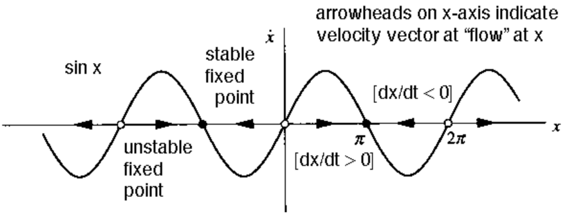
Can be solved exactly by seprating the variables and integrating:

$$t = \ln \left| \frac{\csc x + \cot x}{\csc x_0 + \cot x_0} \right|$$

where $x = x_0$ at $t = 0$

Difficult to interpret the closed form solution

A more physical picture ...



Flow stops when $\dot{x} = 0$: Fixed Points

- Solid black dots represent Stable Fixed Points : Attractors or Sinks — as flow is toward them
- Open circles represent Unstable Fixed Points : Repellers or Sources — as flow moves away from them

Fixed Points

Defined by

$$f(x^*) = 0$$

- Correspond to points where the flow stops : Equilibrium (Steady State) solutions
- If $x = x^*$ initially, then $x(t) = x^*$ for all time
- Stable fixed point : local flow is toward it – all sufficiently small disturbances away from it damp out in time
- Unstable fixed point : local flow is away from it – even small disturbances grow in time

Linear Stability Analysis: Quantitative measure of Stability

Let x^* be a fixed point, and $\eta(t) = x(t) - x^*$ be a small perturbation away from x^*

By differentiation, Taylor expanding to first order and noting that x^* is a constant, we derive a linear differential equation for η:

$$\frac{d\eta}{dt} = \eta \; f'(x^*)$$

- Perturbation $\eta(t)$ grows exponentially if $f'(x^*) > 0$: UNSTABLE
- Perturbation $\eta(t)$ decays exponentially if $f'(x^*) < 0$: STABLE

So slope $f'(x^*)$ at the fixed point x^* determines its stability

- Negative slope at a stable fixed point
- Positive slope at unstable fixed point

Magnitude of $f'(x^*)$ is measure of how stable/unstable a fixed point is: determines the rate of the exponential growth or decay

Characteristic time scale, determining the time required for $x(t)$ to vary significantly in the neighborhood of x^*: $\frac{1}{|f'(x^*)|}$

Impossibility of Oscillations

Only things that can happen for a vector field on the real line:

- All trajectories either approach a fixed point, or diverge to $\pm\infty$ in first-order systems
- Trajectories can only increase or decrease monotonically, or remain constant
- Geometrically, the phase point never reverses direction
- No periodic solutions to $\dot{x} = f(x)$
- Fundamentally topological in origin : you flow monotonically on a line

Bifurcations

- The qualitative structure of the flow can change as parameters are varied
- Transitions and instabilities as some control parameter is varied : Bifurcations
- Fixed points can be created or destroyed, or their stability can change
- The parameter values at which the qualitative change occurs are called bifurcation points

Saddle-Node Bifurcation

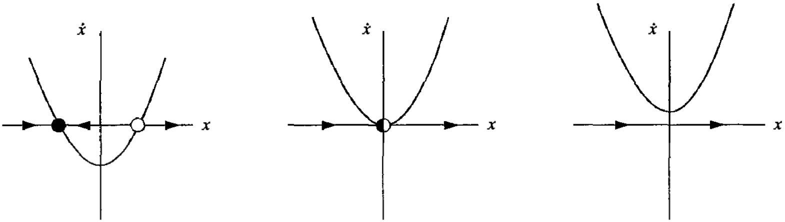
Basic mechanism by which fixed points are created and destroyed

The prototypical example of a saddle-node bifurcation is given by the first-order system

$$\frac{dx}{dt} = r + x^2$$

Parameter r : may be positive, negative, or zero

When r is negative, there are two fixed points : one stable and one unstable

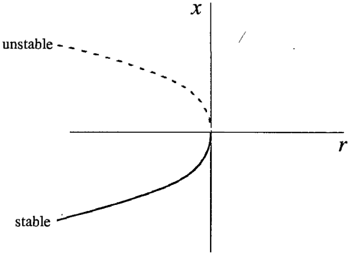


(a) $r<0$ (b) $r=0$ (c) $r>0$

As parameter r is varied: two fixed points move toward each other, collide, and mutually annihilate

At the bifurcation point $r=0$, the x-axis (i.e. $\dot{x}=0$ line) is tangent to the curve

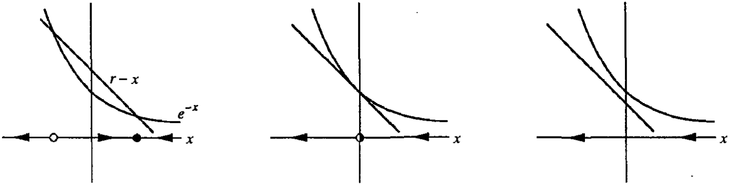
Bifurcation Diagram



- The fixed points x^* are solutions of $\dot{x} = f(x) = r + x^2 = 0$: $x^* = \pm\sqrt{-r}$
- There are two fixed points for $r < 0$, and none for $r > 0$

- Linear Stability: $f'(x^*) = 2x^*$
- So $x^* = -\sqrt{-r}$ is stable, since $f'(x^*) = -2\sqrt{-r} \; < \; 0$
- So $x^* = \sqrt{-r}$ is unstable, since $f'(x^*) = 2\sqrt{-r} \; > \; 0$
- At the bifurcation point $r = 0$: $f'(x^*) = 0$
- Similarly for $\dot{x} = f(x) = r - x^2 = 0$: $x^* = \pm\sqrt{r}$
- In this case, there are two fixed points for $r > 0$, and none for $r < 0$

- Fixed points satisfy $f(x) = r - x - e^{-x} = 0$
- Trick: Plot $r - x$ and e^{-x}



- Intersection of line $r - x$ and curve e^{-x} occurs when $r - x = e^{-x}$ and so $f(x) = 0$

- At some critical value $r = r_c$, the line becomes tangent to the curve, and we have a saddle-node bifurcation

