Recent advances on the Beilinson-Bloch-Kato conjecture

Yifeng Liu

Institute for Advanced Study in Mathematics
Zhejiang University

ICTS Program on Elliptic Curves and the Special Values of L-functions Bangalore, India

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$$T_{i,\ell} \coloneqq \operatorname{\mathsf{Sym}}^{n_i-1}_{\mathbb{Z}_\ell} \mathrm{T}_\ell(A_i), \quad T_\ell \coloneqq (T_{0,\ell} \otimes_{\mathbb{Z}_\ell} T_{1,\ell})(1-n), \quad V_\ell \coloneqq T_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

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Theorem (L.–Tian–Xiao–Zhang–Zhu + Newton–Thorne)

In the above situation, suppose that

- (1) F is a solvable CM field;
- (2) $[F:\mathbb{Q}] \geqslant 4$ if $n \geqslant 3$;
- (3) both A_0 and A_1 can be defined over \mathbb{Q} ;
- (4) neither A_0 nor A_1 has complex multiplication over \overline{F} ;
- (5) A_0 and A_1 are not isogenous over \overline{F} .
- If $L(0, V) \neq 0$, then $H^1_{\ell}(F, V_{\ell}) = 0$ for all but finitely many ℓ .



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Step 1: Take a non-torsion element $s \in \mathrm{H}^1_f(F, T_\ell)$ of zero divisibility. Then $\mathrm{loc}_v(s) \in \mathrm{H}^1(F_v, T_\ell)$ is torsion for every nonarchimedean place v of F above Σ . For every $m \geqslant 1$, denote by s_m the image of s in $\mathrm{H}^1(F, T_\ell \otimes \mathbb{Z}/\ell^m)$.

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Step 2: For each m, find sufficiently many primes $\mathfrak p$ of F^+ inert in F and not above $\Sigma \cup \{\ell\}$ such that $\mathrm{H}^1_{\mathrm{unr}}(F_{\mathfrak p}, T_\ell \otimes \mathbb Z/\ell^m)$ is a free $\mathbb Z/\ell^m$ -module of rank 1 and that $\mathrm{loc}_{\mathfrak p}(s_m)$ is an element of $\mathrm{H}^1_{\mathrm{unr}}(F_{\mathfrak p}, T_\ell \otimes \mathbb Z/\ell^m)$ of zero divisibility.

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It is the construction of c_m in Step 3 that uses the condition $L(0, V) \neq 0$. Combining with the three steps, one sees that the Tate duality pairing $\langle s_m, c_m \rangle$ is nonzero when m is large enough.

Let F^+ be the maximal totally real subfield of F. Regard A_0 and A_1 as elliptic curves over \mathbb{Q} . Let Σ be the set of prime factors of the discriminant of F and the conductors of A_0 and A_1 . We take a prime number ℓ satisfying

(L1) ℓ does not belong to Σ .

For a nonzero element x in a finitely generated \mathbb{Z}_{ℓ} -module X, we define the **divisibility** of x to be the largest integer $d \geqslant 0$ such that $x \in \ell^d X$.

We explain a strategy for showing the vanishing of the Selmer group, which is originally due to Kolyvagin. Suppose on the contrary that $H^1_\ell(F,V_\ell) \neq 0$.

Step 1: Take a non-torsion element $s \in \mathrm{H}^1_f(F, \mathcal{T}_\ell)$ of zero divisibility. Then

 $\operatorname{loc}_{v}(s) \in \operatorname{H}^{1}(F_{v}, \mathcal{T}_{\ell})$ is torsion for every nonarchimedean place v of F above Σ . For every $m \geqslant 1$, denote by s_{m} the image of s in $\operatorname{H}^{1}(F, \mathcal{T}_{\ell} \otimes \mathbb{Z}/\ell^{m})$.

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To simply our lectures, from now on, we assume $F^+ \neq \mathbb{Q}$ and make a further assumption that F contains an imaginary quadratic field F_0 .

