

These is a part of [Bangalore School on Statistical Physics - XII](#)

There will be 5 lectures (2:00pm-3:30pm) and 3 tutorials (6:00pm-7:30pm) of 1.5 hours each. The notes will be updated soon.

Instructor: [Manas Kulkarni](#), manas.kulkarni@icts.res.in

Tutor: Hari Kumar Yadalam, hari.kumar@icts.res.in

Lecture 1: (July 5, Monday, 2021)

- Motivation and general construction for Open Quantum Systems
- System Reservoir Approach

Lecture 2: (July 6, Tuesday, 2021)

- Quantum Master Equation (QME) : General setup
- Application to damped quantum Harmonic oscillator

Lecture 3: (July 7, Wednesday, 2021)

- Quantum Langevin Equation (QLE)
- Comparisons between QLE and QME
- Transport through a system coupled to multiple reservoirs

Lecture 4 (July 8, Thursday, 2021)

- Open two level / multi level systems: Perturbative and some exact results
- Jaynes - Cummings Model: Exact Solutions

Lecture 5 (July 9, Friday, 2021)

- Dicke Model and quantum phase transitions
- Spectral signatures in closed and open Dicke model
- Connections to Hermitian and non-Hermitian Random Matrix Theory

Tutorial 1 (July 5, Monday, 2021)

- Density matrix approach to Quantum Mechanics.
- Quantum Mechanics of Composite Systems.
- Partial traces and reduced density matrices for sub-systems.

Tutorial 2 (July 8, Thursday, 2021)

- Numerical algorithm for the solution of the Driven Dissipative Jaynes-Cummings model.
- Numerical implementation using MATLAB.

Tutorial 3 (July 9, Friday, 2021)

- Numerical algorithm for finding the spectrum of the Liouvillian of Dissipative Dicke model.
- Numerical implementation using MATLAB.

Lecture - 1

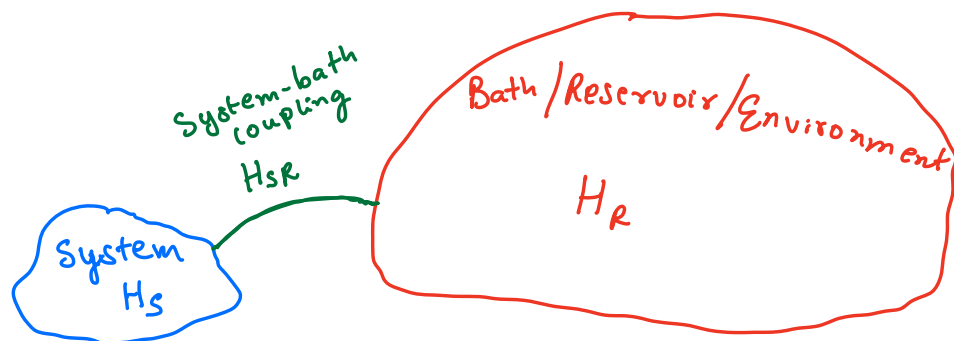
July 5, 2021

→ Motivation and general construction for Open Quantum Systems

→ System - Reservoir Approach

The aim here is to deal with a situation where we have a "quantum system" coupled to "Reservoirs / Baths / Environment"

These quantum systems (that are often assumed to be isolated / closed) are now open to the environment. This is what gives rise to Open Quantum Systems.



Some natural questions

- How to deal with dissipation in quantum mechanics?
- What are the elementary systems that provide the basic building blocks for dissipation in quantum mechanics?
- How to incorporate drive in quantum mechanics?
- What exactly do we mean by environment?