- Since $r - x$ and curve e^{-x} intersect tangentially, both the function and their derivatives are equal at the point of intersection:

$$\frac{d(r-x)}{dx} = \frac{de^{-x}}{dx}$$

- So $-e^{-x} = -1$, i.e. $x = 0$

- Also since $r - x = e^{-x}$ at $x = 0$, $r = r_c = 1$

Normal Forms

Close to a saddle-node bifurcations, the dynamics typically look like $\dot{x} = r - x^2$ or $\dot{x} = r + x^2$: prototypical of saddle-node bifurcations

For instance, for $\dot{x} = r - x - e^{-x}$, using Taylor expansion for e^{-x} to leading order in x, about the bifurcation point at $x = 0$, we find

$$\frac{dx}{dt} = r - x - e^{-x} = r - x - \left(1 - x + \frac{x^2}{2!} + \dots\right) = (r-1) - \frac{x^2}{2} + \dots$$

i.e. the same algebraic form (with some rescalings) as $\dot{x} = r - x^2$

More generally, consider f as a function of both x and r

Near the bifurcation at $x = x^*$ and $r = r_c$, Taylor series expansion gives:

$$\frac{dx}{dt} = f(x,r) = f(x^*, r_c) + (x - x^*)\frac{\partial f}{\partial x} + (r - r_c)\frac{\partial f}{\partial r} + \frac{1}{2}(x - x^*)^2\frac{\partial^2 f}{\partial x^2} + \ldots$$

where the partial derivatives are evaluated at (x^*, r_c)

- $f(x^*, r) = 0$ since x^* is a fixed point
- $\frac{\partial f}{\partial x}\big|_{(x^*, r_c)} = 0$ by the tangency condition

So

$$\dot{x} = a(r - r_c) + b(x - x^*)^2 + \ldots$$

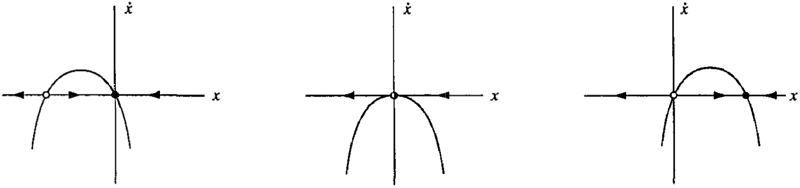
where $a = \frac{\partial f}{\partial r}$ and $b = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}$

Transcritical Bifurcation

Fixed point changes stability as the parameter is varied

The normal form for a transcritical bifurcation is

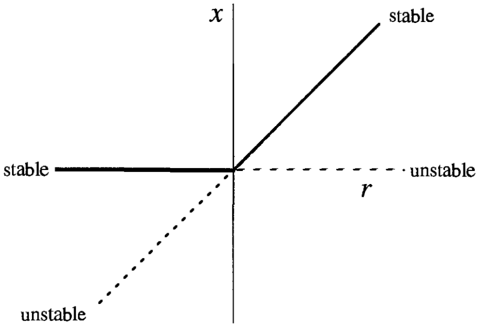
$$\frac{dx}{dt} = r\ x\ -\ x^2$$



(a) $r < 0$ (b) $r = 0$ (c) $r > 0$

- $x^* = 0$: always a fixed point
- Other fixed point : $x^* = r$

Bifurcation diagram for transcritical bifurcation



Fixed points $x^* = 0$ and $x^* = r$: Exchange Stability

Example of Transcritical Bifurcation

$$\dot{x} = x(1 - x^2) \;-\; a(1 - e^{-bx})$$

$x^* = 0$ is a fixed point for all values of parameters a and b

For x close to zero:

$$1 - e^{-bx} = 1 - [1 - bx + \frac{1}{2}b^2x^2 + O(x^3)] = b\;x - \frac{1}{2}b^2\;x^2\;+\;O(x^3)$$

So

$$\dot{x} = x - a(bx - \frac{1}{2}b^2x^2 + O(x^3)) = (1 - ab)\;x\;+\;(\frac{1}{2}ab^2)\;x^2\;+\;O(x^3)$$

So a transcritical bifurcation occurs when $ab = 1$: bifurcation curve

Now consider another example:

$$\dot{x} = r \ \ln x \ + \ x \ - \ 1$$

- $x^* = 1$: Fixed point for all values of r

- Introduce a new variable $u = x - 1$, where u is small
- Then

$$\dot{u} \ = \ \dot{x} \ = \ r \ln(1+u) + u$$
$$= \ r[u - \frac{1}{2}u^2 + O(u^3)] + u \approx (r+1) \ u \ - \ \frac{1}{2} r \ u^2 \ + \ O(u^3)$$

- So a transcritical bifurcation occurs at $r_c = -1$

Putting this into normal form:

Let $u = a\ v$, where a will be chosen later to make the coefficient of the quadratic term 1 :

$$\dot{v} \ = \ (r+1)\ v \ - \ (\frac{1}{2} r a)\ v^2 \ + \ O(v^3)$$

Choosing $a = 2/r$ gives:

$$\dot{v} \ = \ (r+1) v \ - \ v^2 \ + \ 0(v^3)$$

Setting $R = r + 1$ and $X = v$ gives the approximate normal form:

$$\dot{X} \ \approx \ RX \ - \ X^2$$

where $X = \frac{u}{a} = \frac{1}{2} r (x - 1)$

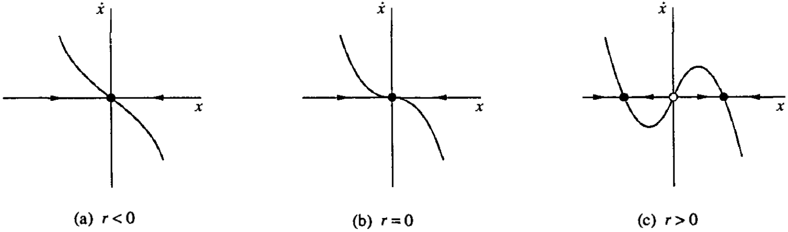
Pitchfork Bifurcation

Supercritical Pitchfork Bifurcation with normal form:

$$\dot{x} \ = \ r\,x \ \ - \ \ x^3$$

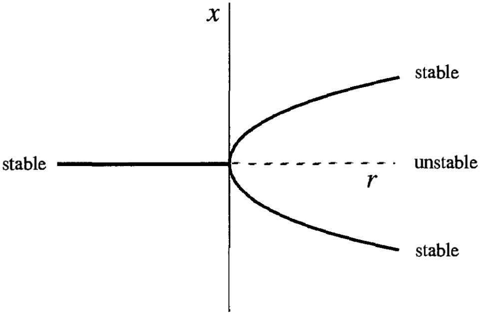
Symmetry: Invariant under the change of variables $x \to -x$

