

Anomalies and the Berry Phase

From $\pi^0 \rightarrow \gamma\gamma$ to the geometry of the fermion vacuum

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ICTS-TIFR

Phases, Caustics and Chaos: Classical and Quantum Aspects

Celebrating the science of Michael Berry

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Plan of the talk

1. Prelude (1949–1969)

The π^0 puzzle; the axial anomaly and its non-abelian form

2. Some uses of anomalies in particle physics

Anomaly cancellation, the $U(1)$ problem, anomaly matching, the WZW term

3. From QCD to the WZW term (1983–85)

Integrating out quarks; the phase of the fermion determinant; baryon quantum numbers

4. The Berry phase connection

Adiabatic phases and monopoles; Berry's phase and the quark determinant

5. The WZW term as a Berry–Wilson loop in configuration space

Current algebra, Schwinger terms and a $U(1)$ connection; the monopole potential

6. Epilogue

Topics not covered; summary

One-line theme

An anomaly is the statement that the fermion vacuum, transported around a loop in background-field space, does not come back to itself: a Berry phase.

Prelude: a PCAC puzzle

The puzzle: $\pi^0 \rightarrow \gamma\gamma$

- **1949, Steinberger:** computes $\pi^0 \rightarrow \gamma\gamma$ through a proton loop (pseudoscalar coupling). Answer surprisingly close to experiment.
- **1967, Sutherland, Veltman:** Current algebra + PCAC,

$$\partial_\mu j_{5,3}^\mu = f_\pi m_\pi^2 \pi^0,$$

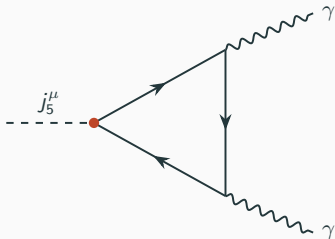
together with gauge invariance imply that the amplitude **vanishes** as $m_\pi \rightarrow 0$.

- But the decay is observed, with $\Gamma \approx 7.8$ eV.
Either PCAC fails badly here, or something in the formal argument is wrong.

The formal step that fails

Naive manipulation assumes $\partial_\mu j_5^\mu = 2im \bar{\psi} \gamma_5 \psi$ holds as an operator identity in the quantum theory. It does not.

Bell–Jackiw and Adler (1969)



AVV triangle: linearly divergent, routing dependent

- **Bell & Jackiw** (σ -model, Pauli–Villars) and **Adler** (spinor QED): no regulator can preserve both vector and axial current conservation.
- Keeping gauge invariance:

$$\partial_\mu j_5^\mu = 2im \bar{\psi} \gamma_5 \psi + \frac{e^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

- A classical symmetry broken by quantization: the **ABJ anomaly**.
- The extra term is a *total derivative*, $\epsilon FF = \partial_\mu K^\mu$, yet it has physical consequences.
- **Anticipated by Schwinger (1951)**: defining the axial current gauge invariantly, by point splitting with the phase $\exp\left(ie\int_x^{x+\epsilon} A \cdot dx\right)$, he found exactly this extra term.

Resolution of the puzzle, and a bonus: colour

- The anomaly term in the Standard Model gives the $\pi^0 \rightarrow \gamma\gamma$ amplitude directly, exact in the chiral limit:

$$\Gamma(\pi^0 \rightarrow \gamma\gamma) = \frac{\alpha^2 m_\pi^3}{64\pi^3 f_\pi^2} \left[N_c (Q_u^2 - Q_d^2) \right]^2 = \frac{\alpha^2 m_\pi^3}{64\pi^3 f_\pi^2} \left(\frac{N_c}{3} \right)^2.$$

- With $f_\pi \simeq 92$ MeV: $\Gamma \simeq 7.7$ eV for $N_c = 3$, versus ≈ 0.86 eV for $N_c = 1$. Since $\Gamma = \hbar/\tau$: lifetime $\tau \approx 8.5 \times 10^{-17}$ s.
- Experiment: $\Gamma \approx 7.8$ eV. One of the early pieces of evidence that quarks come in 3 colours (Nambu).

Adler–Bardeen theorem (1969)

The anomaly coefficient is **one-loop exact**: no corrections from higher orders in perturbation theory. An early hint that the anomaly is topological, not dynamical.

