Recent advances on the Beilinson–Bloch–Kato conjecture

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$$T_{i,\ell} \coloneqq \mathrm{Sym}^{n_i-1}_{\mathbb{Z}_\ell} \mathrm{T}_\ell(A_i), \quad T_\ell \coloneqq (T_{0,\ell} \otimes_{\mathbb{Z}_\ell} T_{1,\ell})(1-n), \quad V_\ell \coloneqq T_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

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Theorem (L.–Tian–Xiao–Zhang–Zhu + Newton–Thorne)

In the above situation, suppose that

(1) F is a solvable CM field;

(2) $[F : \mathbb{Q}] \geqslant 4$ if $n \geqslant 3$;

(3) both A_0 and A_1 can be defined over $\mathbb{Q}$;

(4) neither A_0 nor A_1 has complex multiplication over $\overline{F}$;

(5) A_0 and A_1 are not isogenous over $\overline{F}$.

If $L(0, V) \neq 0$, then $\mathrm{H}^1_f(F, V_\ell) = 0$ for all but finitely many ℓ.

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It is the construction of c_m in Step 3 that uses the condition $L(0, V) \neq 0$.

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It is the construction of c_m in Step 3 that uses the condition $L(0, V) \neq 0$. Combining with the three steps, one sees that the Tate duality pairing $\langle s_m, c_m \rangle$ is nonzero when m is large enough.

Bounding Selmer groups

Let F^+ be the maximal totally real subfield of F. Regard A_0 and A_1 as elliptic curves over $\mathbb{Q}$. Let Σ be the set of prime factors of the discriminant of F and the conductors of A_0 and A_1. We take a prime number ℓ satisfying

(L1) ℓ does not belong to Σ.

For a nonzero element x in a finitely generated $\mathbb{Z}_\ell$-module X, we define the **divisibility** of x to be the largest integer $d \geqslant 0$ such that $x \in \ell^d X$.

We explain a strategy for showing the vanishing of the Selmer group, which is originally due to Kolyvagin. Suppose on the contrary that $\mathrm{H}^1_f(F, V_\ell) \neq 0$.

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To simply our lectures, from now on, we assume $F^+ \neq \mathbb{Q}$ and make a further assumption that F contains an imaginary quadratic field F_0.

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We say that a prime p is a **level-raising prime** with respect to ℓ^m if

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In (P4), S_j are polynomials defined inductively by the formulae $S_0(x) = 2$, $S_1(x) = x^2 - 2p$ and $S_j(x) = x^{2j} - \sum_{k=1}^{j} \binom{2j}{k} p^k S_{j-k}(x)$ in general.

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(1) For (effectively) sufficiently large ℓ, there exist infinitely many level-raising primes with respect to ℓ^m with positive density for each $m > 0$.

(2) For every prime $\mathfrak{p}$ of F^+ above a level-raising prime p with respect to ℓ^m, the $\mathbb{Z}/\ell^m$-modules $\mathrm{H}^1_{\mathrm{unr}}(F_\mathfrak{p}, T_\ell \otimes \mathbb{Z}/\ell^m)$ and hence $\mathrm{H}^1_{\mathrm{sing}}(F_\mathfrak{p}, T_\ell \otimes \mathbb{Z}/\ell^m)$ are both free of rank 1.

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Combining the modularity of rational elliptic curves, recent breakthrough of Newton–Thorne on the automorphy of symmetric power of modular forms and the cyclic automorphic base change, we have for $i = 0, 1$ a unique up to isomorphism cuspidal automorphic representation Π_{n_i} of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$ satisfying

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Moreover, we know that

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  For $N \in \{n, n+1\}$, the Satake parameters of Π_N give a homomorphism

$$\phi_N \colon \mathbb{T}_N^\Sigma \to \mathbb{Z},$$

in which $\mathbb{T}_N^\Sigma$ denotes the abstract spherical Hecke algebra of the unitary group over $O_F[\Sigma^{-1}]/O_{F^+}[\Sigma^{-1}]$ of rank N.

