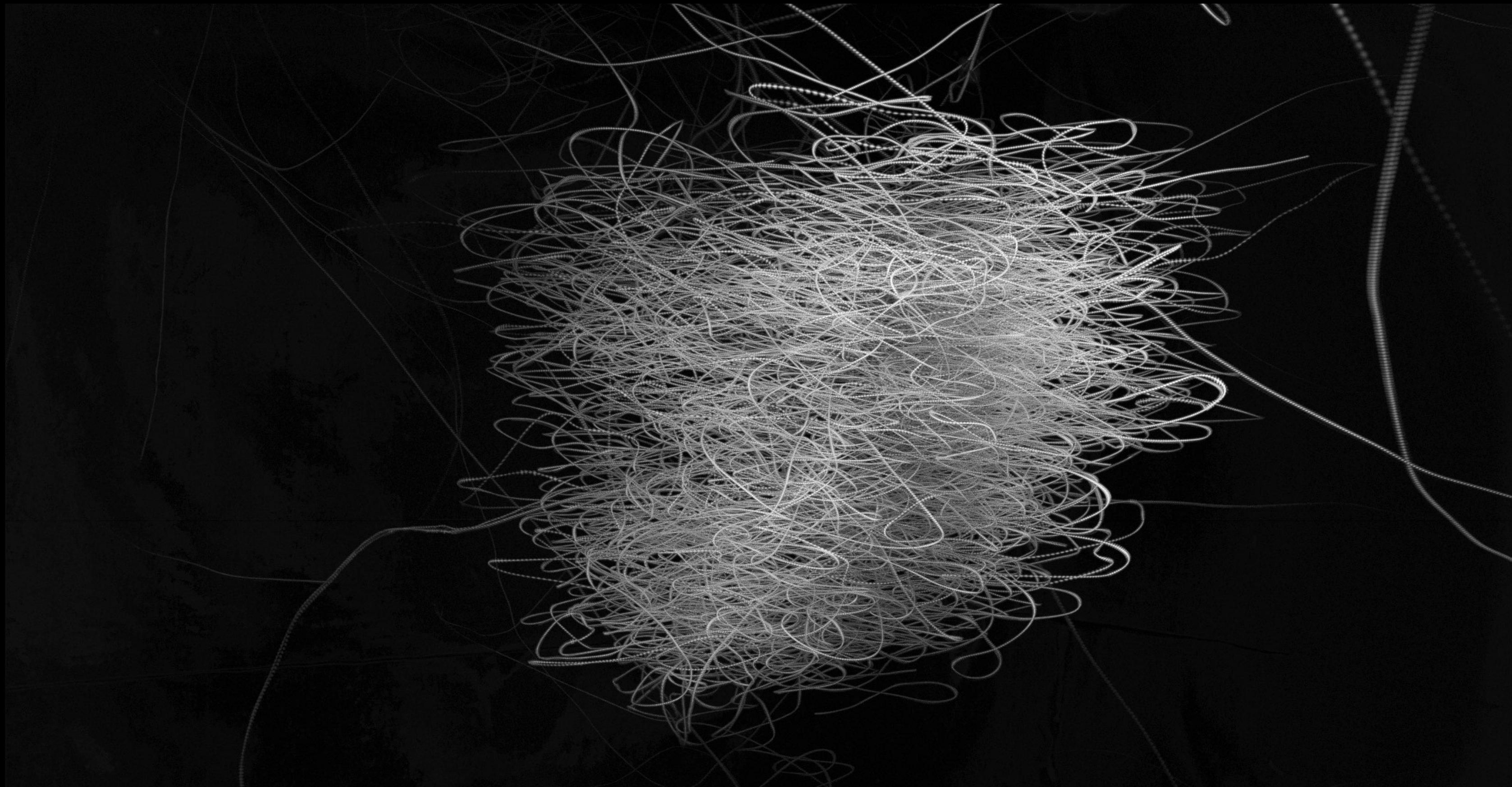
Natural Swarms in 3.99 Dimensions



Andrea Cavagna
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Rome

correlation

scaling

renormalization group

universality

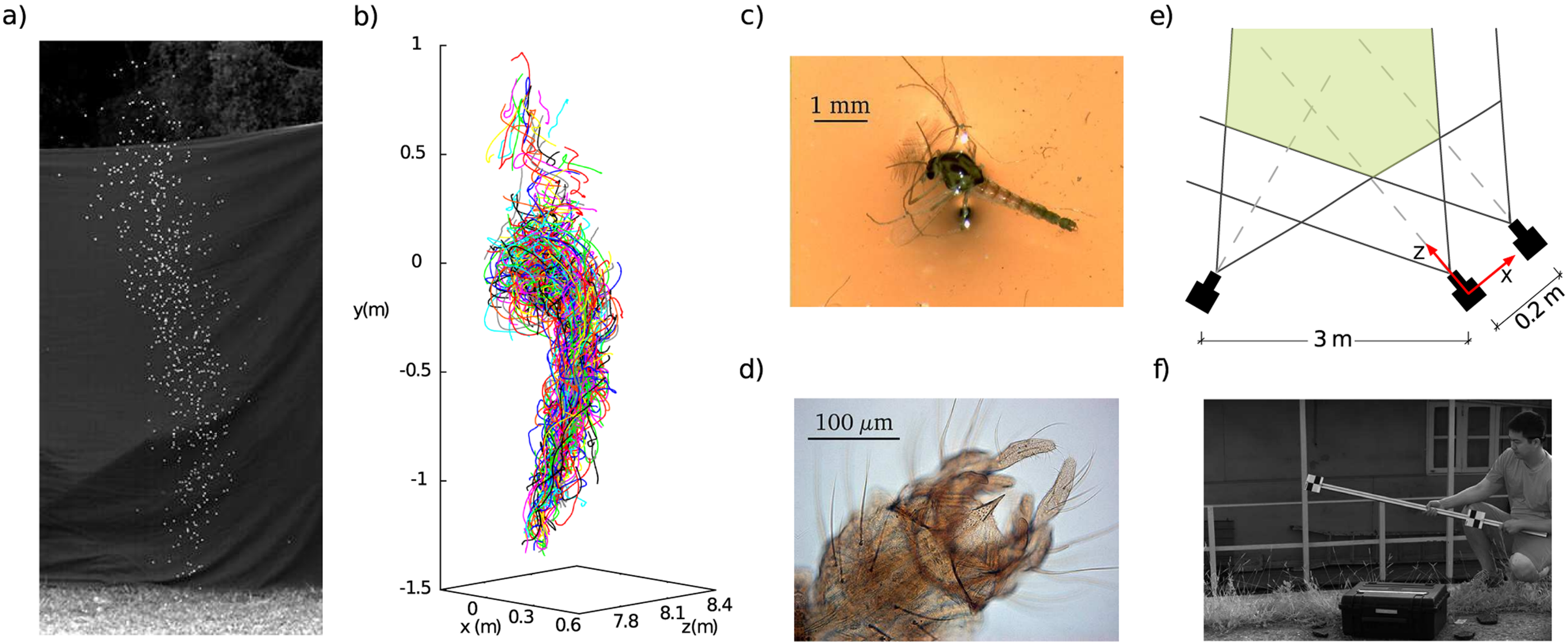
correlation ✓

scaling ✓

renormalization group

universality