Bill Bardeen (1969): the non-abelian anomaly

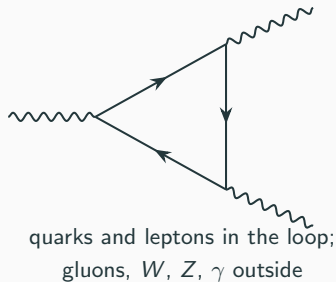
- Bill Bardeen computed the general anomaly for fermions coupled to non-abelian vector and axial-vector fields: triangle, square and pentagon graphs.
- In modern form, for a left-handed Weyl fermion in a gauge field $A = A_\mu^a T_a$ (factors of i and signs are convention dependent):

$$(D_\mu J^\mu)_a = \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} \left[T_a \partial_\mu \left(A_\nu \partial_\rho A_\sigma + \frac{1}{2} A_\nu A_\rho A_\sigma \right) \right].$$

- This is the anomaly that the effective action of the pions must reproduce, and that Dhar–Wadia and Dhar–Shankar–Wadia later recover from the quark determinant.

Some uses of Anomalies in Particle Physics

Anomaly cancellation in the Standard Model



- A gauge anomaly would destroy gauge invariance, and with it unitarity and renormalisability.
- Each of the three quark–lepton generations, running around a loop coupled to the external gauge fields of the Standard Model, has quantum numbers such that **all gauge anomalies cancel**.
- Quarks alone or leptons alone would be anomalous; the cancellation needs both, and $N_c = 3$.
- This is crucial for the **renormalisability** of the Standard Model.
- A profound prediction: once the τ (1975) and the b quark (1977) were found, the third generation needed a partner, the **top quark**, finally discovered at Fermilab in 1995 (CDF and DØ), mass ≈ 173 GeV.

Instantons and the $U(1)$ problem

- In QCD, $\partial_\mu j_5^\mu \propto \text{Tr } G \tilde{G}$ and gauge fields with nonzero $\frac{1}{32\pi^2} \int \text{Tr } G \tilde{G} \in \mathbb{Z}$ exist.
- Fermion zero modes $\Rightarrow \Delta Q_5 = 2N_f \nu$: the axial $U(1)$ is not a symmetry, no ninth Goldstone boson.

Anomaly matching

- Gauge a global symmetry weakly with spectator fermions to cancel its anomaly.
- The UV anomaly must be reproduced at every lower energy scale to be able to cancel the anomaly of the weakly coupled spectator fermions.
- 't Hooft's argument shows that anomalies are **invariant under the renormalisation group flow**: the anomaly computed in the UV equals the anomaly of the IR degrees of freedom.

Witten (1983): global aspects of current algebra

- Wess and Zumino (1971) constructed the effective action of the pions that reproduces the anomaly; Witten gave it a **global, geometric formulation**.
- The chiral Lagrangian $\frac{f_\pi^2}{4} \text{Tr } \partial_\mu U \partial^\mu U^\dagger$ has spurious symmetries (e.g. $U \rightarrow U^T$) forbidding $K^+ K^- \rightarrow \pi^+ \pi^- \pi^0$. The Wess–Zumino term breaks them.
- It cannot be written as a 4D local Lagrangian. Instead, extend U to a 5-disc D^5 with $\partial D^5 = S^4$ (spacetime):

$$\Gamma_{\text{WZW}} = -\frac{i}{240\pi^2} \int_{D^5} d^5x \epsilon^{ijklm} \text{Tr} [L_i L_j L_k L_l L_m], \quad L_i = U^\dagger \partial_i U.$$

- Two extensions differ by an integral over S^5 ; $\pi_5(SU(3)) = \mathbb{Z}$ gives $2\pi \times \text{integer}$. For $e^{iN\Gamma}$ to be well defined: **$N \in \mathbb{Z}$** .
- Gauging the WZW term reproduces $\pi^0 \rightarrow \gamma\gamma$ and fixes $N = N_c$.
- Skyrmions are fermions when N_c is odd: the WZW term gives the soliton its spin.

Analogy Witten himself drew

A charged particle moving on S^2 around a Dirac monopole: an action defined only via a 2-disc, single valued only if eg is quantised. **Keep this picture for Part IV.**

From QCD to the WZW term

Dhar–Wadia, PRL 52 (1984) 959;

Dhar–Shankar–Wadia, PRD 31 (1985) 3256.

