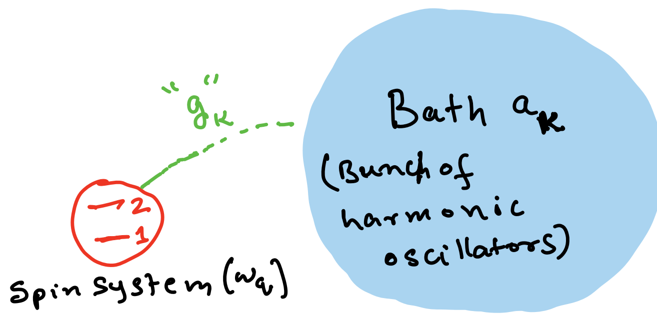
Lecture-4

→ Open two level / multi-level systems (Exact results and perturbative solutions)

→ Jaynes-Cummings Model: Exact solutions.

$2 \equiv \uparrow \equiv e$
$1 \equiv \downarrow \equiv g$



We start with the traditional version of spin-boson model

$$H = \underbrace{\frac{\omega_q}{2}\sigma_z}_{\substack{\text{system}\\ H_S}} + \underbrace{\sum_k \omega_k a_k^\dagger a_k}_{\text{Bath } H_R} + \underbrace{\sum_k \left[g_k a_k^\dagger + g_k^* a_k\right]\sigma_z}_{H_{SR}\text{ System-bath interaction}} \quad \hookrightarrow \boxed{\text{Eq.1}}$$

We are interested in the dynamics and steady state of reduced density matrix of the two level system.

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \quad \rightarrow \boxed{\text{Eq.2}}$$

Therefore we need to trace over bath of the full density matrix $\chi(t)$ to get above Eq.2 for the reduced system.

We have $\chi(t) = e^{-iHt} \chi(0)\, e^{iHt}$. We will perform the Polaron transformation to diagonalize the Hamiltonian.

We choose $$S = \sum_k \left(\frac{g_k}{\omega_k} a_k^+ - \frac{g_k^*}{\omega_k} a_k \right) \sigma^z \quad \rightarrow \boxed{Eq.3}$$

$\rightarrow \boxed{Eq.4}$

Using Eq.3, we get

$$\hat{H} = e^S H e^{-S} = \frac{\omega_q}{2} \sigma^z + \sum_k \omega_k a_k^+ a_k - \sum_k \frac{|g_k|^2}{\omega_k}$$

Irrelevant constant for dynamics.

We then have

$$\chi(t) = e^{-S} \left[e^S e^{-iHt} e^{-S} e^S \chi(0) e^{-S} e^S e^{iHt} e^{-S} \right] e^S$$
$$= e^{-S} \left[e^{-i\hat{H}t} e^S \chi(0) e^{-S} e^{i\hat{H}t} \right] e^S \quad \rightarrow \boxed{Eq.5}$$

Let us consider a general initial state of the form

$$\chi(0) = \sum_{\substack{n,m=0 \\ (\sigma,\sigma' = \downarrow,\uparrow)}}^{\infty} |\sigma, n\rangle\langle\sigma', m|\, \chi_{\sigma n, \sigma' m}^{(0)} \quad \rightarrow \boxed{Eq.6}$$

Labelling the reservoir fock states

$$\chi(t) = \sum_{\substack{n,m=0 \\ (\sigma,\sigma' = \downarrow,\uparrow)}}^{\infty} \chi_{\sigma n, \sigma' m}^{(0)} \left\{ e^{-S_\sigma} e^{-i\bar{H}_\sigma t} e^{S_\sigma} |\sigma, n\rangle\langle\sigma', m|\, e^{-S_{\sigma'}} e^{i\bar{H}_{\sigma'} t} e^{S_{\sigma'}} \right\}$$

$\rightarrow \boxed{Eq.7}$

Note the in above Eq. 7

S_σ = S with σ_z replaced by 1 for $\uparrow$

= S with σ_z replaced by −1 for $\Downarrow$

(Similary for $\hat{H}_\sigma$)

We now look at the reduced density matrix of the qubit.