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- put $H_N := \mathrm{Res}_{F^+/\mathbb{Q}} \mathrm{U}(\Lambda_N \otimes_{O_F} F)$, where $\Lambda_{n+1} := \Lambda_n \oplus O_F \cdot 1$,
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In what follows, we explain the construction of the class c_m in Step 3. Fix a choice of the pair (Λ_n, D) as above.

Moduli spaces associated with unitary groups

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From now on, we fix an embedding $\tau_0 \colon F \hookrightarrow \mathbb{C}$. Put $\Phi := \{\tau \colon F \to \mathbb{C} \mid \tau|_{F_0} = \tau_0|_{F_0}\}$, which is a CM type of F.

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For the simplicity of the lectures, we assume that there exists (and we fix such) a triple (A_0, i_0, λ_0) in which A_0 is an abelian scheme over $O_F[\Sigma^{-1}]$, $i_0 \colon O_F \xrightarrow{\sim} \mathrm{End}(A_0)$ is a complex multiplication of CM type Φ, and λ_0 is a principal polarization of A_0 that is compatible with i_0.

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For $N \in \{n, n+1\}$, we define a moduli space $\mathcal{M}_N = \mathcal{M}_{N,\mathfrak{p}}$ over $O_{F,(\mathfrak{p})}$ classifying quadruples $(A, \iota, \lambda, \eta)$ where

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For the simplicity of the lectures, we assume that there exists (and we fix such) a triple (A_0, i_0, λ_0) in which A_0 is an abelian scheme over $O_F[\Sigma^{-1}]$, $i_0 \colon O_F \xrightarrow{\sim} \mathrm{End}(A_0)$ is a complex multiplication of CM type Φ, and λ_0 is a principal polarization of A_0 that is compatible with i_0.

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- η is a level-D structure, that is, an isometry

$$\eta\colon \Lambda_N \otimes \mathbb{Z}/D \xrightarrow{\sim} \mathrm{Hom}(A_0[D], A[D]).$$

Here, the right-hand side is equipped with a pairing that sends (x, y) to the composite morphism

$$A_0[D] \xrightarrow{x} A[D] \xrightarrow{\lambda} A^\vee[D] \xrightarrow{y^\vee} A_0^\vee[D] \xrightarrow{\lambda_0^{-1}} A_0[D]$$

regarded as an element in $\mathrm{End}_{O_F}(A_0[D]) = O_F \otimes \mathbb{Z}/D$.

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Denote by $M'_N := \mathcal{M}_N \otimes_{O_{F,(\mathfrak{p})}} F$ its generic fiber and $M_N := \mathcal{M}_N \otimes_{O_{F,(\mathfrak{p})}} O_F/\mathfrak{p}$ its special fiber.

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For a point $(A, \iota, \lambda, \eta) \in M_N(S)$, the Hodge sequence

$$0 \to \omega_{A^\vee/S} \to \mathrm{H}_1^{\mathrm{dR}}(A/S) \to \mathrm{Lie}_{A/S} \to 0$$

admits a direct sum decomposition over $\mathrm{Hom}(F, \mathbb{C})$ via the action ι (and $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$):

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For future use, put $S_N^\circ := H_N(\mathbb{Q})\backslash H_N(\mathbb{A}^\infty)/K_N^D$, regarded as a discrete scheme over $O_F/\mathfrak{p}$ according to the context; and we have a similar map

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- The closed subscheme $M_N^\dagger$ of M_N° is a Fermat hypersurface (of degree $p+1$).
- Let V_N' be the unique (up to isomorphism) F/F^+-hermitian space that has signature $(N-1,1)$ at $\tau_0|_{F^+}$ and is isomorphic to $V_N := \Lambda_N \otimes_{O_F} F$ away from $\tau_0|_{F^+}$ and $\mathfrak{p}$. Then M_N' is a Shimura variety associated with the unitary group $\mathrm{Res}_{F^+/\mathbb{Q}}\,\mathrm{U}(V_N')$ of a certain level that is maximal away from Σ (together with a functorial diagram as above).