$\dot{x} = 0$ gives Fixed Point solutions: $x^* = 0$ and $x^* = \pm\sqrt{r}$



- $r < 0$: only (stable) fixed point $x^* = 0$
- $r > 0$: origin become unstable; Two new stable fixed points appear on either side of the origin, symmetrically located at $x^* = \pm\sqrt{r}$

- At $r = 0$, linearization vanishes
- But the fixed point at $x^* = 0$ is still weakly stable
- Perturbations no longer decay exponentially fast, instead the decay is a much slower algebraic function of time
- Critical Slowing Down

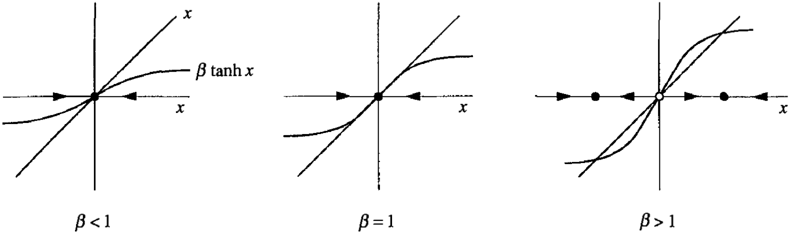


Analogous to a continuous or second-order phase transition

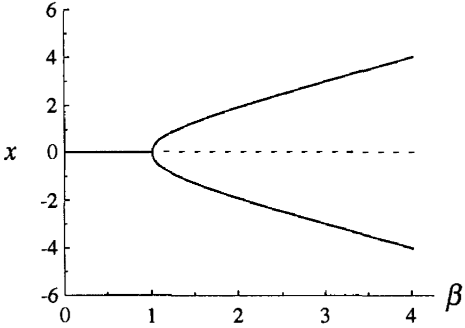
Model relevant to Neural Networks

$$\dot{x} = -x \; + \; \beta \tanh x$$

Plot graphs of $y = x$ and $y = \beta \tanh x$: intersections correspond to fixed points



- As β increases, the $\tanh$ curve becomes steeper at the origin, as its slope there is β
- For $\beta < 1$ the origin is the only fixed point
- Pitchfork bifurcation occurs at $\beta = 1$, $x^* = 0$
- For $\beta > 1$, two new stable fixed points appear, and the origin becomes unstable

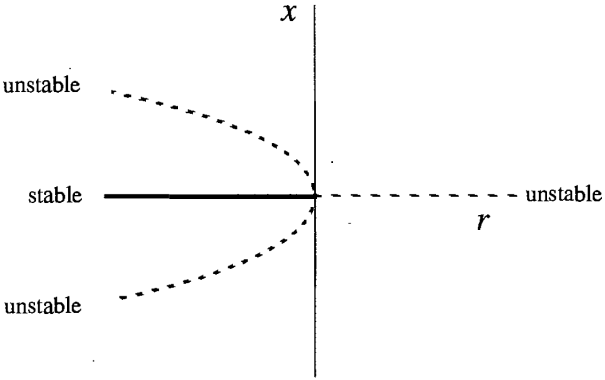

6
4
2
x
0
-2
-4
-6
0
1
2
3
4
β

Subcritical Pitchfork Bifurcation

Prototypical example has a positive destabilizing cubic term:

$$\dot{x} = r\,x + x^3$$

- $x^* = 0$ is stable for $r < 0$ and unstable for $r > 0$
- For $r < 0$ nonzero fixed points $x^* = \pm\sqrt{-r}$ exist and are unstable
- For $r > 0$, trajectories go to infinity: blow-up $x(t) \to \pm\infty$ in finite time

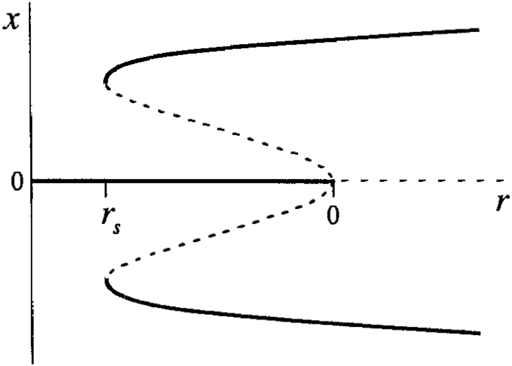

x
unstable
stable
unstable
unstable
r

In real physical systems, such an explosive instability is usually opposed by the stabilizing influence of higher-order terms

Assuming that the system is still symmetric under $x \to -x$, the first stabilizing term must be x^5

Prototypical example of a system with a subcritical pitchfork bifurcation is:

$$\dot{x} = r\,x + x^3 - x^5$$

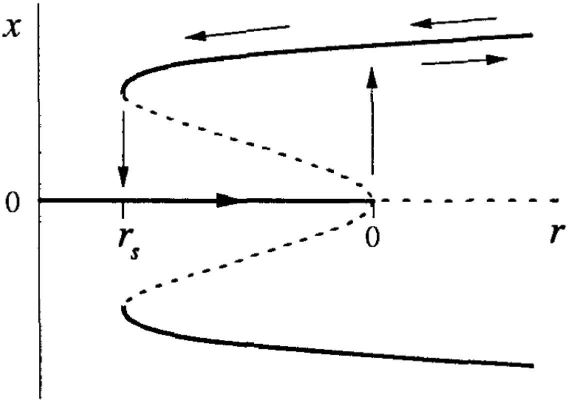


Analogous to a discontinuous or first-order phase transition : jump from zero to large amplitude

At r_s : Saddle-node bifurcation, in which stable and unstable fixed points are created

- For small x: x^* is stable for $r < 0$, and two branches of unstable fixed points after $r = 0$
- The unstable branches turn around and become stable at $r = r_s$, where $r_s < 0$
- These stable large-amplitude branches exist for all $r > r_s$.