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We say that a prime p is a **level-raising prime** with respect to ℓ^m if

- (P1) p is odd, inert in F_0 and splits completely in F^+ ;
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In (P4), S_j are polynomials defined inductively by the formulae $S_0(x)=2$, $S_1(x)=x^2-2p$ and $S_j(x)=x^{2j}-\sum_{k=1}^j\binom{2j}{k}p^kS_{j-k}(x)$ in general.

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We have two assertions concerning level-raising primes.

- (1) For (effectively) sufficiently large ℓ , there exist infinitely many level-raising primes with respect to ℓ^m with positive density for each m > 0.
- (2) For every prime $\mathfrak p$ of F^+ above a level-raising prime p with respect to ℓ^m , the $\mathbb Z/\ell^m$ -modules $\mathrm{H}^1_{\mathrm{unr}}(F_{\mathfrak p}, \mathcal T_\ell \otimes \mathbb Z/\ell^m)$ and hence $\mathrm{H}^1_{\mathrm{sing}}(F_{\mathfrak p}, \mathcal T_\ell \otimes \mathbb Z/\ell^m)$ are both free of rank 1.

Combining the modularity of rational elliptic curves, recent breakthrough of Newton–Thorne on the automorphy of symmetric power of modular forms and the cyclic automorphic base change, we have for i=0,1 a unique up to isomorphism cuspidal automorphic representation Π_{n_i} of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$ satisfying

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For $N \in \{n, n+1\}$, the Satake parameters of Π_N give a homomorphism

$$\phi_N \colon \mathbb{T}_N^{\Sigma} \to \mathbb{Z},$$

in which \mathbb{T}_N^{Σ} denotes the abstract spherical Hecke algebra of the unitary group over $O_F[\Sigma^{-1}]/O_{F^+}[\Sigma^{-1}]$ of rank N.



Due to the recent breakthrough on the Gan–Gross–Prasad conjecture by Beuzart-Plessis–L.–Zhang–Zhu, there exist

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- \diamond put $H_N := \operatorname{Res}_{F^+/\mathbb{O}} \mathrm{U}(\Lambda_N \otimes_{O_F} F)$, where $\Lambda_{n+1} := \Lambda_n \oplus O_F \cdot 1$,
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In what follows, we explain the construction of the class c_m in Step 3. Fix a choice of the pair (Λ_n, D) as above.

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From now on, we fix an embedding $\tau_0 \colon F \hookrightarrow \mathbb{C}$. Put $\Phi \coloneqq \{\tau \colon F \to \mathbb{C} \mid \tau|_{F_0} = \tau_0|_{F_0}\}$, which is a CM type of F.

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For the simplicity of the lectures, we assume that there exists (and we fix such) a triple (A_0,i_0,λ_0) in which A_0 is an abelian scheme over $O_F[\Sigma^{-1}]$, $i_0\colon O_F\overset{\sim}{\longrightarrow} \operatorname{End}(A_0)$ is a complex multiplication of CM type Φ , and λ_0 is a principal polarization of A_0 that is compatible with i_0 .

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- $\diamond \eta$ is a level-D structure, that is, an isometry

$$\eta \colon \Lambda_N \otimes \mathbb{Z}/D \xrightarrow{\sim} \operatorname{\mathsf{Hom}}(A_0[D], A[D]).$$

Here, the right-hand side is equipped with a pairing that sends (x, y) to the composite morphism

$$A_0[D] \xrightarrow{x} A[D] \xrightarrow{\lambda} A^{\vee}[D] \xrightarrow{y^{\vee}} A_0^{\vee}[D] \xrightarrow{\lambda_0^{-1}} A_0[D]$$

regarded as an element in $\operatorname{End}_{O_F}(A_0[D]) = O_F \otimes \mathbb{Z}/D$.