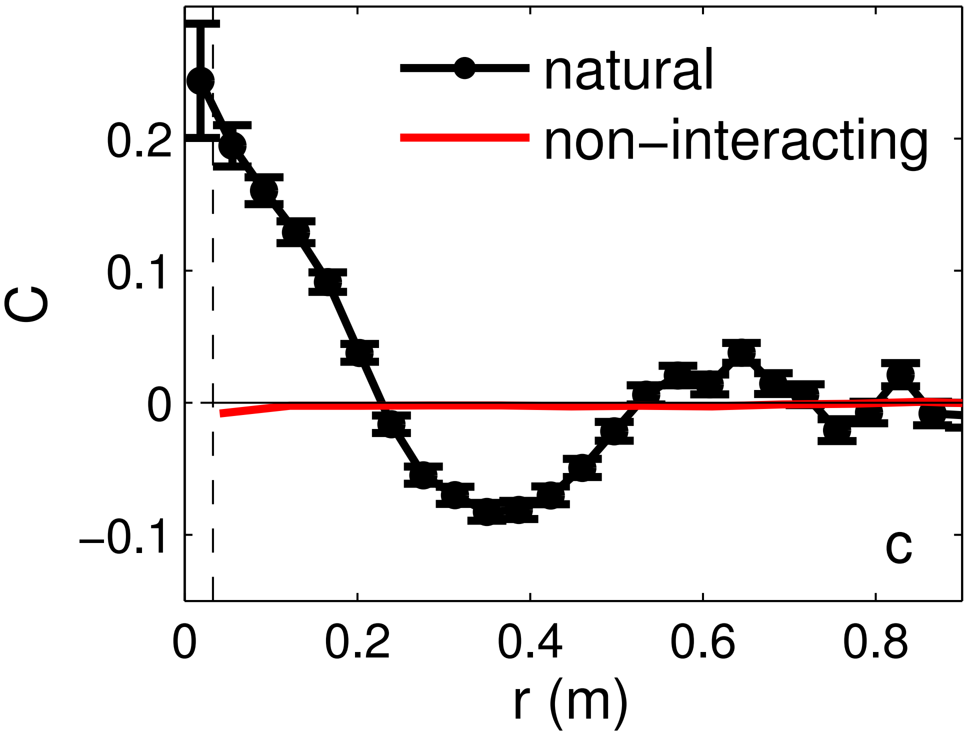
oh, btw — yes, we do experiments!

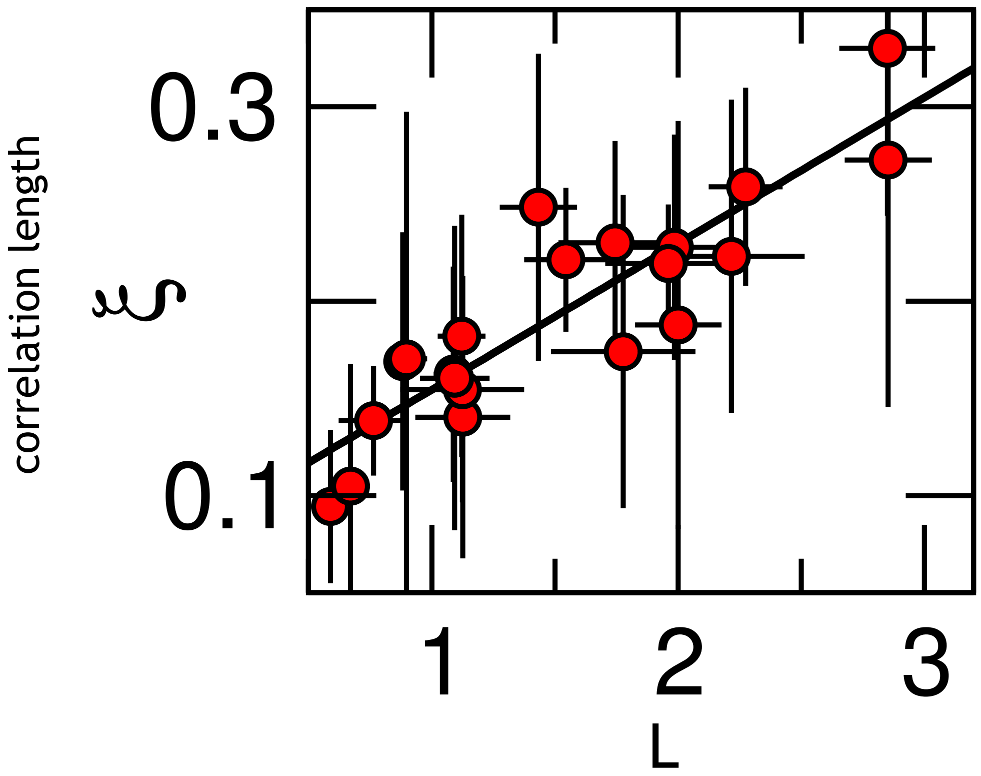


spatial correlations

equal-time velocity correlation function

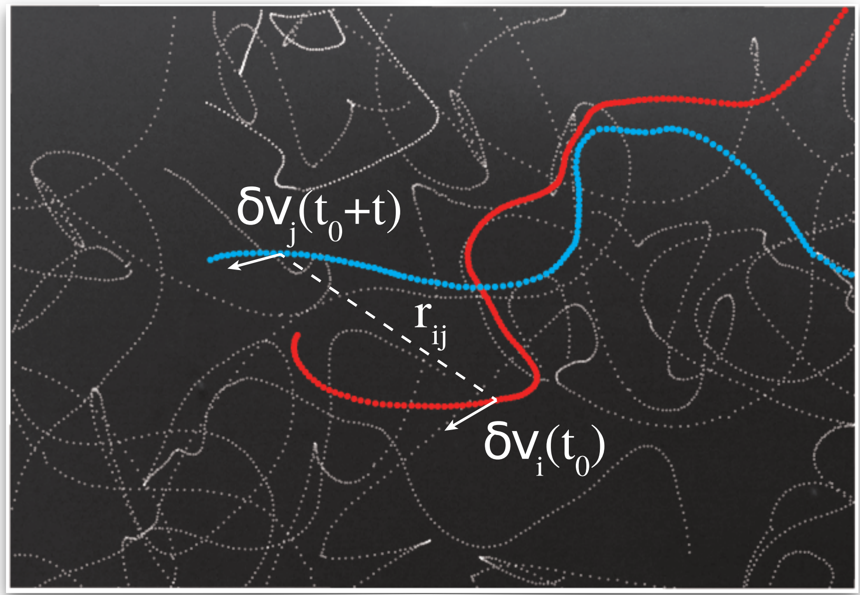
$$C(r) = \langle \delta\vec{v}(x_0, t_0) \cdot \delta\vec{v}(x_0 + r, t_0) \rangle$$