Hysteresis



- In the range $r_s < r < 0$: two qualitatively different stable states coexist, namely the origin and the large-amplitude fixed points
- Initial condition determines which fixed point is approached
- Origin is stable to small perturbations, but not to large ones: so it is locally stable, but not globally stable
- The existence of different stable states allows for the possibility of jumps and hysteresis as r is varied
- So there is lack of reversibility as parameter is varied

Flows on a Circle

$$\frac{d\theta}{dt} = f(\theta)$$

- Angle or Phase θ : point on the circle
- $\dot{\theta}$: velocity vector at that point determined by one-dimensional equation $\dot{\theta} = f(\theta)$
- Provides the most basic model of systems that can oscillate
- No amplitude information

A vector field on the circle assigns a unique velocity vector to each point on the circle

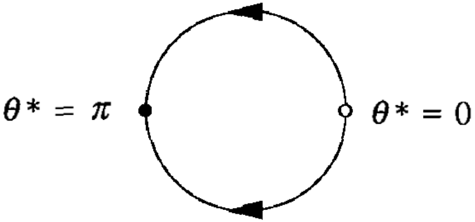
Function $f(\theta)$ is a smooth real-valued, 2π-periodic function :

$$f(\theta + 2\pi) = f(\theta)$$

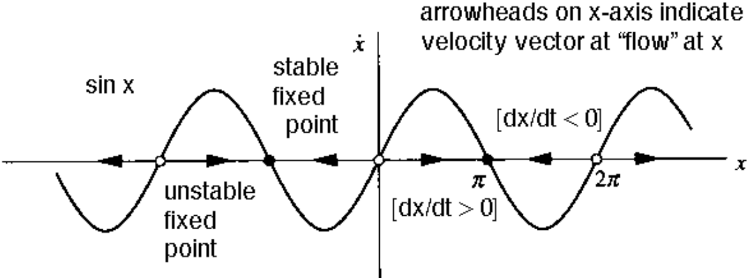
for all real θ

$$\frac{d\theta}{dt} = f(\theta) = \sin(\theta)$$

Fixed Points: $\theta^* = 0$ and $\theta^* = \pi$



$$\dot{x} = \sin x$$



Flow on a Line

Uniform Oscillator

Simplest oscillator is one in which the angle θ changes uniformly:

$$\frac{d\theta}{dt} \;=\; f(\theta) \;=\; \omega$$

where ω is a constant

Solution:

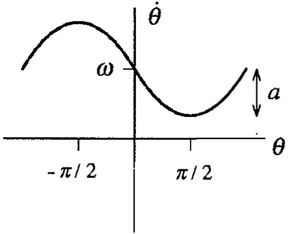
$$\theta(t) \;=\; \omega t \;+\; \theta_0$$

- Corresponds to uniform motion around the circle at an angular frequency ω
- Solution is periodic: returns to the same point on the circle after a time $T = \frac{2\pi}{\omega}$
- T : period of oscillation

Non Uniform Oscillator

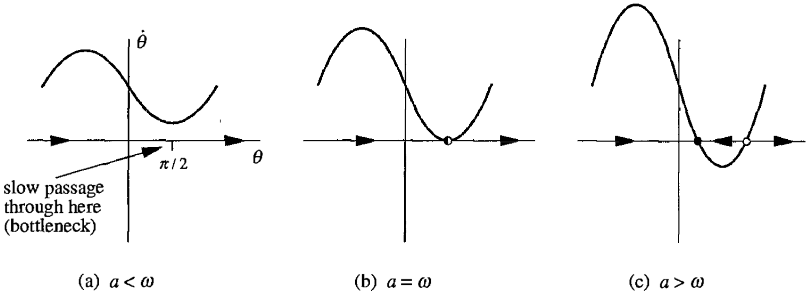
$$\frac{d\theta}{dt} = f(\theta) = \omega - a \sin\theta$$

with $\omega > 0$ and $a > 0$



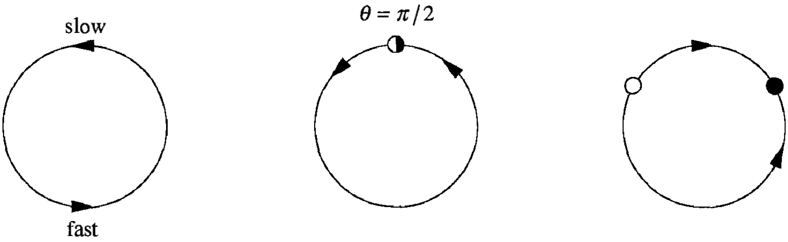
- Parameter a gives the nonuniformity in the flow:
  Fastest at $\theta = -\pi/2$; Slowest at $\theta = \pi/2$ (Bottleneck)
- $a = 0$: Uniform Oscillator

Saddlenode Bifurcation



Taylor expansion about $\theta = \pi/2$, using $\phi = \theta - \pi/2$, where ϕ is small, gives the normal form:

$$\dot{\phi} = \omega - a \ \sin(\phi + \frac{\pi}{2}) = \omega - a \ \cos\phi = (\omega - a) \ + \ \frac{1}{2}a\phi^2 + \dots$$



(a) $a < \omega$ (b) $a = \omega$ (c) $a > \omega$

For $a > \omega$, fixed points satisfy $\dot{\theta} \; = = \omega - a \;\; \sin(\theta) \;\; = \;\; 0$

So $\sin \theta^* = \omega / a$

Linear Stability Analysis for $a > \omega$:

$$f'(\theta^*) = -a \ \cos\theta^*$$

So fixed points with $cos\theta^* > 0$ is stable

Oscillation Period for $a < \omega$: time required for θ to change by 2π

$$T = \int dt = \int_0^{2\pi} \frac{dt}{d\theta} d\theta = \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta} = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$