From QCD to a theory of non-linear fermions

- One fundamental fact about QCD is that the pure gauge sector has a mass gap. To study chiral symmetries in QCD in the conceptual framework of the Kadanoff–Wilson renormalisation group, we imagine integrating over the short-wavelength degrees of freedom of the gauge field, up to length scales of order of the inverse glueball mass. The resulting theory is a theory of non-linear fermions that generalises the Nambu–Jona-Lasinio model:

$$\begin{aligned}
 -i\mathcal{L} = & \bar{\psi}_{L\alpha j} i\not{\partial} \psi_{L\alpha j} + \bar{\psi}_{R\alpha j} i\not{\partial} \psi_{R\alpha j} + g_1^2 (\bar{\psi}_{L\alpha j} \psi_{R\alpha k}) (\bar{\psi}_{R\beta k} \psi_{L\beta j}) \\
 & - i \frac{g_2^2}{4} \left[(\bar{\psi}_{L\alpha j} \gamma_\mu \psi_{L\alpha k}) (\bar{\psi}_{L\beta k} \gamma_\mu \psi_{L\beta j}) + L \leftrightarrow R \right] + \dots
 \end{aligned}$$

$\alpha, \beta = 1, \dots, N$ colour; $j, k = 1, \dots, n$ flavour; g_1, g_2 of mass dimension -1 ; cutoff $\Lambda \ll$ glueball mass.
Global $U(n) \times U(n)$ chiral symmetry.

- We can rewrite the non-linear fermion theory in terms of auxiliary gauge-invariant (flavoured) collective fields: a complex scalar M and vector fields L_μ, R_μ coupling to left- and right-handed quarks,

$$\begin{aligned}
 -i\mathcal{L} = & \bar{\psi}_L i\not{\partial} \psi_L + \bar{\psi}_R i\not{\partial} \psi_R + \bar{\psi}_L M \psi_R + \bar{\psi}_R M^\dagger \psi_L + i(\bar{\psi}_L \not{L} \psi_L + \bar{\psi}_R \not{R} \psi_R) \\
 & + \frac{i}{g_1^2} \text{tr} M^\dagger M + \frac{i}{g_2^2} \text{tr}(L_\mu^2 + R_\mu^2).
 \end{aligned}$$

The Gaussian integral over M, L_μ, R_μ gives back the non-linear fermion theory. Under $(V_1, V_2) \in U(n) \times U(n)$: $\psi_L \rightarrow V_1 \psi_L$, $\psi_R \rightarrow V_2 \psi_R$, $M \rightarrow V_1 M V_2^\dagger$, $L_\mu \rightarrow V_1 L_\mu V_1^\dagger$, $R_\mu \rightarrow V_2 R_\mu V_2^\dagger$.

The setup: quarks in a slowly varying chiral background

- In this background the quarks see the Dirac operator

$$D = i\not{D} - \not{L}P_L - \not{R}P_R + iMP_L + iM^\dagger P_R.$$

- At large N , for $\Lambda^2 g_1^2 > 8\pi^2/N$, chiral symmetry is broken: $M = \bar{H} U$, with $\bar{H} \sim \langle \bar{\psi}\psi \rangle$ a constituent mass and $U(x)$ the pion field. **The quarks are gapped** (Nambu–Jona-Lasinio).
- Integrate out the quarks and expand in long wavelengths, $k \ll \bar{H}$:

The central object

$$N \ln \det D = \underbrace{N \ln |\det D|}_{\text{modulus}} + i N \underbrace{\Delta}_{\text{phase}}$$

Which effective action for U (and the vector mesons) does each piece produce?

Two pieces of the determinant, two kinds of physics

$\ln \det D$ is computed by the **heat-kernel** (Schwinger proper-time) method,

$\ln \det D = -\frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \text{Tr} e^{-s\hat{D}^2}$, $\hat{D} = i\gamma_5 D$, expanded in derivatives of the background fields.

Modulus $\ln |\det D|$: dynamics

- pion kinetic term $\frac{1}{2} F_\pi^2 \text{tr}(L^\mu L_\mu)$
- four-derivative terms, including the Skyrme term $\frac{N(d_4-d_6)}{16} \text{tr}[L_\mu, L_\nu]^2$ and a term $\frac{Nd_5}{16} \text{tr}(L_\mu L^\mu)^2$: a generalised Skyrme model, with soliton solutions
- vector and axial-vector mesons, e.g. $m_A^2 - m_V^2 = 6\bar{H}^2$
- coefficients depend on the cutoff and on the regulator: **not universal**

Phase Δ : topology

- the Wess–Zumino–Witten term $N\Gamma_{WZW}$
- coefficient independent of $\bar{H}$, the cutoff and the regulator: **universal**
- fixed entirely by the chiral anomaly
- carries baryon number and all anomalous processes

The phase is determined by the anomaly

- The phase is not invariant under *local* chiral rotations, because the fermion measure is not (Fujikawa).
- Since our calculation **conserves the vector currents**, the result agrees with Bardeen's anomaly:

$$G_a(i\Gamma) = \frac{1}{48\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr}(t_a L_\mu L_\nu L_\rho L_\sigma), \quad L_\mu = i(\partial_\mu U)U^\dagger,$$

where G_a generates $U(n)$ rotations on the space of configurations $U(x)$.