$$\rho(t) = Tr_R[\chi(t)] = \sum_{\sigma\sigma'} |\sigma\rangle\langle\sigma'| \rho_{\sigma\sigma'}(t) \quad \rightarrow \boxed{Eq. 8}$$

where

$$\rho_{\sigma\sigma'}(t) = Tr_R\left[e^{-S_\sigma} e^{-i\hat{H}_\sigma t} e^{S_\sigma} \chi_{\sigma\sigma'}(0)\ e^{-S_{\sigma'}} e^{i\hat{H}_{\sigma'} t} e^{S_{\sigma'}}\right] \quad \rightarrow \boxed{Eq. 9}$$

$$\text{where } \chi_{\sigma\sigma'}(0) = \sum_{n,m=0}^{\infty} |n\rangle\langle m|\, \chi_{\sigma n, \sigma' m} \quad \rightarrow \boxed{Eq. 10}$$

If $\sigma = \sigma'$ then $\rho_{\sigma\sigma'}(t) = Tr_R[\chi_{\sigma\sigma'}(0)]$ which says that the population of qubits remain same.

We are only left we calculating the off-diagonal element. $\rho_{\uparrow\downarrow}(t)$. From Eq 9 we have,

$$\rho_{\uparrow\downarrow}(t) = Tr_R\left[e^{-S_\downarrow} e^{i\hat{H}_\downarrow t} e^{S_\downarrow} e^{-S_\uparrow} e^{-i\hat{H}_\uparrow t} e^{S_\uparrow} \chi_{\uparrow\downarrow}(0)\right] \quad \rightarrow \boxed{Eq 11}$$

$$\text{Let us define } \mathcal{S} = \sum_k \left[\frac{g_k}{\omega_k} a_k^+ - \frac{g_k^*}{\omega_k} a_k\right] \quad \rightarrow \boxed{Eq. 12}$$

→ Please note this $\mathcal{S}$ is different from S in Eq. 3

After some algebra, we get

$$\rho_{\uparrow\downarrow}(t) = e^{-i\omega_q t}\, Tr_R\left[e^{S(0)}\, e^{-2S(t)}\, e^{S(0)}\, \chi_{\uparrow\downarrow}(0) \right] \quad \boxed{Eq.13}$$

where
$$S(t) = \sum_k \left(\frac{g_k}{\omega_k} e^{i\omega_k t} a_k^{+} - \frac{g_k^{*}}{\omega_k} e^{-i\omega_k t} a_k \right) \rightarrow \boxed{Eq.14}$$

Simplifying further Eq.13 becomes

$$\rho_{\uparrow\downarrow}(t) = e^{-i\omega_q t}\, Tr_R\left[e^{2\sum_k \left[\frac{g_k}{\omega_k}(1-e^{i\omega_k t})\, a_k^{+} - \frac{g_k^{*}}{\omega_k}(1-e^{-i\omega_k t})\, a_k \right]}\, \chi_{\uparrow\downarrow}(0) \right] \quad \boxed{Eq.15}$$

Note that in above Eq. 15, the trace over reservoirs is still pending. Let us choose

$$\chi(0) = \rho^{system}(0) \otimes R_0 \rightarrow \boxed{Eq.16}$$

Bath thermal density operator

$$R_0 = \prod_k \frac{e^{-\beta \omega_k a_k^{+} a_k}}{(1-e^{-\beta\omega_k})^{-1}}$$

Substituting Eq. 16 and performing trace over reservoirs in Eq 15, we get

$$\rho_{\uparrow\downarrow}(t) = e^{-i\omega_q t}\, e^{-4\sum_k \left| \frac{g_k}{\omega_k} \right|^2 (1-\cos\omega_k t)\coth\left(\frac{\beta\omega_k}{2}\right)}\, \rho_{\uparrow\downarrow}(0) \rightarrow \boxed{Eq17}$$

oscillating part

This is expected to provide decay mechanism. (dephasing in this case)

Eq17 is an exact result valid for all times and all system bath couplings. (The above is generalizable to multi-level system under some conditions)