When $a = 0$ (uniform oscillator): $\quad T = 2\pi/\omega$

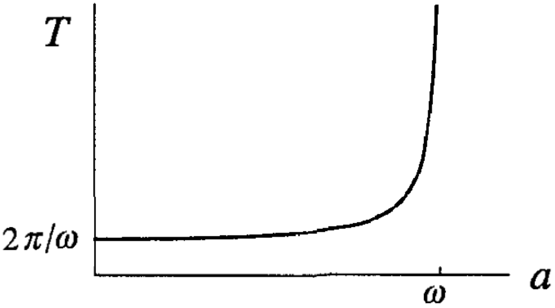
Time period increases with a and diverges as $a \to \omega^-$

$$\sqrt{\omega^2 - a^2} = \sqrt{\omega + a}\sqrt{\omega - a} \approx \sqrt{2\omega}\sqrt{\omega - a}$$

So

$$T \approx \left(\frac{\pi\sqrt{2}}{\sqrt{\omega}}\right) \frac{1}{\sqrt{\omega - a}}$$

Square-root scaling law: very general feature of systems close to a saddle-node bifurcation


T
2π/ω
ω
a

Normal form for saddle-node bifurcations at $r = 0$:

$$\dot{x} = r + x^2$$

Parameter r ($0 < r << 1$) is proportional to the distance from the bifurcation

To estimate the time spent in the bottleneck, we calculate the time taken for x to go from $-\infty$ to ∞:

$$T_{bottleneck} \approx \int_{-\infty}^{\infty} \frac{dx}{r + x^2} = \frac{\pi}{\sqrt{r}}$$

Synchronization : Example of Fireflies

Simple model of the firefly's flashing rhythm in response to stimuli

In the absence of stimuli, firefly goes through a cycle at a frequency ω, according to $\dot{\theta} = \omega$, where $\theta(t)$ is the phase of the firefly's flashing rhythm

Now suppose there's a periodic stimulus whose phase Θ satisfies $\dot{\Theta} = \Omega$

Model of the firefly's response to this stimulus:

$$\dot{\theta} = \omega + A\sin(\Theta - \theta)$$

where $A > 0$: resetting strength reflecting fireflies ability to modify its frequency

Dynamics of Phase Difference $\phi = \Theta - \theta$ is given by the nonuniform oscillator equation:

$$\dot{\phi} = \dot{\Theta} - \dot{\theta} = \Omega - \omega - A\sin\phi$$

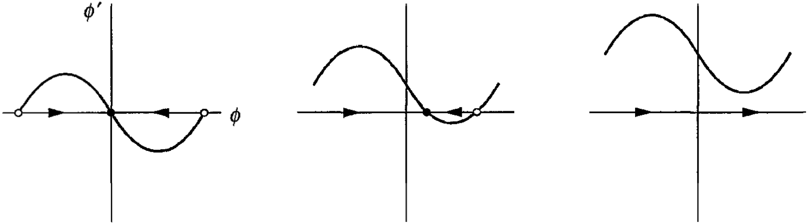
Non-dimensionalize it by introducing:

$$\tau \; = \; A\, t, \qquad \mu \; = \; \frac{\Omega - \omega}{A}$$

Then:

$$\phi' \; = \; \mu \; - \; \sin\phi$$

where $\phi' = d\phi/d\tau$, and the dimensionless group of parameters μ measures the frequency difference, relative to the resetting strength



(a) $\mu = 0$ (b) $0 < \mu < 1$ (c) $\mu > 1$

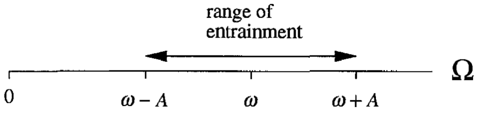
- Stable fixed point at $\phi^* = 0$, implying zero phase difference, when $\mu = 0$ (i.e. $\Omega = \omega$)
- When $0 < \mu < 1$: stable fixed point at $\phi^* > 0$
- For $\mu > 1$: No fixed points

- $0 < \mu < 1$: stable fixed point ϕ^*, i.e. the phase difference is a constant
- This phenomenon is called Phase-locking or Entrainment
- Positive ϕ^* implies that the stimulus flashes ahead of the firefly in each cycle: firefly lags in phase by a constant amount ϕ^*
- Saddle-node bifurcation at $\mu = 1$
- No fixed points for $\mu > 1$: phase difference $\phi(t)$ increases indefinitely, corresponding to phase drift

Entrainment

Entrainment is predicted to occur within a symmetric interval of driving frequencies:

$$\omega - A \;\leq\; \Omega \;\leq\; \omega + A$$



As $\sin\phi^* \;=\; \frac{\Omega-\omega}{A}$, and so the magnitude of $\frac{\Omega-\omega}{A}$ must be bounded by 1

Nonlinear Second-Order Systems

General form of a vector field on the phase plane:

$$\begin{aligned} \dot{x_1} &= f_1(x_1, x_2) \\ \dot{x_2} &= f_2(x_1, x_2) \end{aligned}$$

In vector notation:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

where phase point $\mathbf{x} = (x_1, x_2)$ and $\mathbf{f}(\mathbf{x}) = (f_1(x_1, x_2), f_2(x_1, x_2))$

Solution $\mathbf{x}(t)$: Trajectory

Consequence of the uniqueness of solutions: Trajectories never intersect

- In two-dimensional phase spaces: strong topological consequences
- For example, suppose there is a closed orbit C in the phase plane: then any trajectory starting inside C is trapped in there forever
- Poincaré-Bendixson theorem : If a trajectory is confined to a closed, bounded region and there are no fixed points in the region, then the trajectory must be eventually approach a closed orbit

Fixed Points and Linearization

Consider the system

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned}$$

Suppose that (x^*, y^*) is a fixed point, i.e.

$$f(x^*, y^*) = 0, \qquad g(x^*, y^*) = 0$$

Let $u = x - x^*, \quad v = y - y^*$ denote the components of a small disturbance from the fixed point

Task is now to find evolution equations for u and v

$$\dot{u} = \dot{x} \quad \text{(since } x^* \text{ is a constant)}$$

$$= f(x^* + u, y^* + v) \quad \text{(by substitution)}$$

$$= f(x^*, y^*) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv) \quad \text{(Taylor Series)}$$

$$= u\,\frac{\partial f}{\partial x} + v\,\frac{\partial f}{\partial y} + O(u^2, v^2, uv) \quad \text{(since } f(x^*, y^*) = 0\text{)}$$

where the partial derivates are evaluated at the fixed point (x^*, y^*)