- As Witten showed, this cannot be integrated to a local 4D action.
- But Wess–Zumino consistency is a **zero-curvature condition** on configuration space, so the equation can be integrated along any path in configuration space, with a path-independent answer. The configuration space is a Riemannian manifold with metric $ds^2 = \int d^4x \text{tr}(dU dU^\dagger)$.

Integrating the anomaly along a path in configuration space

- Take a path $U(x, s)$, $s \in [0, 1]$, from $U = 1$ to the configuration of interest, with $\Gamma(0) = 0$. Along it, $L_s = i(\partial_s U)U^\dagger$ and

$$\frac{d\Gamma}{ds} = \frac{1}{48\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} \text{tr}(L_s L_\mu L_\nu L_\rho L_\sigma).$$

- (s, x) sweep out a 5-disc D^5 bounded by spacetime S^4 , and the result is Witten's formula:

$$\Gamma = -\frac{1}{240\pi^2} \int_{D^5} d\Sigma^{ijklm} \text{tr}(L_i L_j L_k L_l L_m).$$

- The integrand is closed, so the answer is path independent; $\pi_5(SU(3)) = \mathbb{Z}$ makes $e^{iN\Gamma}$ single valued only for integer N , and here N is the number of colours.
- Real representations are anomaly free: no WZW term for two flavours.

Low-energy action

$$S = \int d^4x \left[\frac{1}{2} F_\pi^2 \text{tr}(L^\mu L_\mu) + \frac{N(d_4 - d_6)}{16} \text{tr}[L_\mu, L_\nu]^2 + \frac{Nd_5}{16} \text{tr}(L_\mu L^\mu)^2 \right] + N\Gamma_{\text{WZW}} + \dots$$

The d_i are computed from the quark loop as functions of $\bar{H}^2/\Lambda^2$.

The WZW term is **derived** from the quarks, not postulated.

What the phase carries

- **Baryon number.** The baryon current is $B_\mu = -\frac{i}{N} \text{tr} \left(\frac{\delta}{\delta L_\mu} + \frac{\delta}{\delta R_\mu} \right) \ln \det D$. The modulus is vector-gauge invariant, so **only the phase contributes**. At long wavelengths

$$B_\mu = -\frac{1}{24\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr}(\partial^\nu U U^{-1} \partial^\rho U U^{-1} \partial^\sigma U U^{-1}),$$

a topological current: baryon number = winding number, and **baryons are solitons**.

- **Anomalous processes.** Gauging the quark charge Q automatically gives the gauged WZW term: $\pi^0 \rightarrow \gamma\gamma$ and the other current-algebra anomalies, as in Witten.
- **A modified Gauss law.** The anomaly term changes the constraint on configuration space, a first sign that the phase acts like a connection.
- **Quantum numbers of baryons.** Restricted to the collective coordinates of the soliton, the phase fixes spin, statistics and flavour multiplets: Jain–Wadia.

Everything about the quarks that survives at long distances, with a universal coefficient, sits in the **phase** of the determinant.

Collective Coordinates and Baryon Quantum Numbers

S. Jain and S. R. Wadia, *Large- N baryons: collective coordinates of the topological soliton in the $SU(3)$ chiral model*, Nucl. Phys. B258 (1985) 713.

- $S(U) = N[S_0(U) + \Gamma(U)]$: Skyrme plus WZW. $1/N$ is a WKB parameter; Gervais–Sakita collective coordinates (Gregor Wentzel).
- $U(t, \mathbf{x}) = \Omega(t) U_0(\mathbf{x}) \Omega^{-1}(t)$, U_0 the hedgehog (baryon number = 1). $\Omega \sim \Omega h$, $h = e^{i\epsilon Y}$, $Y = \frac{1}{3}\text{diag}(1, 1, -2)$: collective space $SU(3)/U(1)$, treated as $SU(3)$ with a **local** $U(1)$.
- Substituting this U into $S(U)$ gives the collective-coordinate Lagrangian

$$L = -\frac{1}{2}I_{cd} \text{Tr}(t_c \omega_R) \text{Tr}(t_d \omega_R) - \frac{N}{48\pi^2} \text{Tr}(\omega_R K), \quad \omega_R = \Omega^{-1} \dot{\Omega},$$

(I_{cd} : moments of inertia; $K \propto \text{diag}(1, 1, 0)$, built from the baryon density), together with the **$U(1)$ gauge condition** $Y_R = N/3$ on the wave functions.