(break point)

Explicit reciprocity law

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$$\alpha(M'_{\mathrm{diag}}) \in \mathrm{H}^1(F, \mathrm{H}^{2n-1}(P' \otimes_F \overline{F}, \mathbb{Z}_\ell(n))/\mathfrak{m}^{\ell^m})$$

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Denote by

$$\partial_\mathfrak{p} \colon \mathrm{H}^1(F_\mathfrak{p}, -) \to \mathrm{H}^1_{\mathrm{sing}}(F_\mathfrak{p}, -) \coloneqq \mathrm{H}^1(F_\mathfrak{p}, -)/\mathrm{H}^1_{\mathrm{unr}}(F_\mathfrak{p}, -)$$

the natural quotient map. In particular, we have the element

$$\partial_\mathfrak{p}(\mathrm{loc}_\mathfrak{p}(\alpha(M'_{\mathrm{diag}}))) \in \mathrm{H}^1_{\mathrm{sing}}(F_\mathfrak{p}, \mathrm{H}^{2n-1}(P' \otimes_F \overline{F}, \mathbb{Z}_\ell(n))/\mathfrak{m}^{\ell^m}).$$

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There exists a positive integer $\ell_{A_0,A_1,F,n,\Lambda_n,D}$ *depending only on the subscripts such that for every prime number* $\ell \geqslant \ell_{A_0,A_1,F,n,\Lambda_n,D}$ *(which includes (L1–3)) and every* $m \geqslant 1$*, if* p *(the underlying prime number of* $\mathfrak{p}$*) is a level-raising prime with respect to* ℓ^m*, then the following statements hold:*

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$$(\partial_\mathfrak{p}(\mathrm{loc}_\mathfrak{p}(\alpha(M'_{\mathrm{diag}}))), f_n \otimes f_{n+1}) = \gamma \sum_{h \in H_n(\mathbb{Q})\backslash H_n(\mathbb{A}^\infty)/K_n^D} f_n(h) f_{n+1}(h).$$

Theorem

There exists a positive integer $\ell_{A_0,A_1,F,n,\Lambda_n,D}$ depending only on the subscripts such that for every prime number $\ell \geqslant \ell_{A_0,A_1,F,n,\Lambda_n,D}$ (which includes (L1–3)) and every $m \geqslant 1$, if p (the underlying prime number of $\mathfrak{p}$) is a level-raising prime with respect to ℓ^m, then the following statements hold:

(1) *As a $(\mathbb{Z}/\ell^m)[\mathrm{Gal}(\overline{F}/F)]$-module, $\mathrm{H}^{2n-1}(P' \otimes_F \overline{F}, \mathbb{Z}_\ell(n))/\mathfrak{m}^{\ell^m}$ is isomorphic to a finite copy of $T_\ell \otimes \mathbb{Z}/\ell^m$.*

(2) *There is a canonical isomorphism*

$$\mathrm{H}^1_{\mathrm{sing}}(F_\mathfrak{p}, \mathrm{H}^{2n-1}(P' \otimes_F \overline{F}, \mathbb{Z}_\ell(n))/\mathfrak{m}^{\ell^m}) \simeq \mathbb{Z}[S_n^\circ]/\mathfrak{m}_n^{\ell^m} \otimes \mathbb{Z}[S_{n+1}^\circ]/\mathfrak{m}_{n+1}^{\ell^m}.$$

(3) *There exists a constant $\gamma = \gamma_{A_0,A_1,n,p} \in \mathbb{Z}_{(\ell)}^\times$ such that under the natural pairing $\mathbb{Z}[S_n^\circ]/\mathfrak{m}_n^{\ell^m} \otimes \mathbb{Z}[S_{n+1}^\circ]/\mathfrak{m}_{n+1}^{\ell^m} \times (\mathbb{Z}/\ell^m)[S_n^\circ][\mathfrak{m}_n^{\ell^m}] \otimes (\mathbb{Z}/\ell^m)[S_{n+1}^\circ][\mathfrak{m}_{n+1}^{\ell^m}] \to \mathbb{Z}/\ell^m$, we have*

$$(\partial_\mathfrak{p}(\mathrm{loc}_\mathfrak{p}(\alpha(M'_{\mathrm{diag}}))), f_n \otimes f_{n+1}) = \gamma \sum_{h \in H_n(\mathbb{Q}) \backslash H_n(\mathbb{A}^\infty)/K_n^D} f_n(h) f_{n+1}(h).$$

In what follows, we take a prime $\ell \geqslant \ell_{A_0,A_1,F,n,\Lambda_n,D}$ and assume that p is a level-raising prime with respect to ℓ^m for some $m \geqslant 1$.