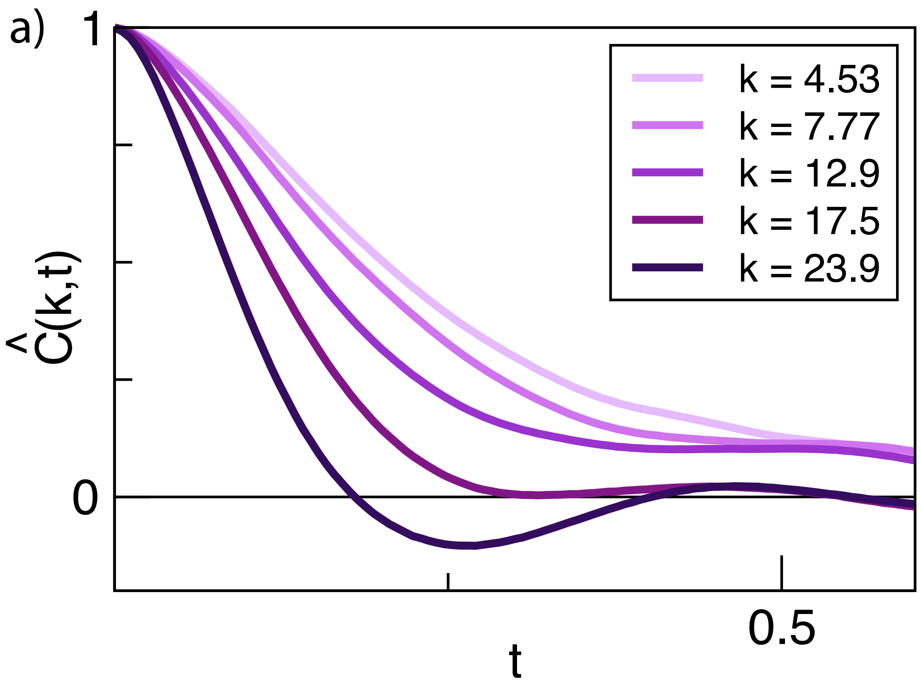




temporal correlations

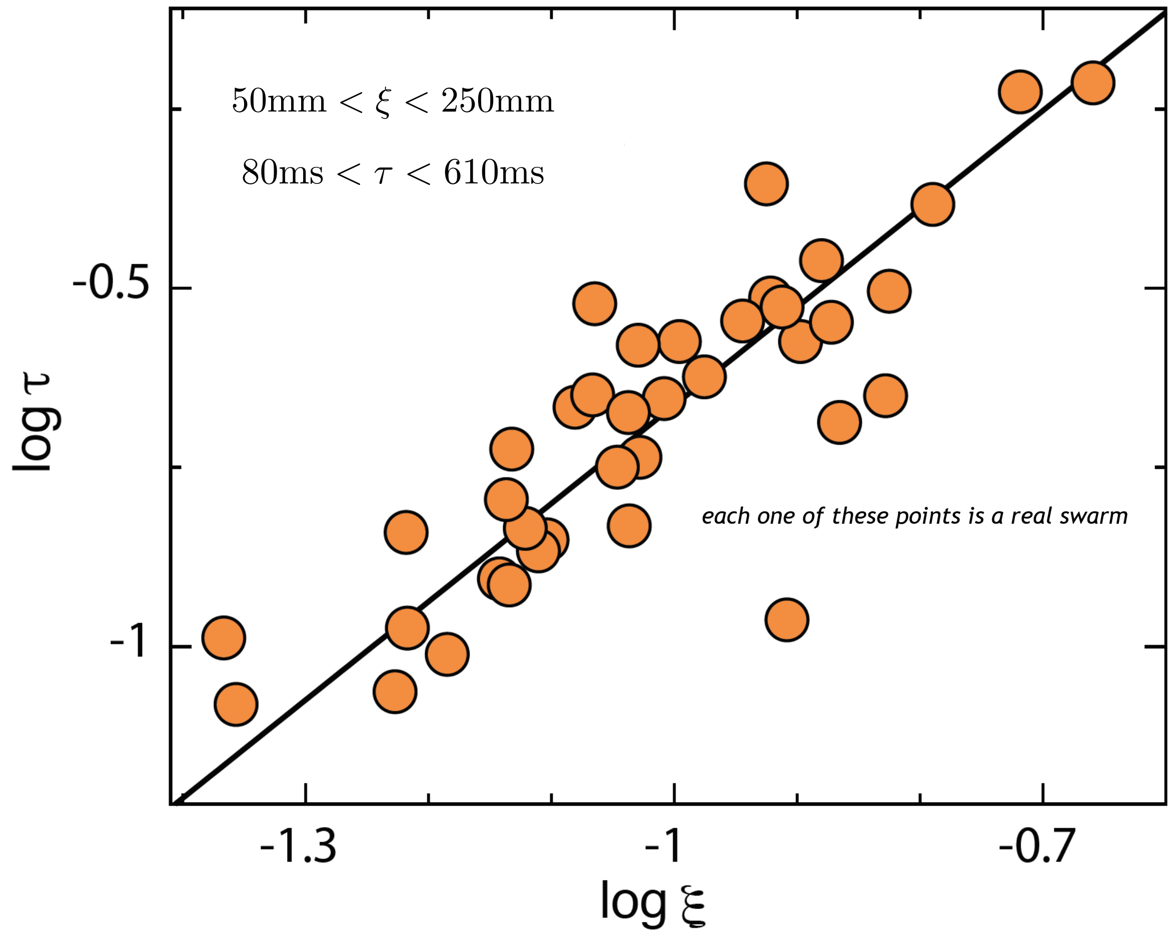
space-time correlation function





$$C(r,t) = \langle \delta\vec{v}(x_0,t_0) \cdot \delta\vec{v}(x_0+r,t_0+t) \rangle$$

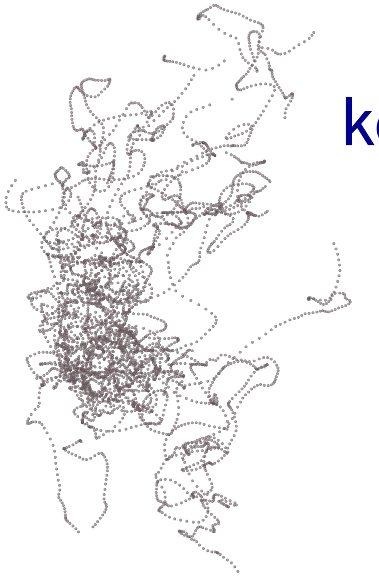
critical slowing down



$$\tau \sim \xi^z$$

dynamical critical exponent z

$$z = 1.37 \pm 0.11$$



key experimental facts about natural swarms:

- scale-free correlations, $\xi \sim L$
- critical slowing down, $\tau \sim \xi^z$
- dynamical critical exponent, $z = 1.37 \pm 0.11$

theory

ingredient #1

imitation aka ferromagnetism

simple ferromagnets

$$\frac{d\boldsymbol{\sigma}_i}{dt} = J \sum_j n_{ij}\, \boldsymbol{\sigma}_j + \boldsymbol{\zeta}_i$$

noise

$$|\boldsymbol{\sigma}_i| = 1$$

coarse-graining →

Model A $$\frac{\partial \boldsymbol{\psi}(x,t)}{\partial t} = -\Gamma \frac{\delta \mathscr{H}}{\delta \boldsymbol{\psi}} + \boldsymbol{\theta}(x,t)$$

Landau-Ginzburg Hamiltonian: $$\mathscr{H} = \int d^d x\ \{(\nabla \boldsymbol{\psi})^2 + r\boldsymbol{\psi}^2 + u\boldsymbol{\psi}^4\}$$

RG flow (on the critical manifold, because $\xi \sim L$)

$$z \approx 2$$

Wilson, Fisher (1972)

Halperin, Hohenberg, Ma (1972)

RG - ferromagnetism

$$z \approx 2$$

Halperin, Hohenberg, Ma (1972)

something is missing

$$z_{\mathrm{exp}} = 1.37 \pm 0.11$$

experiments

what a low critical exponent is telling us?

$$\tau \sim \xi^z \qquad \omega \sim k^z$$

$$\text{space} \sim \text{time}^{1/z}$$

the smaller is z, the more effective is the transport of fluctuations across the system

$z = 1.37$ vs $z \approx 2$

an exponent $z \ll 2$ implies that fluctuations propagate much more effectively than mere diffusion ingredient #2

activity

active ferromagnets: the Vicsek model

$$\begin{cases} \dfrac{d\boldsymbol{\sigma}_i}{dt} = J \sum_j n_{ij}(t)\, \boldsymbol{\sigma}_j + \boldsymbol{\zeta}_i \\ \dfrac{d\boldsymbol{r}_i}{dt} = v_0\, \boldsymbol{\sigma}_i \end{cases}$$

$v_0\, \boldsymbol{\sigma}_i = \boldsymbol{v}_i$ is the velocity

Model A meets Navier-Stokes: Toner-Tu field theory

Vicsek
$$\begin{cases} \dfrac{d\boldsymbol{\sigma}_i}{dt} = J \sum_j n_{ij}(t)\,\boldsymbol{\sigma}_j + \boldsymbol{\zeta}_i \\ \dfrac{d\boldsymbol{r}_i}{dt} = v_0\,\boldsymbol{\sigma}_i \end{cases}$$

coarse-graining →

Toner-Tu
$$\begin{cases} D_t\boldsymbol{v}(x,t) = -\Gamma\dfrac{\delta\mathscr{H}}{\delta\boldsymbol{v}} - \nabla P + \boldsymbol{\theta}(x,t) \\ \dfrac{\partial\rho(x,t)}{\partial t} + \nabla(\rho\,\boldsymbol{v}(x,t)) = 0 \end{cases}$$

$$\mathscr{H} = \int d^d x\ \{(\nabla\boldsymbol{v})^2 + r\boldsymbol{v}^2 + u\boldsymbol{v}^4\}$$

material derivative

$$D_t\,\boldsymbol{v}(x,t) = \partial_t\boldsymbol{v} + \lambda(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}$$