- “The Wess–Zumino term, being linear in the time derivative of U , imposes a crucial boundary condition on the baryon wave functions.”

Baryon quantum numbers

- Solving the collective dynamics of the previous slide, including the $U(1)$ constraint $Y_R = N/3$ that the WZW term imposes on the wave functions, the baryon wave functions are **monopole harmonics**: sections of a line bundle of charge $N/3$ over $SU(3)/U(1)$ (Wu–Yang).
- Consequences:
 - single-valuedness requires $N \in \mathbb{Z}$;
 - the baryons are **fermions for odd N** ;
 - for $N = 3$: the **octet with spin $\frac{1}{2}$** and the **decuplet with spin $\frac{3}{2}$** .
- This generalises **Wentzel's collective-coordinate model** (1940) of the pion–nucleon system, whose internal symmetry group was isospin $SU(2)$, to flavour $SU(3)$ with the WZW term.

The Berry phase connection

Berry (1984): the geometric phase

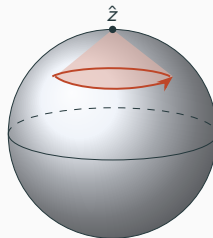
- $H(\mathbf{R})$ with slowly varying parameters $\mathbf{R}(t)$; nondegenerate level $|n, \mathbf{R}\rangle$.
- Adiabatic evolution around a closed loop C :

$$|\psi(T)\rangle = e^{i\gamma_n(C)} e^{-\frac{i}{\hbar} \int_0^T E_n dt} |\psi(0)\rangle$$

$$\gamma_n(C) = \oint_C \mathcal{A}_n, \quad \mathcal{A}_n = i\langle n | \nabla_{\mathbf{R}} n \rangle \cdot d\mathbf{R}$$

- $\mathcal{A}_n$ is a $U(1)$ connection on parameter space; $|n\rangle \rightarrow e^{i\chi(\mathbf{R})}|n\rangle$ is a gauge transformation.
- γ_n is gauge invariant mod 2π and equals the flux of the curvature $\mathcal{F}_n = d\mathcal{A}_n$.

M. V. Berry, Proc. Roy. Soc. Lond. A392 (1984) 45.



$$\gamma = -s\Omega(C)$$

Dictionary: Berry's phase and the quark determinant

Berry's adiabatic phase	Quarks in a chiral background
$H(\mathbf{R})$, slowly varying parameters $\mathbf{R}(t)$	Dirac operator $D[U]$, slowly varying $U(\mathbf{x}, t)$
Parameter space of $\mathbf{R}$	Configuration space $\mathcal{M} = \text{Map}(S^3, SU(N_f))$
Nondegenerate level $ n, \mathbf{R}\rangle$	Fermion vacuum $ \Omega[U]\rangle$ (gapped by $\bar{H}$)
Connection $\mathcal{A}_n = i\langle n \nabla_{\mathbf{R}} n \rangle$	$U(1)$ connection on $\mathcal{M}$
Curvature $\mathcal{F}_n = d\mathcal{A}_n$	Schwinger term
Berry phase $\gamma_n = \oint_C \mathcal{A}_n = \iint \mathcal{F}_n$	Phase of $\det D$: the WZW term on D^5
Gauge invariant mod 2π ; quantised flux	$\pi_5(SU(3)) = \mathbb{Z}$; $N_c \in \mathbb{Z}$

The WZW term as a Berry–Wilson loop in configuration space

Current algebra, Schwinger terms and a $U(1)$ gauge connection

S. R. Wadia, *Current algebra, a $U(1)$ gauge theory and the Wess–Zumino–Witten model*, Prog. Theor. Exp. Phys. 2021, doi:10.1093/ptep/ptab021.

- 1+1D, $G = SU(2)$, configuration space $\mathcal{M} = \text{Map}(S^1, G)$. Current algebra with Schwinger terms:

$$[L_0^\alpha, L_0^\beta] = i\lambda^2 f_{\alpha\beta\gamma} L_0^\gamma + i\alpha f_{\alpha\beta\gamma} L_1^\gamma, \quad [L_0^\alpha(x), L_1^\beta(y)] = i\lambda^2 f_{\alpha\beta\gamma} L_1^\gamma + \frac{\lambda^4}{2} \delta^{\alpha\beta} \partial_x \delta(x-y),$$

with $4\pi\alpha\lambda^{-2} = k \in \mathbb{Z}$.