It is useful to analyse the second exponent in Eq. 17 which is expected to provide some decay mechanism (dephasing in this case). Let us define this exponent as $\Gamma(t)$, which means

$$\Gamma(t) = -4 \sum_k \left|\frac{g_k}{\omega_k}\right|^2 (1-\cos\omega_k t)\coth\left(\frac{\beta\omega_k}{2}\right) \rightarrow \boxed{\text{Eq. 18}}$$

$$= 4\int_0^\infty \frac{J(\omega)}{\omega^2}(1-\cos\omega t)\coth\left(\frac{\beta\omega}{2}\right)d\omega$$

If we choose an ohmic bath spectral function $J(\omega) = c\omega$ and also go to a high temperature limit, we get

$$\Gamma(t) = -\frac{8c}{\beta}\int_0^\infty \frac{1-\cos\omega t}{\omega^2}\,d\omega = -\frac{4c\pi t}{\beta} \rightarrow \boxed{\text{Eq 19}}$$

Therefore the decay rate for the off-diagonal element of the density matrix (let us call it dephasing γ_ϕ) is summarized as follows $(\Gamma(t) = -\gamma_\phi t)$

$$\rho_{\uparrow\downarrow}(t) = e^{-i\omega_q t}\, e^{-\gamma_\phi t} \quad \text{where } \gamma_\phi = \frac{4\pi c}{\beta} \rightarrow \boxed{\text{Eq 20}}$$

Eq 20 is a high-temperature approximation using an ohmic bath but atleast it gives us an idea as to how $\Gamma(t)$ in Eq. 18 acts as a decay term.

A natural question to ask is what happens when Eq. 1 is treated with a Lindblad Master Equation.

We will again start with the original Hamiltonian (Eq. 1). Using earlier lecture notes, one can arrive at

$$\frac{d\rho}{dt} = -\frac{i\omega_q}{2}\left[\sigma_z , \rho\right] - \hat{\Gamma}(0)\, L\left[\sigma^z\right] \rightarrow \boxed{\text{Eq 21}}$$

where $$\hat{\Gamma}(0) = \int_{-\infty}^{+\infty} ds\, \Gamma(s) \rightarrow \boxed{\text{Eq. 22}}$$

with $$\Gamma(s) = \sum_k |g_k|^2 \left(n_k e^{i\omega_k t} + (1+n_k) e^{-i\omega_k t}\right) \rightarrow \boxed{\text{Eq 23}}$$

Substituting Eq 23 into Eq 22 and doing some algebra, we get

$$\hat{\Gamma}(0) = \pi \left[J(0)\,(1+2n(0))\right]$$

This should be understood as $\lim_{\omega \to 0} J(\omega)\left[1+2n(\omega)\right]$ where $J(\omega) = c\omega$ (ohmic)

$$= \frac{2\pi c}{\beta} \rightarrow \boxed{\text{Eq 24}}$$

Using Eq. 24 and Eq 21, we get

$$\rho_{12}(t) = e^{i\omega_q t}\, e^{-\frac{4\pi c}{\beta} t}\, \rho_{12}(0) \rightarrow \boxed{\text{Eq 25}}$$

Note that the rate in Eq. 25 is same as Eq. 19 which was high temperature result of ohmic bath. This shown that the exact result of spin-boson model matches the Lindblad equation result in a certain limit.

For vectorization procedure of equation such as Eq 21 and many more generalizations, see tutorial 2.

Jaynes - Cummings Model

In this lecture we dealt with two level system coupled to bath and in previous lectures we discussed bosonic modes coupled to a bath. We will now discuss a combination of the two. This is the famous Jaynes-Cummings (JC) Model.

$$H = \underbrace{\frac{\omega_q}{2}\sigma^z}_{\text{qubit}} + \omega_c \underbrace{a^\dagger a}_{\text{single photon mode}} + \underbrace{g}_{\text{light-matter interaction}} \left(a^\dagger \sigma^- + h.c\right) \rightarrow \boxed{\text{Eq.26}}$$

We will discuss Eq.26 which is still an isolated system (No reservoirs)

The above has been realized in various experimental settings such as cavity-QED, Circuit-QED, Quantum dot cirquit-QED. Here, we will discuss exact solutions to JC model. The model is defined on a direct product of two spaces

$$\begin{array}{ccc} \text{Two-level Hilbert space} & \otimes & \text{Single mode Cavity Hilbert space} \\ \downarrow & & \\ \text{spaned by } |g\rangle, |e\rangle & & \text{spaned by } |n\rangle \;\; n \in W. \end{array}$$