Two fundamental examples

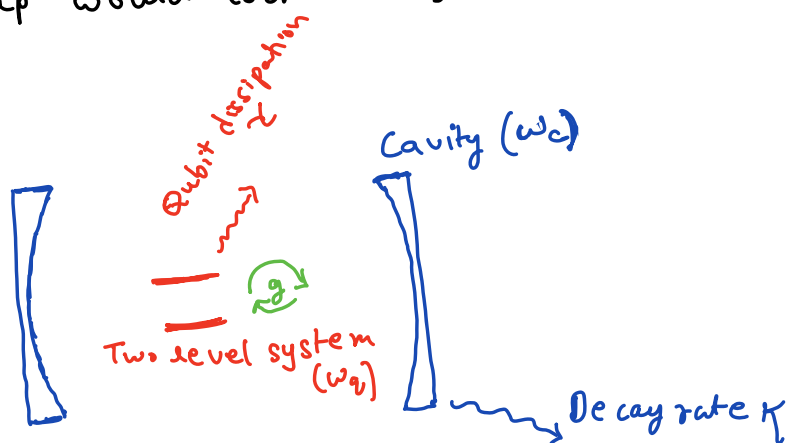
(i) Damped Quantum Harmonic Oscillator:

This describes, for example, a single mode of an electromagnetic field in a lossy cavity (cavity with imperfect mirrors). There are several other state-of-the-art experiments where damped quantum harmonic oscillator (and its generalizations) are relevant.

(ii) Damped two level atoms (or artificial atoms, qubits)

This describes situations where two level systems are subject to imperfections (decay / dephasing) due to its coupling to the environment.

One can envision a combination of (i) and (ii) along with their respective couplings to the environment. A schematic of such a setup would look as follows.



Parameters

Cavity frequency ω_c
 Qubit frequency ω_q
 Cavity decay rate κ
 Qubit dissipation rate γ
 Qubit dephasing rate γ_ϕ
 Cavity-Qubit coupling g
 Temperature T

Approximate Energy scales

GHz
 GHz
 KHz/MHz
 KHz/MHz
 KHz/MHz
 MHz
 mK

Sample Reviews/Survey

- Raimond et al, RMP 73, 565 (2001)
- Haroche et al, Nature Physics 16, 243 (2020)
- Xiang et al, RMP, 85, 623 (2013)
- Clerk et al, Nature Physics, 16, 257 (2020)
- Blais et al, RMP, 93, 025005 (2021)

The above are just approximate experimental energy scales taking into account various Hybrid Quantum Systems,

Open Quantum Systems, Cavity-QED systems.

This is useful to get an idea of hierarchy of energy scales.

It is therefore very important to understand how to deal with system + environment starting from a microscopic point of view.

Note that phenomenological attempts to naively introduce dissipation in quantum mechanics might lead to inconsistencies such as violation of commutation laws and Heisenberg uncertainty.

A nice discussion of this is in the book

[1] H. J. Carmichael, Statistical Methods in Quantum Optics 1: Master Equations and

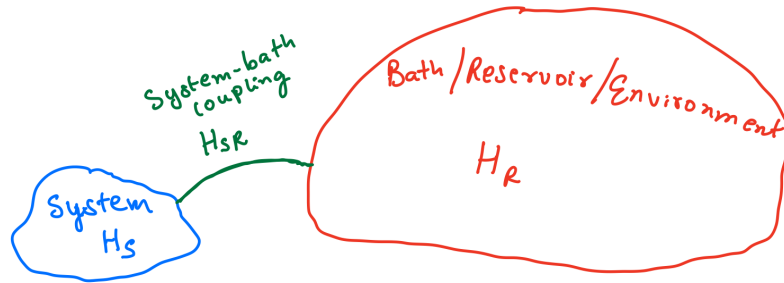
Fokker-Planck Equations.

I would also encourage everyone to read Chapter 1 and 2 of the above book [1] from which some material for initial stages of my lecture series are taken.

System-Reservoir Approach

The philosophy in general is to model environment/reservoir interactions by coupling the undamped (initially isolated) system "S" to a reservoir "R", beginning with a Hamiltonian in a general form

$$H = H_S + H_R + H_{SR} \rightarrow \boxed{\text{Eq. 1}}$$



Here H_S and H_R are Hamiltonians for S and R respectively. H_{SR} is coupling between system and bath.