- Realise the currents with a $U(1)$ connection $A^\alpha(U(x))$ on $\mathcal{M}$:

$$l_0^\alpha(x) = \pi^\alpha(x) - A^\alpha, \quad \pi^\alpha = \text{tr}\left(t^\alpha U \frac{\partial}{\partial U}\right), \quad l_1^\alpha = \text{tr}(t^\alpha \partial_x U U^{-1}), \quad A \rightarrow A + \delta\phi.$$

- Field strength $(\delta_{\alpha x} U(y) = t^\alpha U(y) \delta(x-y))$, and the non-commuting covariant derivatives:

$$F_{\alpha x, \beta y} = i \delta_{\alpha x} A_{\beta y} - i \delta_{\beta y} A_{\alpha x} - f_{\alpha\beta\gamma} A_{\gamma x} \delta(x-y), \quad [l_0^\alpha(x), l_0^\beta(y)] = -i f_{\alpha\beta\gamma} l_0^\gamma \delta(x-y) + i F_{\alpha x, \beta y}.$$

- The current algebra is reproduced if and only if

$$\boxed{F_{\alpha x, \beta y} = \frac{k}{4\pi} f_{\alpha\beta\gamma} l_1^\gamma \delta(x-y)} \quad \Longrightarrow \quad \text{Schwinger term} = \text{curvature on } \mathcal{M}$$

Current algebra, Schwinger terms and a $U(1)$ gauge connection (II)

- **Berry–Wilson loop**, the holonomy of the $U(1)$ connection on $\mathcal{M}$ (a Berry phase in Berry's language, a Wilson loop in gauge theory), around a closed curve $t \mapsto U(x, t)$, then Stokes:

$$\Phi = \oint dt \int_0^{2\pi} dx A_\alpha l_t^\alpha = \int_0^1 ds \oint dt \int dx F_{\alpha\beta} l_t^\alpha l_s^\beta = \frac{k}{12\pi} \int_D d^3x \epsilon^{ijk} \text{tr}(\partial_i U U^{-1} \partial_j U U^{-1} \partial_k U U^{-1})$$

the **WZW term**, normalised so that the flux through any closed surface in $\mathcal{M}$ is $2\pi k$.

- Schrödinger picture: $\Psi[\{U(x)\}, T] \sim \exp(i \int^T dt \int dx A_a \dot{\theta}^a)$; around a closed curve $\Delta\Psi \sim e^{i\Phi}$: a **projective representation** of the current algebra.
- A has a Dirac-like singularity; k is the quantised charge of a “magnetic monopole” on $\mathcal{M}$.
- The field-theory version of the Jain–Wadia collective-coordinate construction; extends to $SU(N)$ and to 3+1D.

Current algebra	$U(1)$ gauge theory on $\mathcal{M}$	Berry
$l_0 = \pi - A$	covariant derivative	adiabatic momentum
Schwinger term	field strength F	Berry curvature
WZW term	Berry–Wilson loop	Berry phase
level k	monopole charge	$2s$ for a spin

Current algebra, Schwinger terms and a $U(1)$ gauge connection (III)

- Euler angles: $U = e^{i\alpha t_3} e^{i\beta t_2} e^{i\gamma t_3}$, $t_a = \sigma_a/2$, $\beta \in [0, \pi]$, $\alpha \in [0, 2\pi)$, $\gamma \in [0, 4\pi)$. The WZ 3-form is the Haar measure, and has a Dirac-type 2-form potential:

$$\frac{ik}{8\pi} \sin \beta \, d\alpha \wedge d\beta \wedge d\gamma = dB, \quad B = \frac{ik}{8\pi} (1 - \cos \beta) \, d\gamma \wedge d\alpha.$$

- The connection on $\mathcal{M}$, $A_a(x) = B(X_a, \partial_x U)$ with $X_a U = it_a U$, is then explicit (primes = ∂_x):

$$\begin{aligned} A_1(x) &= -\frac{ik}{8\pi} \tan \frac{\beta}{2} \cos \alpha (\alpha' + \cos \beta \gamma'), \\ A_2(x) &= +\frac{ik}{8\pi} \tan \frac{\beta}{2} \sin \alpha (\alpha' + \cos \beta \gamma'), \\ A_3(x) &= -\frac{ik}{8\pi} (1 - \cos \beta) \gamma' \end{aligned}$$

regular at $\beta = 0$; singular at $\beta = \pi$: a
Dirac string on the circle $U_{11} = 0$