H is a matrix in the basis $|x\rangle \otimes |n\rangle$ where $x = g, e$ and $n \in W$. It is therefore a $2\infty \times 2\infty$ Matrix. Hence exact solutions might seem very hard to get. However, it turs out that H (Eq. 26) has U(1) symmetry. In other words if $U(\theta) = e^{i\theta(a^\dagger a + \sigma^+\sigma^-)}$ then $U(\theta) H U(\theta) = H$ and this follows from the fact that $N = a^\dagger a + \sigma^+\sigma^-$ commutes with H. Note that N basically counts the total excitation number (qubits + photons). Since $[N, H] = 0$, we can simultaneously diagonalize both. Let us look at the spectra of 'N'

$g \equiv \downarrow \equiv 1$
$e \equiv \uparrow \equiv 2$
We use any of the above notations

$$N\,|g\rangle \otimes |0\rangle = 0\,|g\rangle \otimes |0\rangle$$

$$\vdots$$

$$\left.\begin{aligned} N\,|g\rangle \otimes |n\rangle &= n\,|g\rangle \otimes |n\rangle \\ N\,|e\rangle \otimes |n-1\rangle &= n\,|e\rangle \otimes |n-1\rangle \end{aligned}\right\} \text{Two fold degeneracy of } N \quad \boxed{\text{Eq. 27}}$$

Since $[N, H] = 0$, if $|\psi_p\rangle, |\psi_q\rangle$ correspond to eigenstate of N with eigenvalues p, q then if $p \neq q$ it implies $\langle \psi_p | H | \psi_q \rangle = 0$

In other words H doesnot couple spaces characterized by different excitation numbers.

Therefore

$$\mathbb{H}_{qubit} \otimes \mathbb{H}_{photon} = \text{Span}\{|g\rangle \otimes |0\rangle\} \oplus \text{Span}\{|g\rangle \otimes |1\rangle, |e\rangle \otimes |0\rangle\} \oplus \text{Span}\{|g\rangle \otimes |2\rangle, |e\rangle \otimes |1\rangle\} \oplus \cdots \text{Span}\{|g\rangle \otimes |n\rangle, |e\rangle \otimes |n-1\rangle\} \quad \boxed{\text{Eq. } 28}$$

The JC Hamiltonian (Eq. 26) doesnot couple these different subspaces (Eq. 28). So, to find the spectra, we can simply work in each subspace.

As seen in Eq. 28 the subspace corresponding to eigenvalue 'n' of 'N' is spanned by two states $|g\rangle \otimes |n\rangle$ and $|e\rangle \otimes |n-1\rangle$. H will be a 2x2 matrix.

$$H_n = \begin{bmatrix} \langle g| \otimes \langle n| \; H \; |g\rangle \otimes |n\rangle & \langle g| \otimes \langle n| \; H \; |e\rangle \otimes |n-1\rangle \\ \langle e| \otimes \langle n-1| \; H \; |g\rangle \otimes |n\rangle & \langle e| \otimes \langle n-1| \; H \; |e\rangle \otimes |n-1\rangle \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\omega_a}{2} + \omega_c n & g\sqrt{n} \\ g\sqrt{n} & \frac{\omega_a}{2} + \omega_c (n-1) \end{bmatrix} \quad \boxed{\text{Eq. } 29}$$

Eigenvectors and eigenvalues of above Eq. 29 are

$$|n,+\rangle = \cos\theta_n |g\rangle \otimes |n\rangle + \sin\theta_n |e\rangle \otimes |n-1\rangle$$
$$|n,-\rangle = -\sin\theta_n |g\rangle \otimes |n-1\rangle + \cos\theta_n |e\rangle \otimes |n-1\rangle \quad \rightarrow \boxed{\text{Eq. 30}}$$

$$\mathcal{E}_{n_\pm} = \omega_c \left(n - \tfrac{1}{2}\right) \pm \tfrac{1}{2}\sqrt{(\omega_c - \omega_q)^2 + 4g^2 n}$$

where $\theta_n = \frac{1}{2}\tan^{-1}\left(\frac{2g}{\omega_c - \omega_q}\right) \rightarrow \boxed{\text{Eq 31}}$