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For future use, put $S_N^{\circ} := H_N(\mathbb{Q}) \setminus H_N(\mathbb{A}^{\infty}) / K_N^D$, regarded as a discrete scheme over O_F/\mathfrak{p} according to the context; and we have a similar map

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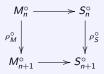
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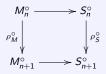
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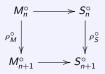
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- \diamond The closed subscheme M_N^{\dagger} of M_N° is a Fermat hypersurface (of degree p+1).
- \diamond Let V_N' be the unique (up to isomorphism) F/F^+ -hermitian space that has signature (N-1,1) at $\tau_0|_{F^+}$ and is isomorphic to $V_N \coloneqq \Lambda_N \otimes_{\mathcal{O}_F} F$ away from $\tau_0|_{F^+}$ and $\mathfrak p$. Then M_N' is a Shimura variety associated with the unitary group $\mathrm{Res}_{F^+/\mathbb Q} \mathrm{U}(V_N')$ of a certain level that is maximal away from Σ (together with a functorial diagram as above).

(break point)

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$$\alpha(M'_{\mathrm{diag}}) \in \mathrm{H}^1(F, \mathrm{H}^{2n-1}(P' \otimes_F \overline{F}, \mathbb{Z}_{\ell}(n))/\mathfrak{m}^{\ell^m})$$

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Denote by

$$\partial_{\mathfrak{p}} \colon \mathrm{H}^{1}(F_{\mathfrak{p}}, -) \to \mathrm{H}^{1}_{\mathrm{sing}}(F_{\mathfrak{p}}, -) \coloneqq \mathrm{H}^{1}(F_{\mathfrak{p}}, -) / \mathrm{H}^{1}_{\mathrm{unr}}(F_{\mathfrak{p}}, -)$$

the natural quotient map. In particular, we have the element

$$\partial_{\mathfrak{p}}(\mathrm{loc}_{\mathfrak{p}}(\alpha(M'_{\mathrm{diag}}))) \in \mathrm{H}^{1}_{\mathrm{sing}}(F_{\mathfrak{p}}, \mathrm{H}^{2n-1}(P' \otimes_{F} \overline{F}, \mathbb{Z}_{\ell}(n))/\mathfrak{m}^{\ell^{m}}).$$



Theorem

There exists a positive integer $\ell_{A_0,A_1,F,n,\Lambda_n,D}$ depending only on the subscripts such that for every prime number $\ell \geqslant \ell_{A_0,A_1,F,n,\Lambda_n,D}$ (which includes (L1–3)) and every $m \geqslant 1$, if p (the underlying prime number of $\mathfrak p$) is a level-raising prime with respect to ℓ^m , then the following statements hold:

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(3) There exists a constant $\gamma = \gamma_{A_0,A_1,n,p} \in \mathbb{Z}_{(\ell)}^{\times}$ such that under the natural pairing $\mathbb{Z}[S_n^{\circ}]/\mathfrak{m}_n^{\ell^m} \otimes \mathbb{Z}[S_{n+1}^{\circ}]/\mathfrak{m}_{n+1}^{\ell^m} \times (\mathbb{Z}/\ell^m)[S_n^{\circ}][\mathfrak{m}_n^{\ell^m}] \otimes (\mathbb{Z}/\ell^m)[S_{n+1}^{\circ}][\mathfrak{m}_{n+1}^{\ell^m}] \to \mathbb{Z}/\ell^m$, we have

$$(\partial_{\mathfrak{p}}(\mathrm{loc}_{\mathfrak{p}}(\alpha(M'_{\mathrm{diag}}))),f_{n}\otimes f_{n+1})=\gamma\sum_{h\in H_{n}(\mathbb{Q})\backslash H_{n}(\mathbb{A}^{\infty})/K_{n}^{D}}f_{n}(h)f_{n+1}(h).$$