- The Berry–Wilson-loop phase becomes a local 2D integral over the strip swept out in (t, x) :

$$\Phi = \frac{k}{8\pi} \int dt \int_0^{2\pi} dx (1 - \cos \beta) (\dot{\gamma} \alpha' - \dot{\alpha} \gamma').$$

A monopole, and a section of a $U(1)$ bundle

$B \propto (1 - \cos \beta) d\gamma \wedge d\alpha$ is the Dirac potential $A_\phi \propto (1 - \cos \theta)$, string on the negative axis. The choice $(-1 - \cos \beta)$ moves the string to $\beta = 0$; the two differ by a gauge transformation on the overlap, so A is a connection on a **non-trivial $U(1)$ bundle** over $\mathcal{M}$. The total flux is $2\pi k$: single-valuedness of $e^{i\Phi}$ requires $k \in \mathbb{Z}$, Dirac quantisation.

Epilogue

Topics not covered

Mathematics, in parallel with ABJ

- **Atiyah–Singer index theorem** (1963, 1968): the anomaly is the local index density.
- **Atiyah–Patodi–Singer** (1975): η -invariant, spectral asymmetry, boundaries.
- **Families index and determinant line bundles** (Atiyah–Singer 1984; Quillen 1985; Bismut–Freed 1986): the Berry connection of the fermion vacuum made rigorous.

Anomalies across dimensions

- **Descent equations** (Stora, Zumino): the anomaly in d from $\text{Tr } F^{n+1}$ in $d + 2$.
- **Anomaly inflow** (Callan–Harvey 1985): a d -dimensional anomaly cancelled by a $(d + 1)$ -dimensional Chern–Simons term.

Gravity and strings

- Gravitational anomalies (Alvarez-Gaumé–Witten 1984).
- **Green–Schwarz** (1984): anomaly cancellation in type I superstring theory singles out $SO(32)$ (and $E_8 \times E_8$): the first superstring revolution.
- Global anomalies: Witten's $SU(2)$ anomaly (1982).

Condensed matter and beyond

- Quantum Hall effect: Berry curvature of Bloch bands (Thouless–Kohmoto–Nightingale–den Nijs 1982); edge modes carry the 1+1D anomaly.
- Weyl semimetals, chiral magnetic effect, anomalous hydrodynamics.
- Anomalies as invertible field theories; generalised symmetries.

Summary

1. The ABJ anomaly solved the π^0 puzzle and counted colours.
2. Anomalies are **topological**, not dynamical: one-loop exact (Adler–Bardeen), constrained by consistency (Wess–Zumino), equal to the index density of the Dirac operator (Fujikawa; Atiyah–Singer), matched between UV and IR ('t Hooft), with quantised coefficients (Witten).
3. Integrating out gapped quarks (Dhar–Wadia, Dhar–Shankar–Wadia) gives the WZW term with universal coefficient N_c : the **phase** of $\det D$, separated from the regulator-dependent modulus.
4. That phase is a Berry phase of the Dirac sea over $\text{Map}(S^3, G)$. On the soliton orbit (Jain–Wadia) it is a $\mathbb{CP}^2$ monopole flux fixing $Y_R = N_c/3$, the spins and the statistics.
5. The Schwinger term is the curvature of a $U(1)$ connection on $\text{Map}(S^1, G)$ and the WZW term is its Berry–Wilson loop (Wadia 2021). Along gauge orbits, this curvature is the gauge anomaly.
6. Quantisation of anomaly coefficients = Dirac quantisation of monopoles in field space.

Anomaly $\longleftrightarrow$ index $\longleftrightarrow$ Chern class $\longleftrightarrow$ Berry phase