Singular quotient via potential map

Singular quotient via potential map

Let $\mathcal{Q}$ be the blow-up of $\mathcal{P} \otimes_{O_{F,(\mathfrak{p})}} O_{F_\mathfrak{p}}$ along the closed subscheme $M_n^\circ \times M_{n+1}^\circ$, with $Q' := \mathcal{Q} \otimes_{O_{F_\mathfrak{p}}} F_\mathfrak{p}$ and $Q := \mathcal{Q} \otimes_{O_{F_\mathfrak{p}}} O_F/\mathfrak{p}$.

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For $r \in \mathbb{Z}$, put

$$B^r(Q) := \ker\left(\delta_0^* \colon \mathrm{H}^{2r}(\overline{Q_0}, \mathbb{Z}_\ell(r)) \to \mathrm{H}^{2r}(\overline{Q_1}, \mathbb{Z}_\ell(r))\right),$$

$$B_r(Q) := \operatorname{coker}\left(\delta_{1!} \colon \mathrm{H}^{2(2n-r-2)}(\overline{Q_1}, \mathbb{Z}_\ell(2n-r-2)) \to \mathrm{H}^{2(2n-r-1)}(\overline{Q_0}, \mathbb{Z}_\ell(2n-r-1))\right),$$

where δ_0^* and $\delta_{1!}$ are suitable alternating sums of restriction and Gysin maps as in the weight spectral sequence.

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$$\Delta^r \colon C_{2n-r}(Q) \to C^r(Q).$$

Singular quotient via potential map

Singular quotient via potential map

Proposition

There is a canonical isomorphism

$$\mathrm{H}^1_{\mathrm{sing}}(F_{\mathfrak{p}}, \mathrm{H}^{2n-1}(P' \otimes_F \overline{F}, \mathbb{Z}_\ell(n))_{\mathfrak{m}^\ell}) \xrightarrow{\sim} \operatorname{coker} \Delta^n_{\mathfrak{m}^\ell}.$$

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Moreover, under the above isomorphism, the element $\partial_{\mathfrak{p}}(\mathrm{loc}_{\mathfrak{p}}(\alpha(M'_{\mathrm{diag}})))$ *coincides with the image of the cycle class of the strict transform of* M_{diag} *in* Q_0 *(regarded as in* $B^n(Q)^0_{\mathfrak{m}^\ell}$*) under the natural map* $B^n(Q)^0_{\mathfrak{m}^\ell} \to C^n(Q)_{\mathfrak{m}^\ell} \to \operatorname{coker} \Delta^n_{\mathfrak{m}^\ell}$.

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- ◇ Write $n = 2r$.
- ◇ Denote by $\sigma\colon \mathcal{Q} \to \mathcal{P}$ the blow-up morphism.
- ◇ Put $P^{\circ\bullet} := M_n^\circ \times M_{n+1}^\bullet$ and denote by $Q^{\circ\bullet}$ its strict transform under σ. Similarly, we have $P^{\circ\circ}$, $P^{\bullet\circ}$, $P^{\bullet\bullet}$, and their versions in Q. In particular, Q_0 is the disjoint union of $Q^{\circ\circ}$, $Q^{\circ\bullet}$, $Q^{\bullet\circ}$ and $Q^{\bullet\bullet}$.

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We now construct a canonical map

$$\nabla\colon \mathrm{H}^{2n}(\overline{Q_0}, \mathbb{Z}_\ell(n)) \to \mathbb{Z}_\ell[S^\circ_n] \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[S^\circ_{n+1}],$$

which turns out to factor through $C^n(Q)$ and induce an isomorphism

$$(\operatorname{coker} \Delta^n)/\mathfrak{m}^{\ell^m} \to \mathbb{Z}[S^\circ_n]/\mathfrak{m}_n^{\ell^m} \otimes \mathbb{Z}[S^\circ_{n+1}]/\mathfrak{m}_{n+1}^{\ell^m}$$

for the quotient.