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(3) There exists a constant $\gamma = \gamma_{A_0,A_1,n,p} \in \mathbb{Z}_{(\star)}^{\times}$ such that under the natural pairing $\mathbb{Z}[S_n^{\circ}]/\mathfrak{m}_n^{\ell^m} \otimes \mathbb{Z}[S_{n+1}^{\circ}]/\mathfrak{m}_{n+1}^{\ell^m} \times (\mathbb{Z}/\ell^m)[S_n^{\circ}][\mathfrak{m}_n^{\ell^m}] \otimes (\mathbb{Z}/\ell^m)[S_{n+1}^{\circ}][\mathfrak{m}_{n+1}^{\ell^m}] \to \mathbb{Z}/\ell^m, \text{ we have }$

$$(\partial_{\mathfrak{p}}(\mathrm{loc}_{\mathfrak{p}}(\alpha(M'_{\mathrm{diag}}))), f_n \otimes f_{n+1}) = \gamma \sum_{h \in H_n(\mathbb{Q}) \backslash H_n(\mathbb{A}^{\infty}) / K_n^{\mathbb{Q}}} f_n(h) f_{n+1}(h).$$

In what follows, we take a prime $\ell \geqslant \ell_{A_0,A_1,F,n,\Lambda_n,D}$ and assume that p is a level-raising prime with respect to ℓ^m for some $m \geqslant 1$.

Let $\mathcal Q$ be the blow-up of $\mathcal P\otimes_{O_{F,(\mathfrak p)}}O_{F_{\mathfrak p}}$ along the closed subscheme $M_n^\circ\times M_{n+1}^\circ$, with $\mathcal Q':=\mathcal Q\otimes_{O_{F_{\mathfrak p}}}F_{\mathfrak p}$ and $\mathcal Q:=\mathcal Q\otimes_{O_{F_{\mathfrak p}}}O_F/\mathfrak p$.

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For $r \in \mathbb{Z}$, put

$$\begin{split} &B^r(Q) \coloneqq \ker \left(\delta_0^* : \mathrm{H}^{2r}(\overline{Q_0}, \mathbb{Z}_\ell(r)) \to \mathrm{H}^{2r}(\overline{Q_1}, \mathbb{Z}_\ell(r)) \right), \\ &B_r(Q) \coloneqq \operatorname{coker} \left(\delta_{1!} : \mathrm{H}^{2(2n-r-2)}(\overline{Q_1}, \mathbb{Z}_\ell(2n-r-2)) \to \mathrm{H}^{2(2n-r-1)}(\overline{Q_0}, \mathbb{Z}_\ell(2n-r-1)) \right), \end{split}$$

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Let $\mathcal Q$ be the blow-up of $\mathcal P\otimes_{\mathcal O_{F, \mathfrak p}} \mathcal O_{F_\mathfrak p}$ along the closed subscheme $M_n^\circ \times M_{n+1}^\circ$, with $\mathcal Q' := \mathcal Q\otimes_{\mathcal O_{F_\mathfrak p}} F_\mathfrak p$ and $\mathcal Q := \mathcal Q\otimes_{\mathcal O_{F_\mathfrak p}} \mathcal O_F/\mathfrak p$. Then $\mathcal Q$ is a projective strictly semistable scheme over $\mathcal O_{F_\mathfrak p}$ such that no three irreducible components of the special fiber have common intersection. Denote by

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$$\Delta^r : C_{2n-r}(Q) \to C^r(Q).$$

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Proposition

There is a canonical isomorphism

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Moreover, under the above isomorphism, the element $\partial_{\mathfrak{p}}(\operatorname{loc}_{\mathfrak{p}}(\alpha(M'_{\operatorname{diag}})))$ coincides with the image of the cycle class of the strict transform of M_{diag} in Q_0 (regarded as in $B^n(Q)^0_{\mathfrak{m}^\ell}$) under the natural map $B^n(Q)^0_{\mathfrak{m}^\ell} \to C^n(Q)_{\mathfrak{m}^\ell} \to \operatorname{coker} \Delta^n_{\mathfrak{m}^\ell}$.