- J. Steinberger, Phys. Rev. 76 (1949) 1180.
- D. G. Sutherland, Nucl. Phys. B2 (1967) 433; M. Veltman, Proc. Roy. Soc. A301 (1967) 107.
- J. Schwinger, Phys. Rev. 82 (1951) 664.
- J. S. Bell, R. Jackiw, Nuovo Cim. A60 (1969) 47.
- S. L. Adler, Phys. Rev. 177 (1969) 2426.
- S. L. Adler, W. A. Bardeen, Phys. Rev. 182 (1969) 1517.
- W. A. Bardeen, Phys. Rev. 184 (1969) 1848.
- D. J. Gross, R. Jackiw, Phys. Rev. D6 (1972) 477; C. Bouchiat, J. Iliopoulos, Ph. Meyer, Phys. Lett. 38B (1972) 519.
- J. Wess, B. Zumino, Phys. Lett. 37B (1971) 95.
- G. 't Hooft, Phys. Rev. Lett. 37 (1976) 8; in *Recent Developments in Gauge Theories* (Cargèse 1979).
- K. Fujikawa, Phys. Rev. Lett. 42 (1979) 1195; Phys. Rev. D21 (1980) 2848.
- J. Goldstone, F. Wilczek, Phys. Rev. Lett. 47 (1981) 986.
- E. Witten, Nucl. Phys. B223 (1983) 422; B223 (1983) 433.
- A. Dhar, S. R. Wadia, Phys. Rev. Lett. 52 (1984) 959 [received 19 Sept 1983].
- A. Dhar, R. Shankar, S. R. Wadia, Phys. Rev. D31 (1985) 3256 [received 18 Dec 1984].

- S. Jain, S. R. Wadia, Nucl. Phys. B258 (1985) 713 [received 17 Feb 1984].
- S. R. Wadia, *Current algebra, a $U(1)$ gauge theory and the Wess–Zumino–Witten model*, Prog. Theor. Exp. Phys. 2021, doi:10.1093/ptep/ptab021.
- C. Vafa, E. Witten, Nucl. Phys. B234 (1984) 173; J. L. Gervais, B. Sakita, Phys. Rev. D11 (1975) 2943.
- A. P. Balachandran, G. Marmo, V. P. Nair, C. G. Trahern, Phys. Rev. D25 (1982) 2713; A. P. Balachandran, V. P. Nair, S. G. Rajeev, A. Stern, Phys. Rev. Lett. 49 (1982) 1124.
- E. Guadagnini, Nucl. Phys. B236 (1984) 35; P. O. Mazur, M. A. Nowak, M. Praszalowicz, Phys. Lett. 147B (1984) 137.
- E. D'Hoker, E. Farhi, Nucl. Phys. B248 (1984) 59, 77.
- T. T. Wu, C. N. Yang, Nucl. Phys. B107 (1976) 365.
- D. I. Diakonov, V. Yu. Petrov, P. V. Pobylitsa, Nucl. Phys. B306 (1988) 809.
- **M. V. Berry**, *Quantal phase factors accompanying adiabatic changes*, Proc. Roy. Soc. Lond. A392 (1984) 45.
- B. Simon, *Holonomy, the quantum adiabatic theorem, and Berry's phase*, Phys. Rev. Lett. 51 (1983) 2167.
- F. Wilczek, A. Zee, Phys. Rev. Lett. 52 (1984) 2111.
- L. D. Faddeev, Phys. Lett. 145B (1984) 81.
- P. Nelson, L. Alvarez-Gaumé, Commun. Math. Phys. 99 (1985) 103.
- H. Sonoda, Nucl. Phys. B266 (1986) 410; A. J. Niemi, G. W. Semenoff, Phys. Rev. Lett. 55 (1985) 927.

- M. Stone, Phys. Rev. D33 (1986) 1191; I. J. R. Aitchison, Acta Phys. Polon. B18 (1987) 207.
- C. G. Callan, J. A. Harvey, Nucl. Phys. B250 (1985) 427.
- M. F. Atiyah, I. M. Singer, Ann. Math. 87 (1968) 484; M. F. Atiyah, V. K. Patodi, I. M. Singer, Math. Proc. Camb. Phil. Soc. 77 (1975) 43.
- M. F. Atiyah, I. M. Singer, Proc. Natl. Acad. Sci. 81 (1984) 2597; D. Quillen, Funct. Anal. Appl. 19 (1985) 31; J.-M. Bismut, D. S. Freed, Commun. Math. Phys. 106 (1986) 159.
- L. Alvarez-Gaumé, E. Witten, Nucl. Phys. B234 (1984) 269; M. B. Green, J. H. Schwarz, Phys. Lett. 149B (1984) 117.
- E. Witten, Phys. Lett. 117B (1982) 324.
- D. J. Thouless, M. Kohmoto, M. P. Nightingale, M. den Nijs, *Quantized Hall conductance in a two-dimensional periodic potential*, Phys. Rev. Lett. 49 (1982) 405.
- R. Jackiw, Comments At. Mol. Phys. 21 (1988) 71.