In most cases reservoir is of less concern and we can specify its properties in very general terms (for e.g. temperature, energy density of states)

We want information of the system S
(and the impact of reservoirs on the system).

We don't worry about the detailed
information about the composite system

$S \otimes R$. Let us define $\chi(t)$ to be

the density operator for $S \otimes R$.

Therefore the reduced density operator $\rho(t)$
is given by

$$\rho(t) = \text{tr}_R [\chi(t)] \quad \rightarrow \text{Eq. 2}$$

Reduced density matrix for the system of interest

partial trace over reservoir states

Total density matrix of the full system and reservoir

Note: Three concepts that we mentioned above
namely density matrices, composite
system and partial traces will
be recapped in today's tutorial.

If \hat{O} is an operator acting on the Hilbert space of S , we can compute its average in the Schrodinger picture if we know $\rho(t)$ without the full $\chi(t)$

$$\langle \hat{O} \rangle = \text{tr}_{S \otimes R} [\hat{O} \chi(t)] = \text{tr}_S [\hat{O} \text{tr}_R [\chi(t)]]$$

$$= \text{tr}_S [\hat{O} \rho(t)]$$

$\hat{O}_S \otimes \mathbb{I}_R$

we need equations for this

↳ Eq. 3

Hence goal is to obtain equation for $\rho(t)$ with the properties of R entering as parameters.

Schrodinger Equation in integro-differential form

The schrodinger equation is, (Liouville-von Neumann equation)

$$\dot{\chi} = \frac{1}{i\hbar} [H, \chi] \rightarrow \text{Eq. 4}$$

Full Hamiltonian $H_S + H_R + H_{SR}$

We transform Eq. 4 into interaction picture which separates the rapid motion generated by $H_S + H_R$ from the slow motion generated by the interaction H_{SR} .

We define

$$\tilde{\chi}(t) = e^{i/\hbar (H_S + H_R)t} \chi(t) e^{-i/\hbar (H_S + H_R)t} \rightarrow \text{Eq. 5}$$

The tilde notation will represent interaction picture.

After some algebra we get (using Eq. 4 and Eq. 5)

$$\dot{\tilde{\chi}} = \frac{1}{i\hbar} \left[\tilde{H}_{SR}(t), \tilde{\chi}(0) \right] - \frac{1}{\hbar^2} \int_0^t dt' \left[\tilde{H}_{SR}(t), \left[\tilde{H}_{SR}(t'), \tilde{\chi}(t') \right] \right] \rightarrow \text{Eq. 6}$$

where because of the tilde notation

we have $\tilde{H}_{SR}(t) = e^{i/\hbar (H_S + H_R)t} H_{SR} e^{-i/\hbar (H_S + H_R)t} \rightarrow \text{Eq. 7}$

Note that Eq. 6 is exact and it is simply rewritten in a convenient form which helps us to explore reasonable approximations.

Born and Markov Approximation

Now, we will discuss two important approximations: Born and Markov approximations.

Let us assume that interaction is turned on at $t=0$ and no correlation exists between S and R at this initial time

Then $\chi(0) = \tilde{\chi}(0)$ factorizes as $\chi(0) = \rho(0) R_0$ where R_0 is an initial reservoir density operator. After partial tracing over R on Eq. 6 we get

$$\dot{\tilde{\rho}} = -\frac{i}{\hbar^2} \int_0^t dt' \text{tr}_R \left\{ [\tilde{H}_{SR}(t), [\tilde{H}_{SR}(t'), \tilde{\chi}(t')]] \right\} \quad \rightarrow \text{Eq. 8}$$

where for simplicity we have eliminated the term $\left(\frac{1}{i\hbar}\right) \text{tr}_R \{ [\tilde{H}_{SR}(t), \chi(0)] \}$ with the

assumption $\text{tr}_R [\tilde{H}_{SR}(t) R_0] = 0 \rightarrow \text{Eq. 9}$

Note that Eq. 9 is guaranteed if reservoir operators (that couple to S) have zero mean in the state R_0 . We will come back to this point later.