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To further simplify the discussion, from now on, we will just consider the case where $n=n_0$, that is, n is **even**. We introduce more notation.

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- \diamond Write n = 2r.
- \diamond Denote by $\sigma \colon \mathcal{Q} \to \mathcal{P}$ the blow-up morphism.
- \diamond Put $P^{\circ \bullet} := M_n^{\circ} \times M_{n+1}^{\bullet}$ and denote by $Q^{\circ \bullet}$ its strict transform under σ . Similarly, we have $P^{\circ \circ}$, $P^{\bullet \circ}$, $P^{\bullet \circ}$, and their versions in Q. In particular, Q_0 is the disjoint union of $Q^{\circ \circ}$, $Q^{\circ \bullet}$, $Q^{\bullet \circ}$ and $Q^{\bullet \circ}$.

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We now construct a canonical map

$$\nabla \colon \mathrm{H}^{2n}(\overline{\mathbb{Q}_0}, \mathbb{Z}_\ell(n)) \to \mathbb{Z}_\ell[S_n^\circ] \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[S_{n+1}^\circ],$$

which turns out to factor through $C^n(Q)$ and induce an isomorphism

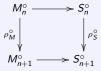
$$(\operatorname{coker} \Delta^n)/\mathfrak{m}^{\ell^m} \to \mathbb{Z}[S_n^\circ]/\mathfrak{m}_n^{\ell^m} \otimes \mathbb{Z}[S_{n+1}^\circ]/\mathfrak{m}_{n+1}^{\ell^m}$$

for the quotient.

To construct ∇ , we will find many cycles contained in Q_0 that are indexed by S_N° (for N=n,n+1). It turns out that the union of those cycles is exactly the basic locus of M_N , that is, the locus where $A[\mathfrak{p}^{\infty}]$ is supersingular.

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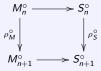
Indeed, the whole M_N° is contained in the basic locus. We also recall that it is a projective bundle over S_N° , which fits into the following diagram



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To study the basic locus on M_N^{\bullet} , we fix an O_F -submodule $\mathfrak{p}\Lambda_n \subseteq \Lambda_n^{\bullet} \subseteq \Lambda_n$ such that $\Lambda_n^{\bullet}/\mathfrak{p}\Lambda_n$ is a Lagrangian subspace of $\Lambda_n/\mathfrak{p}\Lambda_n$. Put $\Lambda_{n+1}^{\bullet} := \Lambda_n^{\bullet} \oplus O_F \cdot 1$. For $N \in \{n, n+1\}$, put

$$S_N^{\bullet} := H_N(\mathbb{Q}) \backslash H_N(\mathbb{A}^{\infty}) / K_N^{D \bullet},$$

where $K_N^{D\bullet}$ is defined similarly as K_N^D using Λ_N^{\bullet} . We also put $S_N^{\dagger} := H_N(\mathbb{Q}) \backslash H_N(\mathbb{A}^{\infty}) / K_N^{D\dagger}$, where $K_N^{D\dagger} := K_N^D \cap K_N^{D\bullet}$.

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$$\begin{array}{ccc} M_n^{\circ} & \longrightarrow & S_n^{\circ} \\ & & & \downarrow \rho_S^{\circ} \\ M_{n+1}^{\circ} & \longrightarrow & S_{n+1}^{\circ} \end{array}$$

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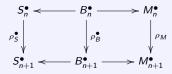
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Similar to ρ_S° , we have the maps

$$\rho_S^{\bullet} \colon S_n^{\bullet} \to S_{n+1}^{\bullet}, \quad \rho_S^{\dagger} \colon S_n^{\dagger} \to S_{n+1}^{\dagger}.$$



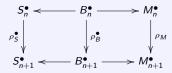
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in which ρ_B^{\bullet} is locally an isomorphism, $B_N^{\bullet} \to S_N^{\bullet}$ is projective smooth of dimension r, and $B_N^{\bullet} \to M_N^{\bullet}$ is a closed immersion when restricted to each connected component of the source.