Next, we study the time dynamics of JC model.
For simplicity we work in the resonant case $\omega_q = \omega_c$
and choose an initial state $|\psi(0)\rangle = |e\rangle \otimes |n-1\rangle$

It is easy to show that the subsequent dynamics is

$$|\psi(t)\rangle = e^{i\omega_c(n-1/2)t}\left[i g \sin(g\sqrt{n}\,t) |g\rangle \otimes |n\rangle + \cos(g\sqrt{n}\,t) |e\rangle \otimes |n-1\rangle \right] \rightarrow \boxed{\text{Eq 32}}$$

From above Eq. 32 we see that after time period $T = \frac{2\pi}{g\sqrt{n}}$ the system returns to its initial state.

The cavity + qubit system periodically exchange energy $|e\rangle \otimes |n-1\rangle \longrightarrow |g\rangle \otimes |n\rangle \longrightarrow |e\rangle \otimes |n-1\rangle$
with time period given above and these are called n-photon rabi. oscillations. Interestingly even when $n=1$ we have $|e\rangle \otimes |0\rangle \rightarrow |g\rangle \otimes |1\rangle \rightarrow |e\rangle \otimes |0\rangle$
and these are called Vacuum Rabi Oscillations.

One may wonder what happens if we choose a more complicate initial state, for e.g coherent state? In other words $|\psi(0)\rangle = |e\rangle \otimes |\alpha\rangle$

where $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} |n\rangle$ and $\alpha \in \mathbb{C}$

It is easy to show that

$$|\psi(t)\rangle = \left[e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i\omega_c(n+\frac{1}{2})t} - i\sin\left(g\sqrt{n+1}\,t\right) \right] |g\rangle \otimes |n-1\rangle$$
$$+ \left[e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i\omega_c(n+\frac{1}{2})t} + \cos\left(g\sqrt{n+1}\,t\right) \right] |e\rangle \otimes |n\rangle$$

$\hookrightarrow \boxed{\text{Eq. 33}}$

From above Eq 33 we can compute the probability that the atom stays in the excited state

$$P_e(t) = \langle \psi(t)|\,|e\rangle\langle e|\,|\psi(t)\rangle$$

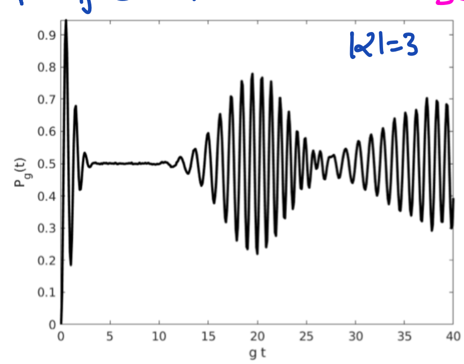
$\hookrightarrow \boxed{\text{Eq 34}}$

(and equivalently for the ground state, P_g)

Clearly P_e or P_g is not periodic because it is a complicated combination of trignometric functions.

We actually get the well-known phenomenon called Collapse and revival. (see figure)

Example of Collapse - Revival



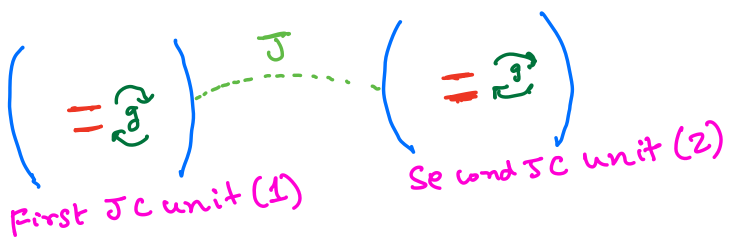
See also below book [2] G.S. Agarwal Quantum optics

In tutorial-2, we will show how to numerically solve an open version (drive and dissipation) of the JC model.

Localization/self-trapping in JC Dimer.