Linearized System

Neglecting quadratic terms:

$$\dot{u} = u\,\frac{\partial f}{\partial x} + v\,\frac{\partial f}{\partial y}$$

$$\dot{v} = u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y}$$

So:

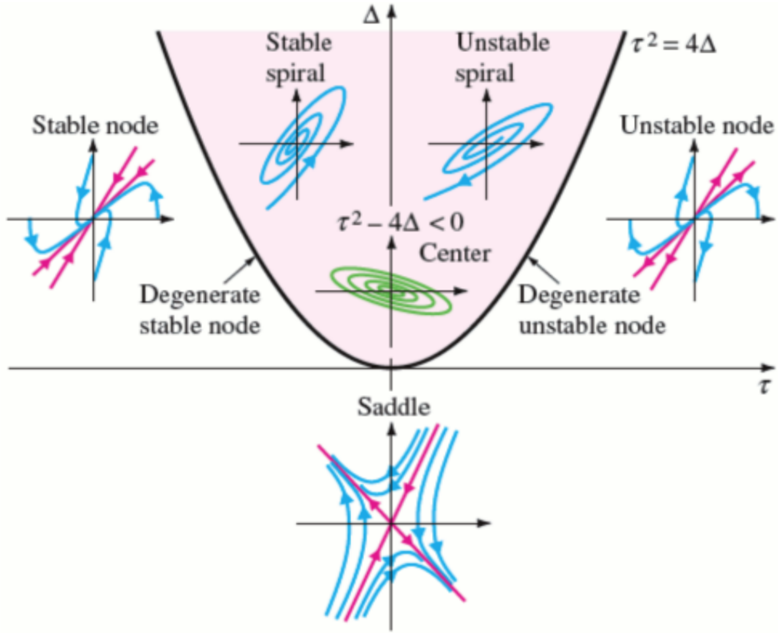
$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}$$

where A is the Jacobian Matrix at the fixed point (x^*, y^*)

Multivariable analog of the derivative $f'(x^*)$

Denoting τ as Trace and Δ is the Determinant of Jacobian Matrix:

$$\lambda_{1,2} = \frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\Delta}) \qquad \tau \ = \ \lambda_1 \ + \ \lambda_2, \qquad \Delta \ = \ \lambda_1 \lambda_2$$

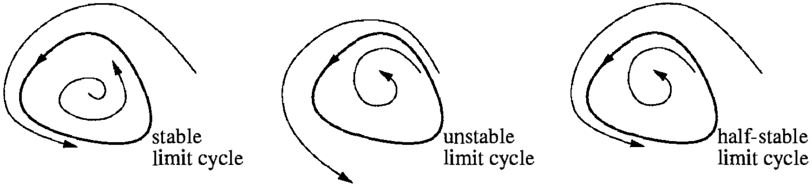


Δ
Stable spiral
Unstable spiral
$\tau^2 = 4\Delta$
Stable node
Unstable node
$\tau^2 - 4\Delta < 0$
Center
Degenerate stable node
Degenerate unstable node
τ
Saddle

Limit Cycle : Nonlinear Phenomenon

A limit cycle is an isolated closed trajectory

Neighboring trajectories are not closed; they spiral either toward or away from the limit cycle



Self-sustained Oscillations

Kuramoto Model

- The Kuramoto model was originally motivated by the phenomenon of collective synchronization, in which an enormous system of oscillators spontaneously locks to a common frequency, despite the inevitable differences in the natural frequencies of the individual oscillators

- Biological examples:
  - Networks of pacemaker cells in the heart
  - Circadian pacemaker cells in the brain
  - Metabolic synchrony in yeast cell suspensions
  - Congregations of synchronously flashing fireflies
  - Crickets that chirp in unison

- Examples from physics and engineering:
  - Arrays of lasers
  - Power Grid
  - Superconducting Josephson junctions

Phase equations of the Kuramoto Model

Long-term dynamics of a system of coupled limit-cycle oscillators:

$$\dot{\theta}_i = \omega_i \; + \; \Sigma_{j=1}^{N} \, \Gamma_{ij}(\theta_j - \theta_i) \qquad i = 1, \ldots N$$

Γ_{ij} : Interaction Functions

This reduction to a phase model represents a tremendous simplification – but these equations are still far too difficult to analyze in general, since the interaction functions could have arbitrarily many Fourier harmonics and the connection topology is unspecified – the oscillators could be connected in a chain, a ring, a random graph, or any other topology

Kuramoto recognized that the mean-field case should be the most tractable

The Kuramoto model corresponds to the simplest possible case of equally weighted, all-to-all, purely sinusoidal coupling:

$$\Gamma_{ij}(\theta_j - \theta_i) \;=\; \frac{K}{N}\sin(\theta_j - \theta_i)$$

where $K \geq 0$ is the Coupling Strength

- Frequencies ω_i are distributed according to some probability density $g(\omega)$
- For simplicity, Kuramoto assumed that $g(\omega)$ is unimodal and symmetric about its mean frequency Ω: $g(\Omega + \omega) \; = \; g(\Omega - \omega)$ for all ω, like a Gaussian distribution
- Due to rotational symmetry in the model, we can set the mean frequency to $\Omega = 0$ by redefining $\theta_i \;\rightarrow\; \theta_i \;+\; \Omega\, t$ for all i, which corresponds to going into a rotating frame at frequency Ω.
- Note that the governing equations

$$\dot{\theta}_i = \omega_i \; + \; \frac{K}{N}\, \Sigma_{j=1}^{N} \sin(\theta_j - \theta_i) \qquad i = 1, \ldots N$$

are invariant under this

- It effectively subtracts Ω from all the ω_i and therefore shifts the mean of $g(\omega)$ to zero
- So from now on, $g(\omega) = g(-\omega)$ for all ω, and the ω_i denote deviations from the mean frequency Ω
- We also suppose that $g(\omega)$ is nowhere increasing on $[0, \infty)$, in the sense that $g(\omega) \geq g(\nu)$ whenever $\omega \leq \nu$; this formalizes what is meant by unimodal.