Basic locus

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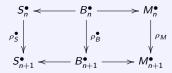
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The fibers of the morphism $B_N^{\bullet} \to S_N^{\bullet}$ are certain Deligne–Lustig varieties. For example, when n=2, the fibers are isomorphic to \mathbb{P}^1 ; when n=4, the fibers are up to purely inseparable morphisms blow-ups of the Fermat surface along all O_F/\mathfrak{p} -points.

For $N \in \{n, n+1\}$, the union of M_N° and the image of $B_N^{\bullet} \to M_N^{\bullet}$ is exactly the basic locus of M_N .

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For $N \in \{n, n+1\}$, the union of M_N° and the image of $B_N^\bullet \to M_N^\bullet$ is exactly the basic locus of M_N . For the intersection between M_N° and B_N^\bullet , we have the commutative diagram

in the category of O_F/\mathfrak{p} -schemes, in which the fibers of the left morphism are isomorphic to \mathbb{P}^{r-1} .

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$$\begin{split} \operatorname{Inc}_{\circ \uparrow} &: \operatorname{H}^{2n}(\overline{Q^{\circ \bullet}}, \mathbb{Z}_{\ell}(n)) \xrightarrow{\sigma_{1}} \operatorname{H}^{2n}(\overline{P^{\circ \bullet}}, \mathbb{Z}_{\ell}(n)) \xrightarrow{\operatorname{Kun}} \operatorname{H}^{2r}(\overline{M_{n}^{\circ}}, \mathbb{Z}_{\ell}(r)) \otimes \operatorname{H}^{2r}(\overline{M_{n+1}^{\circ}}, \mathbb{Z}_{\ell}(r)) \\ &\xrightarrow{\operatorname{res}} \operatorname{H}^{2r}(\overline{M_{n}^{\circ}}, \mathbb{Z}_{\ell}(r)) \otimes \operatorname{H}^{2r}(\overline{M_{n+1}^{\dagger}}, \mathbb{Z}_{\ell}(r)) \xrightarrow{\operatorname{Gys}} \operatorname{H}^{2r}(\overline{M_{n}^{\circ}}, \mathbb{Z}_{\ell}(r)) \otimes \operatorname{H}^{2r+2}(\overline{M_{n+1}^{\circ}}, \mathbb{Z}_{\ell}(r+1)) \\ &\xrightarrow{\operatorname{Lef}} \operatorname{H}^{2(n-1)}(\overline{M_{n}^{\circ}}, \mathbb{Z}_{\ell}(n-1)) \otimes \operatorname{H}^{2n}(\overline{M_{n+1}^{\circ}}, \mathbb{Z}_{\ell}(n)) \xrightarrow{\operatorname{Gys}} \mathbb{Z}_{\ell}[S_{n}^{\circ}] \otimes \mathbb{Z}_{\ell}[S_{n+1}^{\circ}]. \end{split}$$

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$$\begin{split} \operatorname{Inc}_{\bullet \bullet} \colon \operatorname{H}^{2n}(\overline{Q^{\bullet \bullet}}, \mathbb{Z}_{\ell}(n)) &\xrightarrow{\sigma_{1}} \operatorname{H}^{2n}(\overline{P^{\bullet \bullet}}, \mathbb{Z}_{\ell}(n)) \xrightarrow{\operatorname{Kun}} \operatorname{H}^{2r}(\overline{M_{n}^{\bullet}}, \mathbb{Z}_{\ell}(r)) \otimes \operatorname{H}^{2r}(\overline{M_{n+1}^{\bullet}}, \mathbb{Z}_{\ell}(r)) \\ \xrightarrow{\operatorname{res}} \operatorname{H}^{2r}(\overline{B_{n}^{\bullet}}, \mathbb{Z}_{\ell}(r)) \otimes \operatorname{H}^{2r}(\overline{B_{n+1}^{\bullet}}, \mathbb{Z}_{\ell}(r)) \xrightarrow{\operatorname{Gys}} \mathbb{Z}_{\ell}[S_{n}^{\bullet}] \otimes \mathbb{Z}_{\ell}[S_{n+1}^{\bullet}]. \end{split}$$