Theory: Schmidt et al, PRB 82, 100507 (R) (2010)

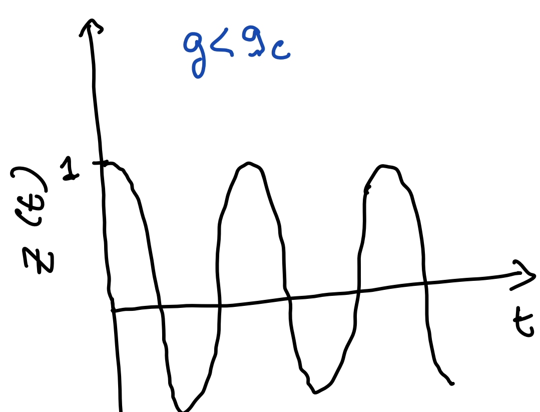
Experiment: Raftery et al, PRX 4, 031043 (2014)



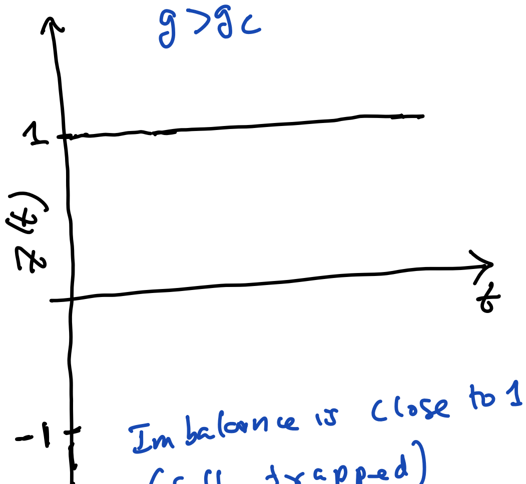
$$
\left.
\begin{aligned}
H = {} & \frac{\omega_q}{2}\sigma_1^z + \omega_c a_1^\dagger a_1 + g(\sigma_1^+ a_1 + h.c) \\
& + \frac{\omega_q}{2}\sigma_2^z + \omega_c a_2^\dagger a_2 + g(\sigma_2^+ a_2 + h.c) \\
& + J(a_1^\dagger a_2 + h.c)
\end{aligned}
\right\}
\text{closed quantum system} \rightarrow \boxed{\text{Eq 35}}
$$

Let us define an imbalance $Z(t) = \dfrac{\langle a_1^\dagger a_1\rangle - \langle a_2^\dagger a_2\rangle}{\langle a_1^\dagger a_1\rangle + \langle a_2^\dagger a_2\rangle}$ $\boxed{\text{Eq 36}}$

Let us prepare an initial state such that first cavity is occupied and second cavity is empty. Hence $Z(0) = 1$. If $Z(t)$ stays close to 1 then we can say the system is self trapped/localized.



Imbalance oscillates (no self-trapping)



Imbalance is close to 1 (self trapped)

Driven-Dissipative Jaynes-Cummings Model

Let us consider a driven JC Model,

$$H = \frac{\omega_q}{2}\sigma^z + \omega_c a^\dagger a + g(a^\dagger\sigma^- + a\sigma^+) + \underbrace{E}_{\text{Drive strength}} \cos \underbrace{\omega_d}_{\text{Drive frequency}} t\,(a+a^\dagger) \quad \boxed{\text{Eq 37}}$$

In addition to this let us say that the cavity is subject to decay (κ) and qubit is subject to both decay/damping (γ) and dephasing (γ_ϕ). This is a rich driven-dissipation quantum system realized in experiments. The Lindblad Master Equation in this case for the system (ie, cavity + qubit) is given by (in Rotating Wave Approximation)

$$\frac{d\rho}{dt} = -i[H,\rho] + \kappa[\bar{n}(\omega_c)+1]\,L[a] + \kappa\bar{n}(\omega_c)\,L[a^\dagger] + \gamma[\bar{n}(\omega_q)+1]\,L[\sigma^-] + \gamma\bar{n}(\omega_q)\,L[\sigma^+] + \gamma_\phi L[\sigma^z] \quad \boxed{\text{Eq38}}$$

where $H = (\omega_c-\omega_d)a^\dagger a + (\omega_q-\omega_d)\frac{\sigma^z}{2} + g(a^\dagger\sigma^- + a\sigma^+) + \frac{E}{2}(a+a^\dagger)$ $\boxed{\text{Eq39}}$

(Time independent Hamiltonian in the Rotating Frame)

L defined in Eq. 16 of lecture 2