Basic locus

Basic locus

To construct ∇, we will find many cycles contained in Q_0 that are indexed by S_N° (for $N = n, n+1$). It turns out that the union of those cycles is exactly the basic locus of M_N, that is, the locus where $A[\mathfrak{p}^\infty]$ is supersingular.

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Indeed, the whole M_N° is contained in the basic locus. We also recall that it is a projective bundle over S_N°, which fits into the following diagram

$$\begin{array}{ccc} M_n^\circ & \longrightarrow & S_n^\circ \\ \downarrow{\scriptstyle \rho_M^\circ} & & \downarrow{\scriptstyle \rho_S^\circ} \\ M_{n+1}^\circ & \longrightarrow & S_{n+1}^\circ \end{array}$$

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To study the basic locus on $M_N^\bullet$, we fix an O_F-submodule $\mathfrak{p}\Lambda_n \subseteq \Lambda_n^\bullet \subseteq \Lambda_n$ such that $\Lambda_n^\bullet/\mathfrak{p}\Lambda_n$ is a Lagrangian subspace of $\Lambda_n/\mathfrak{p}\Lambda_n$. Put $\Lambda_{n+1}^\bullet := \Lambda_n^\bullet \oplus O_F \cdot 1$. For $N \in \{n, n+1\}$, put

$$S_N^\bullet := H_N(\mathbb{Q})\backslash H_N(\mathbb{A}^\infty)/K_N^{D\bullet},$$

where $K_N^{D\bullet}$ is defined similarly as K_N^D using $\Lambda_N^\bullet$. We also put $S_N^\dagger := H_N(\mathbb{Q})\backslash H_N(\mathbb{A}^\infty)/K_N^{D\dagger}$, where $K_N^{D\dagger} := K_N^D \cap K_N^{D\bullet}$.

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Similar to ρ_S°, we have the maps

$$\rho_S^\bullet\colon S_n^\bullet \to S_{n+1}^\bullet, \quad \rho_S^\dagger\colon S_n^\dagger \to S_{n+1}^\dagger.$$

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in which $\rho_B^\bullet$ is locally an isomorphism, $B_N^\bullet \to S_N^\bullet$ is projective smooth of dimension r, and $B_N^\bullet \to M_N^\bullet$ is a closed immersion when restricted to each connected component of the source.

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The fibers of the morphism $B_N^\bullet \to S_N^\bullet$ are certain Deligne–Lustig varieties. For example, when $n = 2$, the fibers are isomorphic to $\mathbb{P}^1$; when $n = 4$, the fibers are up to purely inseparable morphisms blow-ups of the Fermat surface along all $O_F/\mathfrak{p}$-points.

For $N \in \{n, n+1\}$, the union of M_N° and the image of $B_N^\bullet \to M_N^\bullet$ is exactly the basic locus of M_N.

We construct a commutative diagram

$$\begin{array}{ccccc} S_n^\bullet & \longleftarrow & B_n^\bullet & \longrightarrow & M_n^\bullet \\ \downarrow{\scriptstyle \rho_S^\bullet} & & \downarrow{\scriptstyle \rho_B^\bullet} & & \downarrow{\scriptstyle \rho_M} \\ S_{n+1}^\bullet & \longleftarrow & B_{n+1}^\bullet & \longrightarrow & M_{n+1}^\bullet \end{array}$$

in which $\rho_B^\bullet$ is locally an isomorphism, $B_N^\bullet \to S_N^\bullet$ is projective smooth of dimension r, and $B_N^\bullet \to M_N^\bullet$ is a closed immersion when restricted to each connected component of the source.

The fibers of the morphism $B_N^\bullet \to S_N^\bullet$ are certain Deligne–Lustig varieties. For example, when $n = 2$, the fibers are isomorphic to $\mathbb{P}^1$; when $n = 4$, the fibers are up to purely inseparable morphisms blow-ups of the Fermat surface along all $O_F/\mathfrak{p}$-points.