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For $N \in \{n,n+1\}$, the correspondence $S_N^\circ \leftarrow S_N^\dagger \to S_N^\bullet$ of finite sets gives rise to two "transpose" maps

 $\mathtt{T}_N \colon \mathbb{Z}_\ell[S_N^\circ] \to \mathbb{Z}_\ell[S_N^\bullet], \quad \mathtt{T}_N \colon \mathbb{Z}_\ell[S_N^\bullet] \to \mathbb{Z}_\ell[S_N^\circ]$

according to the domain.

$$\begin{split} \operatorname{Inc}_{\bullet \bullet} \colon & \operatorname{H}^{2n}(\overline{Q^{\bullet \bullet}}, \mathbb{Z}_{\ell}(n)) \xrightarrow{\sigma_{1}} \operatorname{H}^{2n}(\overline{P^{\bullet \bullet}}, \mathbb{Z}_{\ell}(n)) \xrightarrow{\operatorname{Kun}} \operatorname{H}^{2r}(\overline{M_{n}^{\bullet}}, \mathbb{Z}_{\ell}(r)) \otimes \operatorname{H}^{2r}(\overline{M_{n+1}^{\bullet}}, \mathbb{Z}_{\ell}(r)) \\ \xrightarrow{\operatorname{res}} & \operatorname{H}^{2r}(\overline{B_{n}^{\bullet}}, \mathbb{Z}_{\ell}(r)) \otimes \operatorname{H}^{2r}(\overline{B_{n+1}^{\bullet}}, \mathbb{Z}_{\ell}(r)) \xrightarrow{\operatorname{Gys}} \mathbb{Z}_{\ell}[S_{n}^{\bullet}] \otimes \mathbb{Z}_{\ell}[S_{n+1}^{\bullet}]. \end{split}$$

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according to the domain. We now define abla to be the sum of the following four maps

$$\begin{split} & (\mathtt{T}_n^2 \otimes \mathtt{T}_{n+1}^2) \circ \mathtt{Inc}_{\circ \dagger}, \quad (\rho+1)^2 (\mathtt{T}_n^2 \otimes \mathtt{T}_{n+1}) \circ \mathtt{Inc}_{\circ \bullet}, \\ & (\rho+1) (\mathtt{T}_n \otimes \mathtt{T}_{n+1}^2) \circ \mathtt{Inc}_{\bullet \dagger}, \quad (\rho+1)^3 (\mathtt{T}_n \otimes \mathtt{T}_{n+1}) \circ \mathtt{Inc}_{\bullet \bullet}. \end{split}$$

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$$(p+1) (\mathtt{T}_n \otimes \mathtt{T}_{n+1}^2) \circ \mathtt{Inc}_{\bullet \dagger}, \quad (p+1)^3 (\mathtt{T}_n \otimes \mathtt{T}_{n+1}) \circ \mathtt{Inc}_{\bullet \bullet}.$$

Proposition

The map $\nabla\colon \mathrm{H}^{2n}(\overline{\mathbb{Q}_0},\mathbb{Z}_\ell(n))\to \mathbb{Z}_\ell[S_n^\circ]\otimes_{\mathbb{Z}_\ell}\mathbb{Z}_\ell[S_{n+1}^\circ]$ defined above factors through $C^n(Q)$ and induce an isomorphism

$$(\operatorname{coker} \Delta^n)/\mathfrak{m}^{\ell^m} \to \mathbb{Z}[S_n^{\circ}]/\mathfrak{m}_n^{\ell^m} \otimes \mathbb{Z}[S_{n+1}^{\circ}]/\mathfrak{m}_{n+1}^{\ell^m}.$$



Thank you for your attention!