We have stated that $\tilde{\chi}$ factorizes at $t=0$. Of course, at later time, coupling between S and R will arise which will in turn give rise to correlations (thereby violating factorisation).

We will assume that this coupling is weak.

This means that at all times $\chi(t)$ should only show deviations of $O(H_{SR})$

from an uncorrelated state.

Note that R is a large system whose state should be basically unaffected by its coupling

to S .

Obviously S will be significantly affected by R .

We can therefore write

$$\tilde{\chi}(t) = \tilde{f}(t) R_0 + O[H_{SR}] \rightarrow \boxed{\text{Eq. 10}}$$

This Eq. 10 is the first major approximation called the Born approximation. Putting Eq. 10 into Eq. 8, we get

$$\tilde{f}_0 = -\frac{1}{\hbar^2} \int_0^t dt' \text{tr}_R \left\{ \left[\tilde{H}_{SR}(t), \left[\tilde{H}_{SR}(t'), \tilde{f}(t') R_0 \right] \right] \right\} \rightarrow \boxed{\text{Eq. 11}}$$

Note that Eq. 11 is still non-Markovian (presence of t' in the integrand). The future of $\tilde{f}(t)$ depends on its past history through integration over $\tilde{f}(t')$ where t' runs from 0 to t . The second major approximation, the Markov approximation, replaces $\tilde{f}(t') \xrightarrow{\text{MARKOV}} \tilde{f}(t)$.

Finally, we get the Born-Markov equation

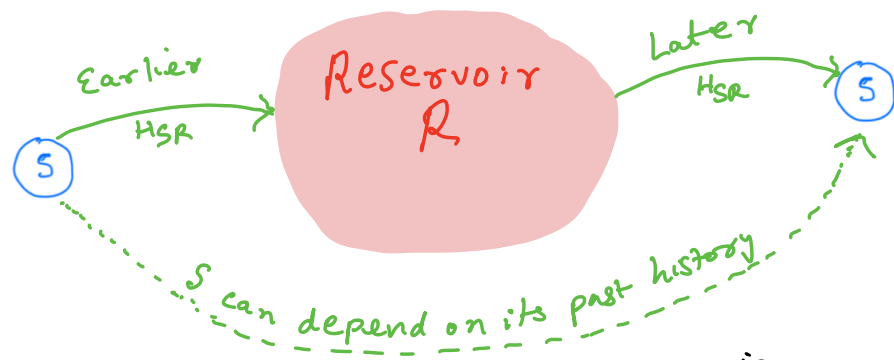
$$\dot{\tilde{\rho}} = -\frac{1}{\hbar^2} \int_0^t dt' \text{tr}_R \left\{ \left[\tilde{H}_{SR}(t), \left[\tilde{H}_{SR}(t'), \tilde{\rho}(t) R_0 \right] \right] \right\}$$

↳ Eq. 12

The above is also known as Redfield Equation.
(Quantum Master Equation)

Note on Markov approximation and Reservoir

Correlations



Markovian behaviour often occurs and is reasonable. Obviously S can in principle depend on its past history. Earlier states of S can change reservoir states R through interaction H_{SR} . This can manifest itself in future evolution of S since it interacts with the changed reservoir. If however, the reservoir is large (say maintained in thermal equilibrium) we certainly do not expect it to

preserve the minor changes due to S .
for very long. Definitely not long
enough to significantly affect the
future evolution of S . Therefore it
becomes a question about reservoir
correlation time (t_R) versus time scale
for significant change in system (t_S)
(For a nice discussion see Ref [1])

Note: Reservoir correlation time depends not only
on reservoir properties (such as its temperature,
chemical potential, density of states) but it also depends
on the details of the reservoir-system coupling.

In the next lecture, we will make
our construction HSR a little more
specific. This will help in further
understanding of the above mentioned
concepts.