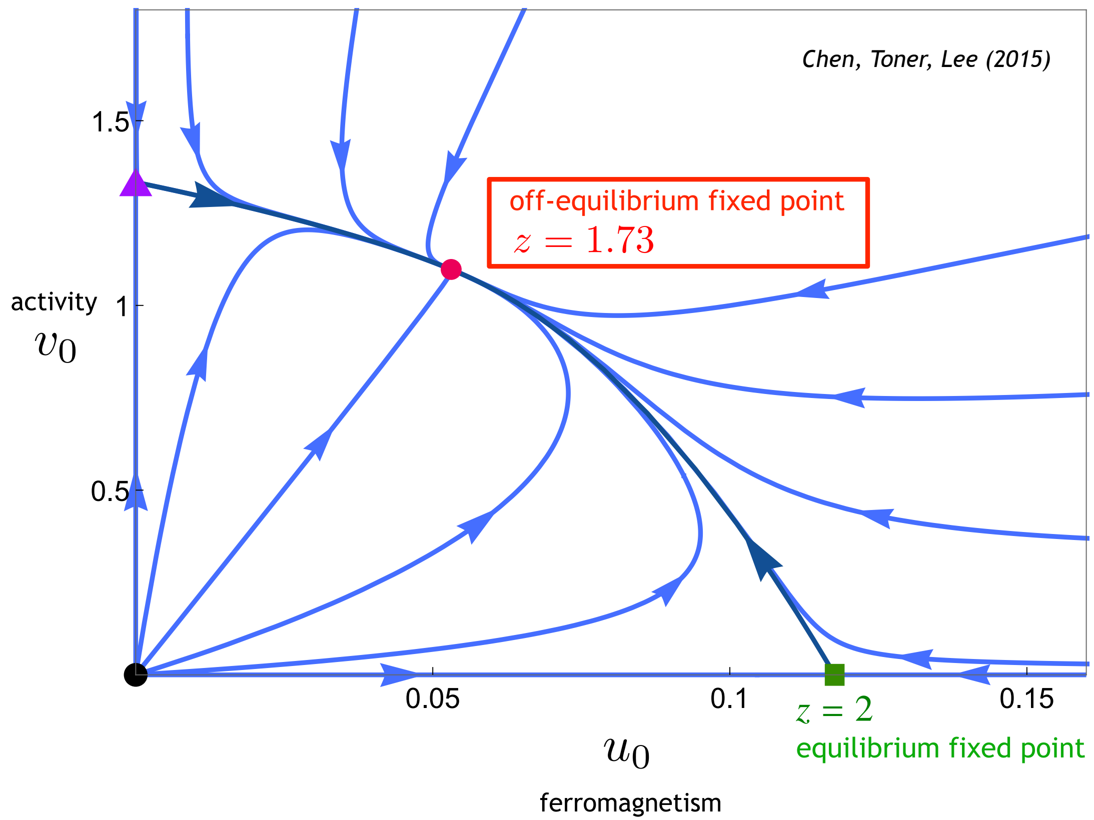
incompressible case: $\rho(x,t) = \rho_0$

$$\begin{cases} D_t\boldsymbol{v}(x,t) = -\Gamma\dfrac{\delta\mathscr{H}}{\delta\boldsymbol{v}} - \nabla P + \boldsymbol{\theta}(x,t) \\ \nabla\cdot\boldsymbol{v} = 0 \end{cases}$$

Chen, Toner, Lee (2015)

see also Forster, Nelson, Stephens (1977)

activity brings the RG flow to a new fixed point



this RG exponent is also confirmed by simulations of the *compressible* case

RG - ferromagnetism

$z \approx 2$

Halperin, Hohenberg, Ma (1972)

RG - ferromagnetism and activity

$z = 1.73$

Chen, Toner, Lee (2015)

something is still missing

$z_{\mathrm{exp}} = 1.37 \pm 0.11$

experiments

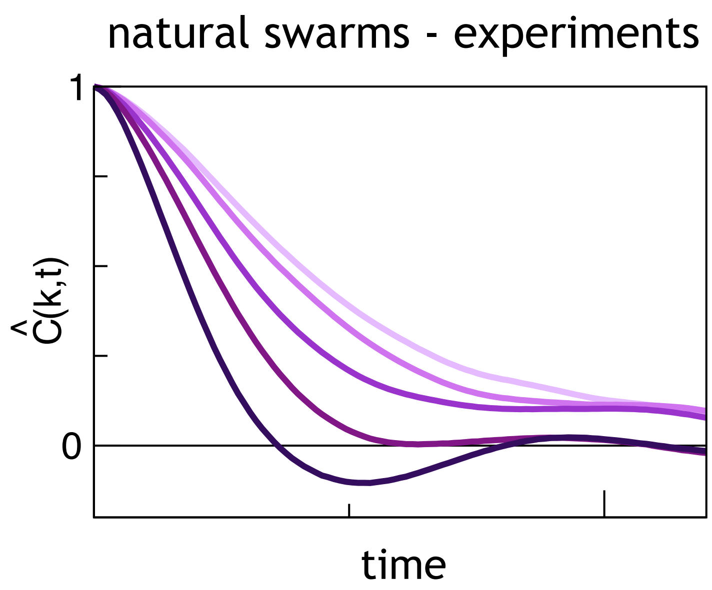
let's go back to the experimental evidence

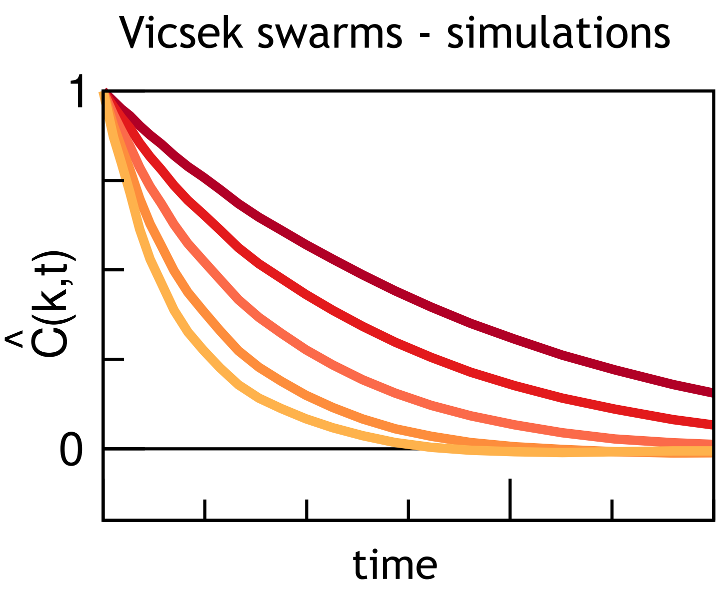
and remember:

a smaller exponent z suggests that

a more efficient propagation mechanism is at play

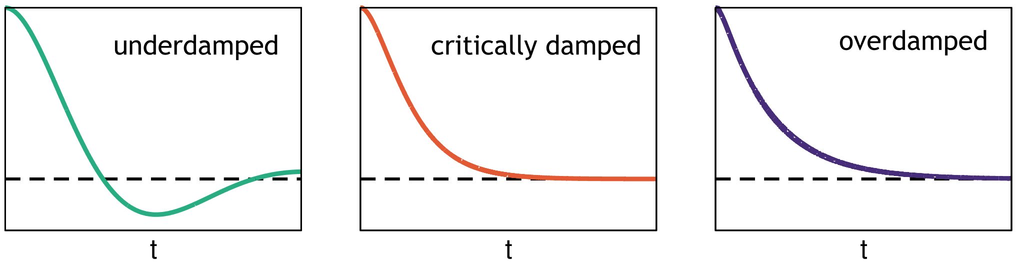
fact: temporal relaxation in natural swarms is underdamped





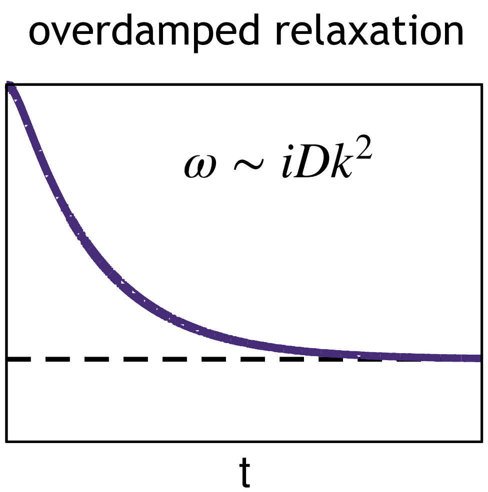
toy example:

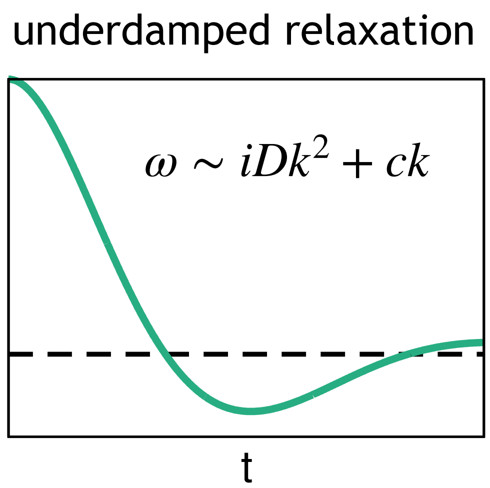
$$m\ddot{q} + \eta\dot{q} + kq = \zeta$$



why this should help?

underdamping requires a real part of the frequency





because alignment requires a Laplacian, a real part of ω indicates there are *two* derivatives in time:

$$\frac{\partial}{\partial t} \sim \nabla^2 \longrightarrow i\omega \sim k^2 \longrightarrow z_{\text{naive}} \sim 2$$

$$\frac{\partial^2}{\partial t^2} \sim \nabla^2 \longrightarrow \omega^2 \sim k^2 \longrightarrow z_{\text{naive}} \sim 1$$

looks promising!

we must go back to underdamped dynamics

conjugate variables and their Poisson relations

$$\{p_\alpha, p_\beta\} = 0$$

$$\{p_\alpha, q_\beta\} = \delta_{\alpha\beta}$$

p is the generator
of the translations of q

$$\begin{cases} \dot{q} = \boxed{p} \\ \dot{p} = \boxed{-\dfrac{\partial H}{\partial q}} \boxed{- \eta p + \theta} \end{cases}$$

reversible | irreversible - relax

overdamped limit →

$$\eta \dot{q} = \boxed{-\frac{\partial H}{\partial q} + \theta}$$

we need to restore the generator of the *rotations* of the polarization field ψ

this is the *internal* angular momentum, aka **spin** s:

$$\{s_\alpha, s_\beta\} = \epsilon_{\alpha\beta\gamma}\, s_\gamma$$

$$\{s_\alpha, \psi_\beta\} = \epsilon_{\alpha\beta\gamma}\, \psi_\gamma$$

$$\begin{cases} \dot{\psi} = \boxed{\psi \times s} \\ \dot{s} = \boxed{-\psi \times \dfrac{\partial H}{\partial \psi}} \boxed{- \eta s + \theta} \end{cases}$$

← back to underdamped

$$\dot{\psi} = \boxed{-\frac{\partial H}{\partial \psi} + \theta}$$

Attanasi et al 2014, Cavagna et al 2015

Theory of dynamic critical phenomena

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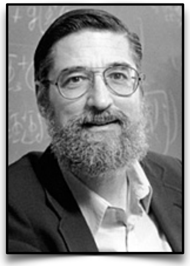


Reviews of Modern Physics, July 1977

CONTENTS

Halperin-Hohenberg's Model G

Model G

$$\begin{cases} \dfrac{\partial \boldsymbol{\psi}(x,t)}{\partial t} = + g\,\boldsymbol{\psi} \times \dfrac{\delta \mathcal{H}}{\delta \boldsymbol{s}} - \Gamma \dfrac{\delta \mathcal{H}}{\delta \boldsymbol{\psi}} + \boldsymbol{\theta}_{\psi}(x,t) \\ \dfrac{\partial \boldsymbol{s}(x,t)}{\partial t} = - g\,\boldsymbol{\psi} \times \dfrac{\delta \mathcal{H}}{\delta \boldsymbol{\psi}} - \Lambda \dfrac{\delta \mathcal{H}}{\delta \boldsymbol{s}} + \boldsymbol{\theta}_{s}(x,t) \end{cases}$$

reversible terms — relaxational irreversible terms

overdamped limit →

Model A

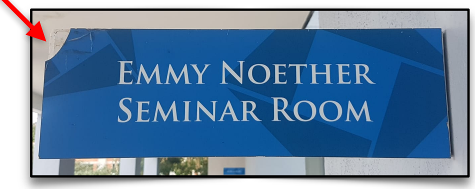
$$\frac{\partial \boldsymbol{\psi}(x,t)}{\partial t} = - \Gamma \frac{\delta \mathcal{H}}{\delta \boldsymbol{\psi}} + \boldsymbol{\theta}(x,t)$$

relaxational irreversible terms

the rotational symmetry of the dynamics implies that the spin $\boldsymbol{s}(x,t)$ *is a conserved quantity*

spin conservation law ⟹ $\omega = iDk^2 \pm ck$ and $z = 1.5$

fix this!



ingredient #3

underdamping - inertia - spin conservation

(but this is not your regular inertia!)

promoting Model G to an active field theory

Equilibrium Model G:

$$\begin{cases} \dfrac{\partial \boldsymbol{v}(x,t)}{\partial t} = +\, g\, \boldsymbol{v} \times \dfrac{\delta \mathcal{H}}{\delta \boldsymbol{s}} - \Gamma \dfrac{\delta \mathcal{H}}{\delta \boldsymbol{v}} + \boldsymbol{\theta}_v(x,t) \\ \dfrac{\partial \boldsymbol{s}(x,t)}{\partial t} = -\, g\, \boldsymbol{v} \times \dfrac{\delta \mathcal{H}}{\delta \boldsymbol{v}} - \Lambda \dfrac{\delta \mathcal{H}}{\delta \boldsymbol{s}} + \boldsymbol{\theta}_s(x,t) \end{cases}$$

go active:

$$\begin{cases} \partial_t \boldsymbol{v} \;\rightarrow\; D_t \boldsymbol{v} = \partial_t \boldsymbol{v} + \gamma_{\boldsymbol{v}} (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} \\ \partial_t \boldsymbol{s} \;\rightarrow\; D_t \boldsymbol{s} = \partial_t \boldsymbol{s} + \gamma_{\boldsymbol{s}} (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{s} \end{cases}$$

Self-Propelled Model G (or Active Model G) - our theory:

$$\begin{cases} D_t \boldsymbol{v}(x,t) = +\, g\, \boldsymbol{v} \times \dfrac{\delta \mathcal{H}}{\delta \boldsymbol{s}} - \Gamma \dfrac{\delta \mathcal{H}}{\delta \boldsymbol{v}} - \nabla P + \boldsymbol{\theta}_v(x,t) \\ D_t \boldsymbol{s}(x,t) = -\, g\, \boldsymbol{v} \times \dfrac{\delta \mathcal{H}}{\delta \boldsymbol{v}} - \Lambda \dfrac{\delta \mathcal{H}}{\delta \boldsymbol{s}} + \boldsymbol{\theta}_s(x,t) \end{cases}$$

to be studied in the swarm phase

plus incompressibility: $\nabla \cdot \boldsymbol{v} = 0$

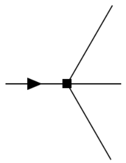
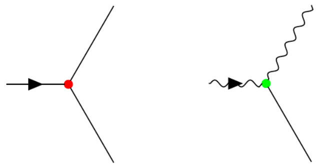
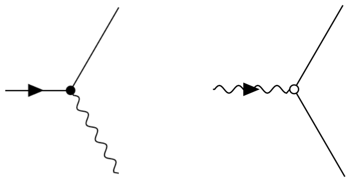
$$\mathcal{H} = \int d^d x \, [(\nabla \boldsymbol{v})^2 + r\boldsymbol{v}^2 + u\boldsymbol{v}^4] + \frac{1}{2}\boldsymbol{s}^2$$