Order Parameter

To visualize the dynamics of the phases, it is convenient to imagine a collection of points running around the unit circle in the complex plane

A complex order parameter can be defined as:

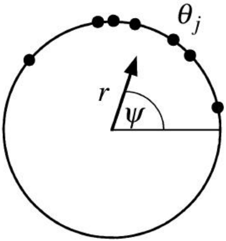
$$r\ e^{i\Psi} \ = \ \frac{1}{N}\Sigma_{j=1}^{N}\ e^{i\theta_j}$$

It is a macroscopic quantity that can be interpreted as the collective rhythm produced by the whole population

It corresponds to the centroid of the phases

Geometric interpretation of the order parameter

The phases θ_j are plotted on the unit circle



Their centroid is given by the complex number $r\ e^{i\Psi}$, shown as an arrow : Radius $r(t)$ measures the Phase Coherence, and $\Psi(t)$ is the Average Phase

- If all the oscillators move in a single tight clump, then $r \approx 1$ and the population acts like a giant oscillator

- On the other hand, if the oscillators are scattered around the circle, then $r \approx 0$ and the individual oscillations add incoherently and no macroscopic rhythm is produced

Kuramoto noticed that the governing equation

$$\dot{\theta}_i = \omega_i \ + \ \frac{K}{N} \ \Sigma_{j=1}^{N} \sin(\theta_j - \theta_i)$$

can be rewritten in terms of the order parameter, as follows:

Multiply both sides of the order parameter equation by $e^{-i\theta}$ to obtain:

$$r \ e^{i(\Psi - \theta_i)} \ = \ \frac{1}{N} \ \Sigma_{j=1}^{N} \ e^{i(\theta_j - \theta_i)}$$

Equating imaginary parts yields:

$$r \ \sin(\Psi - \theta_i) \ = \ \frac{1}{N} \ \Sigma_{j=1}^{N} \ \sin(\theta_j - \theta_i)$$

Thus the Kuramoto dynamics can be written as:

$$\dot{\theta}_i \; = \; \omega_i \; + \; K \; r \; \sin(\Psi - \theta_i), \qquad i = 1, \ldots N$$

In this form, the mean-field character of the model becomes obvious

Each oscillator appears to be uncoupled from all the others, although ofcourse they are interacting, but only through the mean-field quantities r and Ψ

Specifically, the phase θ_i is pulled toward the mean phase Ψ, rather than toward the phase of any individual oscillator

- The effective strength of the coupling is proportional to the coherence r
- This sets up a positive feedback loop between coupling and coherence
- As the population becomes more coherent, r grows and so the effective coupling Kr increases, which tends to bring even more oscillators into the synchronized set
- Winfree was the first to discover this mechanism underlying spontaneous synchronization, but it stands out especially clearly in the Kuramoto model

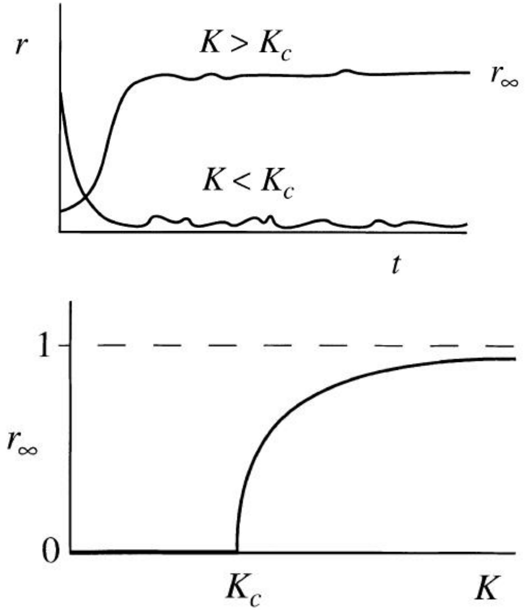
Simulations

If we integrate the model numerically, how does $r(t)$ evolve?

For example, fix $g(\omega)$ to be a Gaussian and vary the coupling K

Simulations show that for all K less than a certain threshold K_c, the oscillators act as if they were uncoupled:

- Phases become uniformly distributed around the circle, starting from any initial condition
- Then $r(t)$ decays to size $O(1/\sqrt{N})$, as expected for any random scatter of N points on a circle
- But when K exceeds K_c, this incoherent state becomes unstable and $r(t)$ grows, reflecting the nucleation of a small cluster of oscillators that are mutually synchronized, thereby generating a collective oscillation.
- Eventually $r(t)$ saturates at some level $r_\infty < 1$, though still with $O(1/\sqrt{N})$ fluctuations.


r
$K > K_c$
r_∞
$K < K_c$
t
1
r_∞
0
K_c
K

The population splits into two groups: the oscillators near the center of the frequency distribution lock together at the mean frequency Ω and co-rotate with the average phase $\Psi(t)$, while those in the tails run near their natural frequencies and drift relative to the synchronized cluster

This mixed state is often called partially synchronized

With further increases in K, more and more oscillators get into the synchronized cluster, and r_∞ grows

The numerics further suggest that r_∞ depends only on K, and not on the initial condition

In other words, it seems there is a globally attracting state for each value of K

Analysis for the critical coupling K_c

Attempt to obtain a broad explanation of the stability of the zero branch below threshold K_c and the bifurcating branch above threshold

Few of these problems have been solved, while the rest remain open...

In his earliest work, Kuramoto analyzed his model without the benefit of simulations - he guessed the correct long-term behavior of the solutions in the limit $N \to \infty$, using symmetry considerations

Specifically, Kuramoto sought steady solutions, where $r(t)$ is constant and $\Psi(t)$ rotates uniformly at frequency Ω

By going into the rotating frame with frequency Ω and choosing the origin of this frame correctly, one can set $\Psi \equiv 0$ without loss of generality. Then the governing equation becomes

$$\dot{\theta}_i \;=\; \omega_i \;-\; K\, r\, \sin\theta_i, \qquad i = 1, \ldots N$$

Since r is assumed constant, all the oscillators are effectively independent in the steady solutions

The strategy now is to solve for the resulting motions of all the oscillators (which will depend on r as a parameter)

These motions in turn imply values for r and Ψ which must be consistent with the values originally assumed.