For $N \in \{n, n+1\}$, the union of M_N° and the image of $B_N^\bullet \to M_N^\bullet$ is exactly the basic locus of M_N. For the intersection between M_N° and $B_N^\bullet$, we have the commutative diagram

$$\begin{array}{ccc} M_N^\circ \times_{M_N} B_N^\bullet & \longrightarrow & M_N^\circ \times B_N^\bullet \\ \downarrow & & \downarrow \\ S_N^\dagger & \longrightarrow & S_N^\circ \times S_N^\bullet \end{array}$$

in the category of $O_F/\mathfrak{p}$-schemes, in which the fibers of the left morphism are isomorphic to $\mathbb{P}^{r-1}$.

Incidence maps

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$$\begin{aligned}
\mathtt{Inc}_{\circ\dagger} \colon \mathrm{H}^{2n}(\overline{Q^{\circ\bullet}}, \mathbb{Z}_\ell(n)) &\xrightarrow{\sigma_!} \mathrm{H}^{2n}(\overline{P^{\circ\bullet}}, \mathbb{Z}_\ell(n)) \xrightarrow{\text{Kün}} \mathrm{H}^{2r}(\overline{M_n^\circ}, \mathbb{Z}_\ell(r)) \otimes \mathrm{H}^{2r}(\overline{M_{n+1}^\bullet}, \mathbb{Z}_\ell(r)) \\
&\xrightarrow{\text{res}} \mathrm{H}^{2r}(\overline{M_n^\circ}, \mathbb{Z}_\ell(r)) \otimes \mathrm{H}^{2r}(\overline{M_{n+1}^\dagger}, \mathbb{Z}_\ell(r)) \xrightarrow{\text{Gys}} \mathrm{H}^{2r}(\overline{M_n^\circ}, \mathbb{Z}_\ell(r)) \otimes \mathrm{H}^{2r+2}(\overline{M_{n+1}^\circ}, \mathbb{Z}_\ell(r+1)) \\
&\xrightarrow{\text{Lef}} \mathrm{H}^{2(n-1)}(\overline{M_n^\circ}, \mathbb{Z}_\ell(n-1)) \otimes \mathrm{H}^{2n}(\overline{M_{n+1}^\circ}, \mathbb{Z}_\ell(n)) \xrightarrow{\text{Gys}} \mathbb{Z}_\ell[S_n^\circ] \otimes \mathbb{Z}_\ell[S_{n+1}^\circ].
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For $N \in \{n, n+1\}$, the correspondence $S_N^\circ \leftarrow S_N^\dagger \rightarrow S_N^\bullet$ of finite sets gives rise to two "transpose" maps

$$\mathtt{T}_N\colon \mathbb{Z}_\ell[S_N^\circ] \to \mathbb{Z}_\ell[S_N^\bullet], \quad \mathtt{T}_N\colon \mathbb{Z}_\ell[S_N^\bullet] \to \mathbb{Z}_\ell[S_N^\circ]$$

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$$(\mathtt{T}_n^2 \otimes \mathtt{T}_{n+1}^2) \circ \mathtt{Inc}_{\circ\dagger}, \quad (p+1)^2(\mathtt{T}_n^2 \otimes \mathtt{T}_{n+1}) \circ \mathtt{Inc}_{\circ\bullet},$$
$$(p+1)(\mathtt{T}_n \otimes \mathtt{T}_{n+1}^2) \circ \mathtt{Inc}_{\bullet\dagger}, \quad (p+1)^3(\mathtt{T}_n \otimes \mathtt{T}_{n+1}) \circ \mathtt{Inc}_{\bullet\bullet}.$$

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Proposition

The map $\nabla\colon \mathrm{H}^{2n}(\overline{Q_0},\mathbb{Z}_\ell(n)) \to \mathbb{Z}_\ell[S_n^\circ] \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[S_{n+1}^\circ]$ defined above factors through $C^n(Q)$ and induce an isomorphism

$$(\operatorname{coker}\Delta^n)/\mathfrak{m}^{\ell^m} \to \mathbb{Z}[S_n^\circ]/\mathfrak{m}_n^{\ell^m} \otimes \mathbb{Z}[S_{n+1}^\circ]/\mathfrak{m}_{n+1}^{\ell^m}.$$

Thank you for your attention!