4 dynamical fields and 5 non-linear couplings

$\mathbf{v}(x,t)$ $\mathbf{s}(x,t)$ 

$\hat{\mathbf{v}}(x,t)$  $\hat{\mathbf{s}}(x,t)$ 

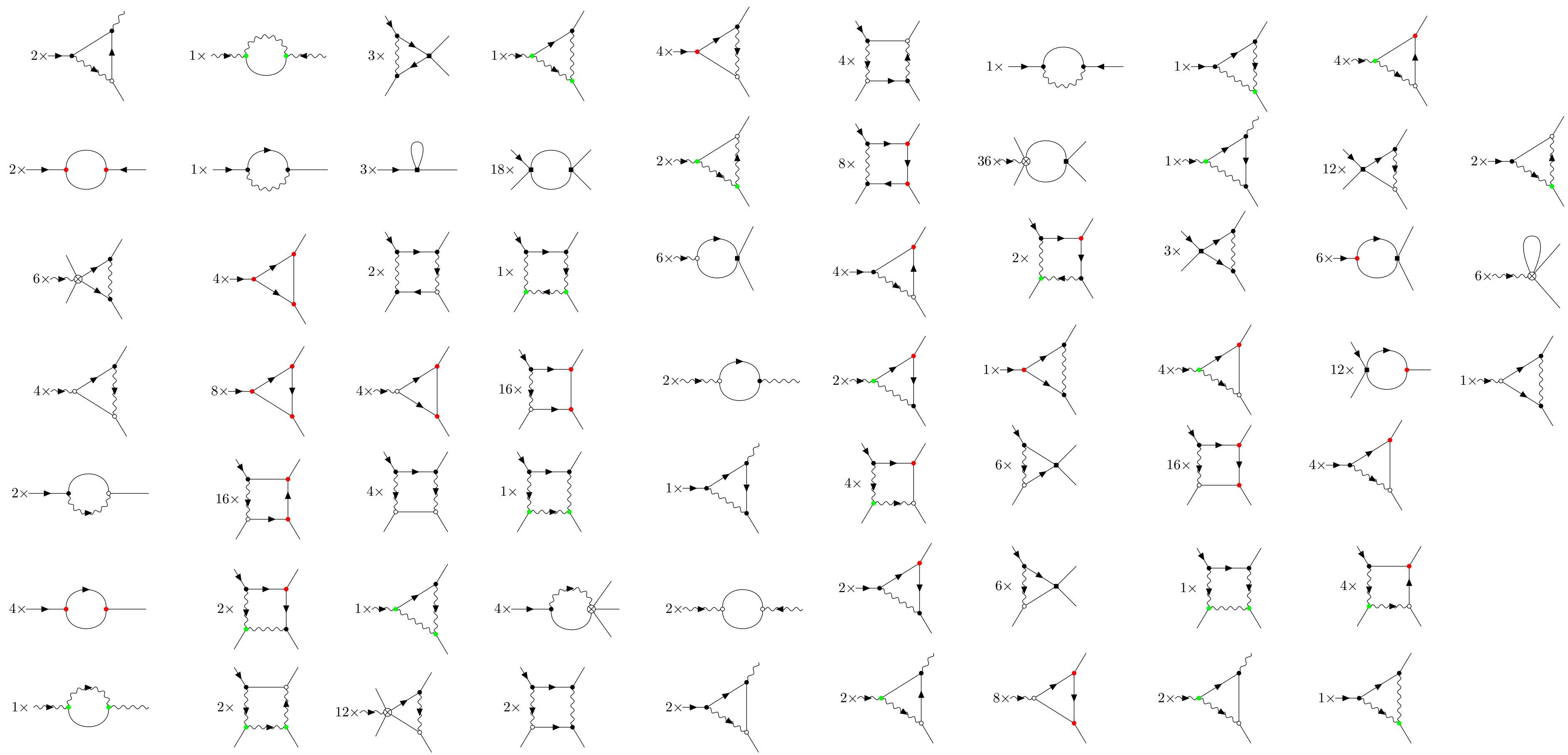
Martin-Siggia-Rose/Janssen-De Dominicis-Peliti

- ferromagnetic interaction: 
- active transport of the velocity and of the spin: 
- inertial spin-velocity couplings: 

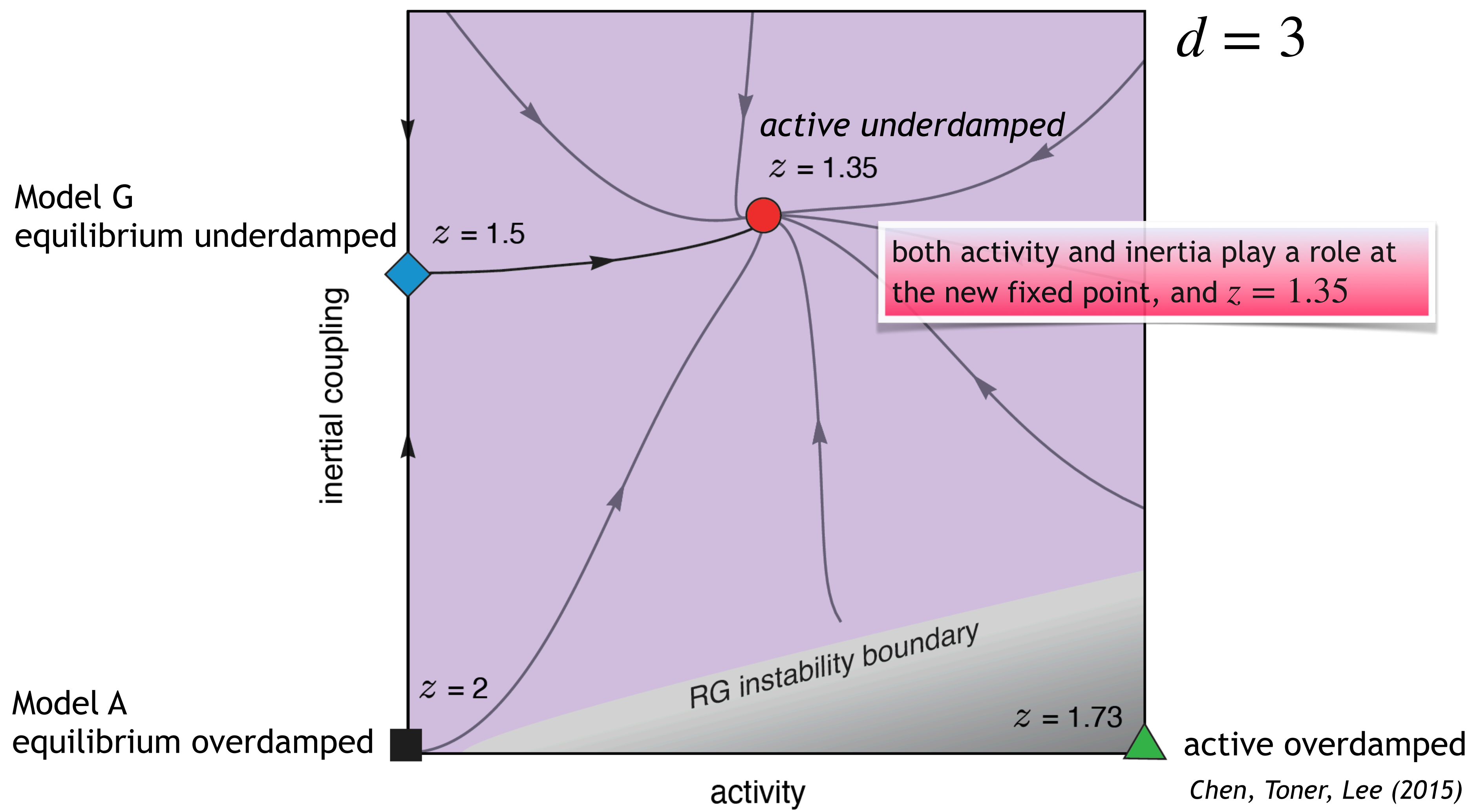
all coupling constants have RG scaling dimension equal to $\varepsilon = 4 - d$, hence:

expansion for $d = 4 - \varepsilon$

a handful of 1-loop diagrams



a novel fixed point emerges



both activity and inertia play a role at the new fixed point, and $z = 1.35$

Chen, Toner, Lee (2015)