This self-consistency condition is the key to the analysis

The solutions exhibit two types of long-term behavior, depending on the size of $|\omega_i|$ relative to Kr

The oscillators with $|\omega_i| \leq Kr$ approach a stable fixed point defined implicitly by

$$\omega_i \quad = \quad K \; r \; \sin\theta_i$$

where $|\theta_i| \leq \pi/2$

These oscillators will be phase-locked at frequency Ω in the original frame

In contrast, the oscillators with $|\omega_i| > Kr$ are drifting and run around the circle in a nonuniform manner

The locked oscillators correspond to the center of $g(\omega)$ and the drifting oscillators correspond to the tails

So this explains why the population splits into two groups

However, the existence of the drifting oscillators would seem to contradict the original assumption that r and Ψ are constant.

How can the centroid of the population remain constant with all those drifting oscillators moving non-uniformly around the circle?

Kuramoto demanded that the drifting oscillators form a stationary distribution on the circle

Then the centroid stays fixed even though individual oscillators continue to move

Let $\rho(\theta, \omega)d\theta$ denote the fraction of oscillators with natural frequency ω that lie between θ and $\theta + d\theta$

Stationarity requires that $\rho(\theta, \omega)$ be inversely proportional to the speed at θ; oscillators pile up at slow places and thin out at fast places on the circle. Hence

$$\rho(\theta, \omega) \;=\; \frac{C}{|\omega - Kr\sin\theta|}$$

The normalization constant C is determined by $\int_{-\pi}^{\pi} \rho(\theta, \omega)d\theta = 1$ for each ω, which gives:

$$C \;=\; \frac{1}{2\pi}\,\sqrt{\omega^2 - (Kr)^2}$$

Invoking the self-consistency condition: the constant value of the order parameter must be consistent with that implied by definition of order parameter

$$r\ e^{i\Psi}\ =\ \frac{1}{N}\Sigma_{j=1}^{N}\ e^{i\theta_j}$$

Using angular brackets to denote population averages, we have

$$\langle e^{i\theta}\rangle = \langle e^{i\theta}\rangle_{lock} + \langle e^{i\theta}\rangle_{drift}$$

Since $\Psi = 0$ by assumption,

$$\langle e^{i\theta}\rangle\ =\ r\ e^{i\Psi}\ =\ r$$

Evaluating the locked contribution:

In the locked state, sin $\theta^* = \omega/Kr$ for all $|\omega| \leq Kr$

As $N \to \infty$, the distribution of locked phases is symmetric about $\theta = 0$ because $g(\omega) = g(-\omega)$, there are just as many oscillators at θ^* as at $-\theta^*$

Hence $\langle \sin\theta \rangle_{lock} = 0$ and

$$\langle e^{i\theta} \rangle_{lock} = \langle \cos\theta \rangle_{lock} = \int_{-Kr}^{Kr} \cos\theta(\omega)\ g(\omega)\ d\omega$$

where $\theta(\omega)$ is defined implicitly by $\omega_i = Kr \sin\theta_i$

Changing variables from ω to θ yields

$$\begin{aligned}\langle e^{i\theta} \rangle_{lock} &= \int_{-\pi/2}^{\pi/2} \cos\theta\ g(Kr\sin\theta)\ Kr\cos\theta\ d\theta \\ &= \ Kr \int_{-\pi/2}^{\pi/2} \cos^2\theta\ g(Kr\sin\theta)\ d\theta\end{aligned}$$

Contribution of the drifting oscillators:

$$\langle e^{i\theta} \rangle_{drift} = \int_{-\pi/2}^{\pi/2} \int_{|\omega|>Kr} \rho(\theta,\omega)\ g(\omega)\ d\omega d\theta$$

It turns out that this integral vanishes

This follows from $g(\omega) = g(-\omega)$ and the symmetry $\rho(\theta + \pi, -\omega) = \rho(\theta, \omega)$ implied by $\rho(\theta,\omega)\ =\ \frac{C}{|\omega - Kr\sin\theta|}$

Therefore, the self-consistency condition reduces to

$$r = Kr \int_{-\pi/2}^{\pi/2} \cos^2\theta \; g(Kr\sin\theta) \; d\theta$$

This always has a trivial zero solution $r = 0$, for any value of K

This corresponds to a completely incoherent state with $\rho(\theta, \omega) = 1/2\pi$ for all θ, ω.

A second branch of solutions, corresponding to partially synchronized states, satisfies

$$1 = K \int_{-\pi/2}^{\pi/2} \cos^2\theta \; g(Kr\sin\theta)\; d\theta$$

This branch bifurcates continuously from $r = 0$ at a value $K = K_c$ obtained by letting $r \to 0^+$

So the critical coupling K_c at the onset of collective synchronization is:

$$K_c \;=\; \frac{2}{\pi\; g(0)}$$

For the special case of a Lorentzian or Cauchy density, with scale parameter γ specifying the half-width at half-maximum (HWHM)

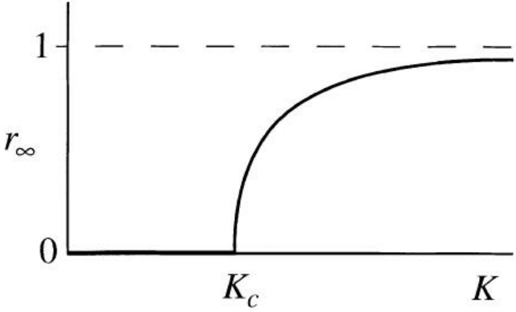
$$g(\omega) \;=\; \frac{\gamma}{\pi(\gamma^2 + \omega^2)}$$

exact integration gives:

$$r \;=\; \sqrt{1 - \frac{K_c}{K}}$$

for all $K \geq K_c$

So near onset, the amplitude of the bifurcating branch obeys the square-root scaling law, with r scaling as the square root of the normalized distance above threshold $(K - K_c)/K$


1
0
r_∞
K_c
K