RG - ferromagnetism

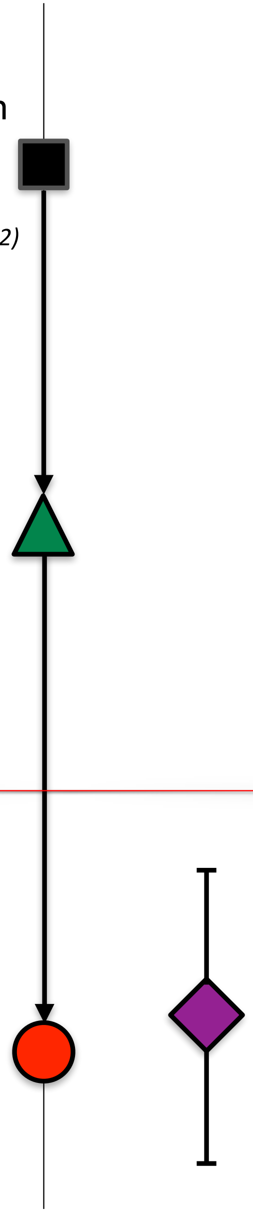
$z \approx 2$

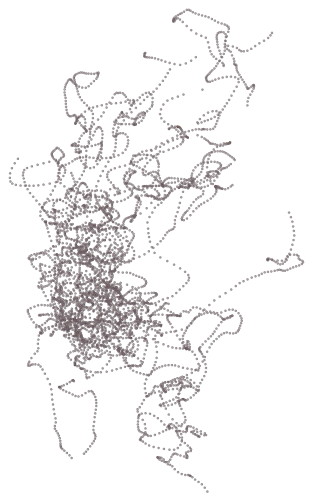
Halperin, Hohenberg, Ma (1972)

RG - ferromagnetism and activity

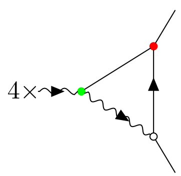
$z = 1.73$

Chen, Toner, Lee (2015)





a fair agreement



RG - ferromagnetism, activity and inertia

$z = 1.35$

this work

$z_{\rm exp} = 1.37 \pm 0.11$

experiments

numerical simulations

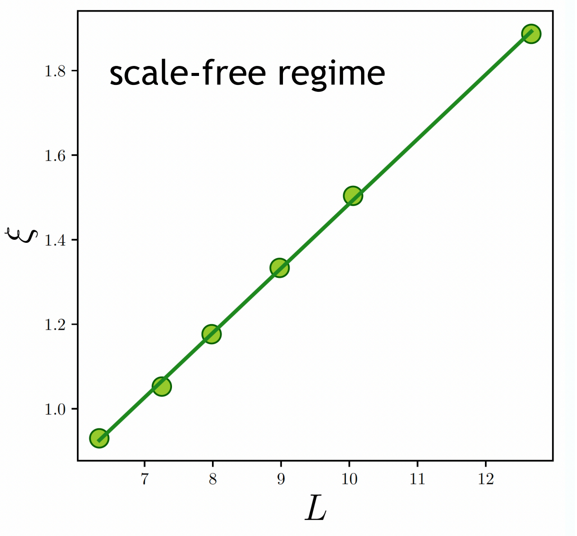
numerical simulations - Inertial Spin Model

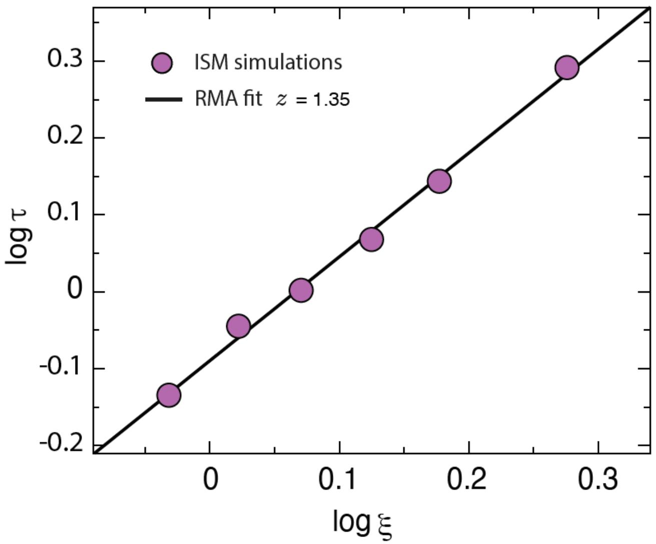
$$\begin{cases} \dfrac{d\boldsymbol{v}_i}{dt} = \dfrac{1}{\chi}\boldsymbol{s}_i \times \boldsymbol{v}_i \\ \dfrac{d\boldsymbol{s}_i}{dt} = \boldsymbol{v}_i \times \dfrac{J}{n_i}\sum_j n_{ij}(t)\boldsymbol{v}_j - \dfrac{\eta}{\chi}\boldsymbol{s}_i + \boldsymbol{v}_i \times \boldsymbol{\zeta}_i \\ \dfrac{d\boldsymbol{r}_i}{dt} = \boldsymbol{v}_i \end{cases}$$

Sriram, notice this... and yet spin cannot be eliminated!

$$\langle \boldsymbol{\zeta}_i(t) \cdot \boldsymbol{\zeta}_j(t') \rangle = 2dT\,\eta\,\delta_{ij}\delta(t-t')$$

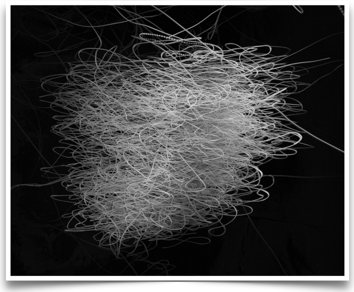
Cavagna et al 2015





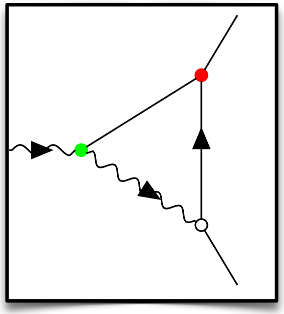
$$z_{\text{sim}} = 1.35 \pm 0.04$$

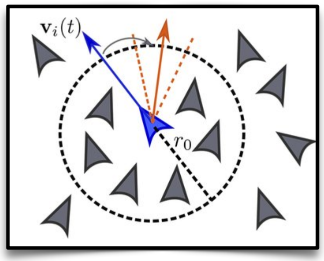
final comparison



$$z_{\mathrm{exp}} = 1.37 \pm 0.11$$

$$z_{\mathrm{RG}} = 1.35$$

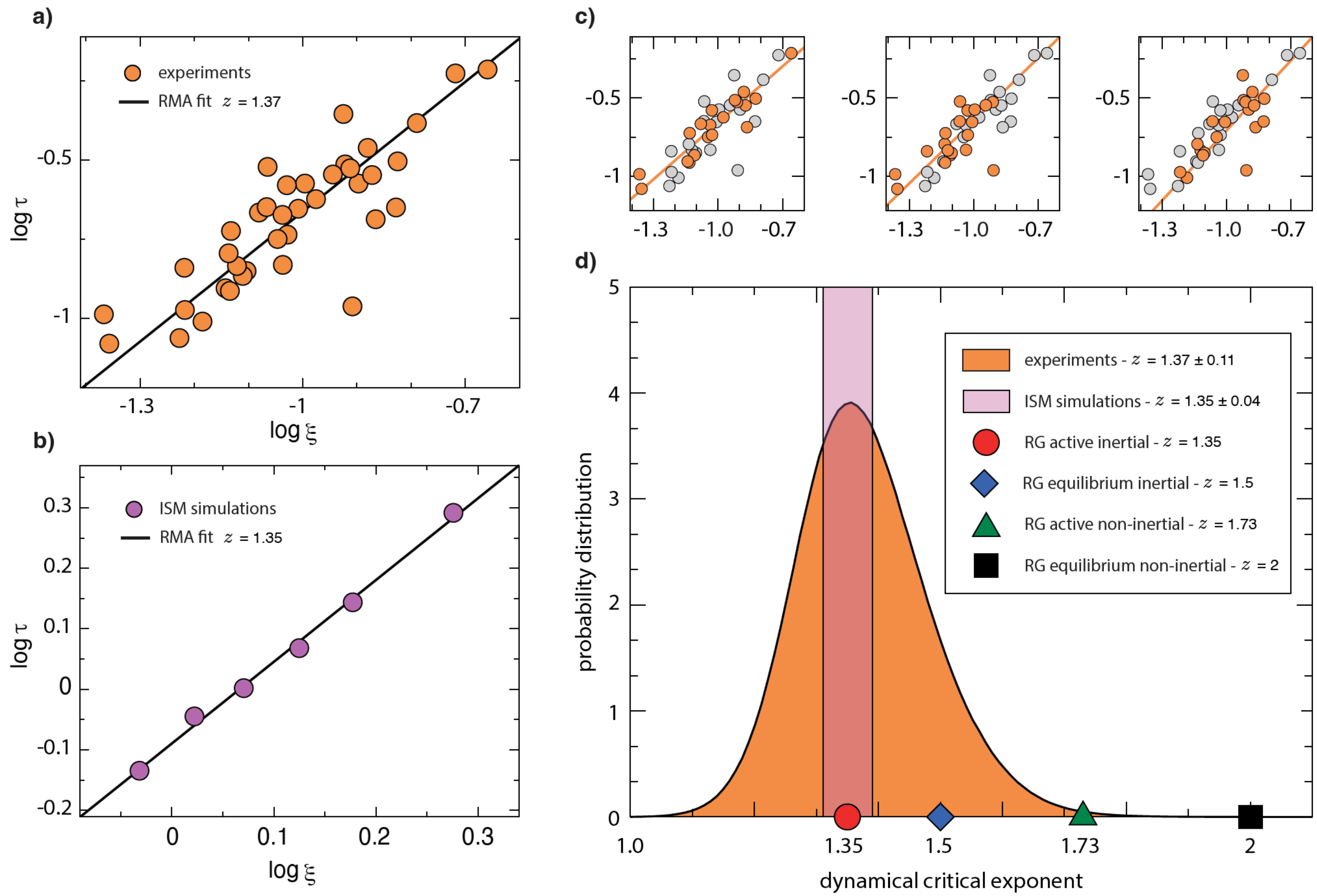




$$z_{\mathrm{sim}} = 1.35 \pm 0.04$$

no free parameters

summary: experiments - simulations - RG theory



too good to be true?

$$z_{\mathrm{RG,1-loop}} = 1.35$$

$$z_{\mathrm{sim}} = 1.35 \pm 0.04$$

$$z_{\mathrm{RG,2-loop}} = ?$$

will the 2-loop corrections be just zero ?!?

$$z_{\mathrm{RG}}^{\mathrm{Model\ G}} = 1.5$$

$$\delta z_{\mathrm{1-loop}} = 0.15$$

$$\delta z_{\mathrm{2-loop}} = ?$$

$$\delta z_{\mathrm{2-loop}}^{\mathrm{Ising}} = 0.02$$

it's *not* to good to be true — we are in line with standard calculations

RG crossover

spin dynamics:

$$\dot{s} = -\Lambda s + \dots$$

conservative:

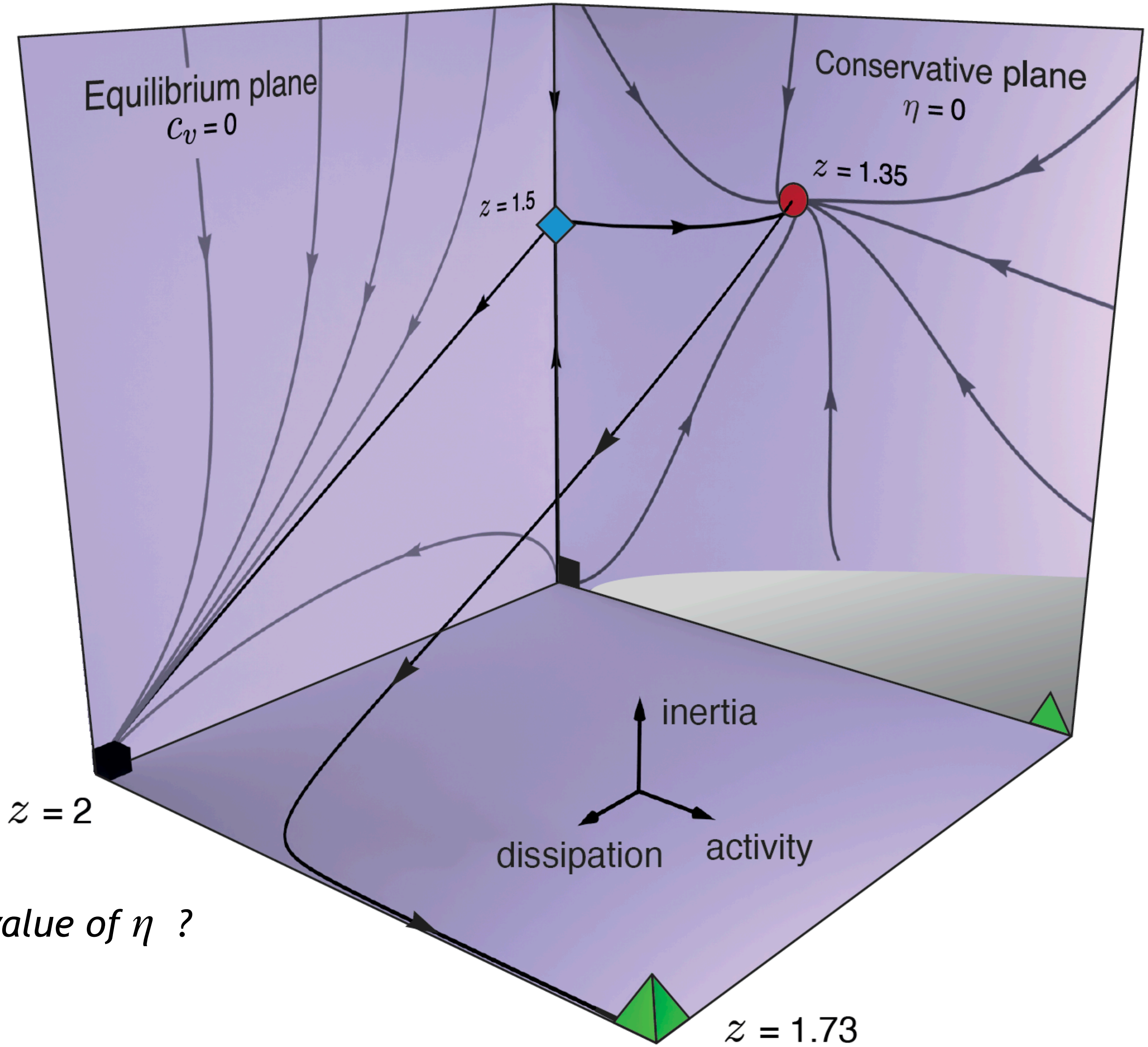
$$\Lambda = \lambda k^2$$

non - conservative:

$$\Lambda = \lambda k^2 + \eta$$

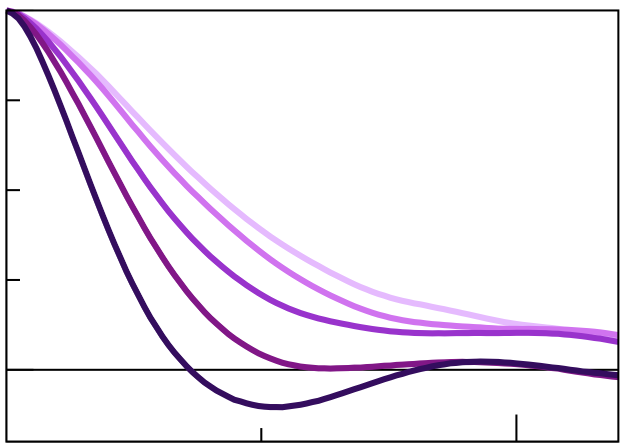
η is an RG-relevant variable

but how large is the physical value of η *?*

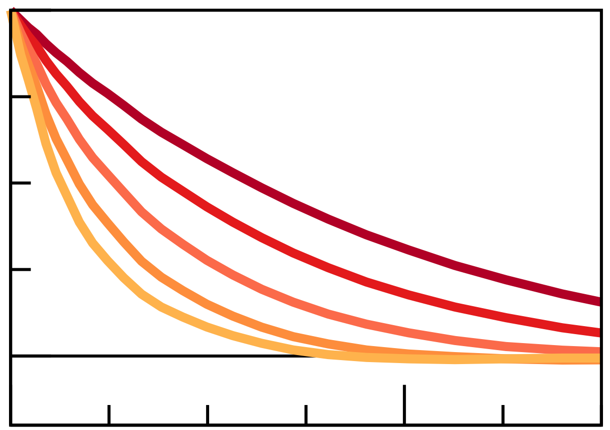


temporal relaxation in natural swarms is clearly underdamped

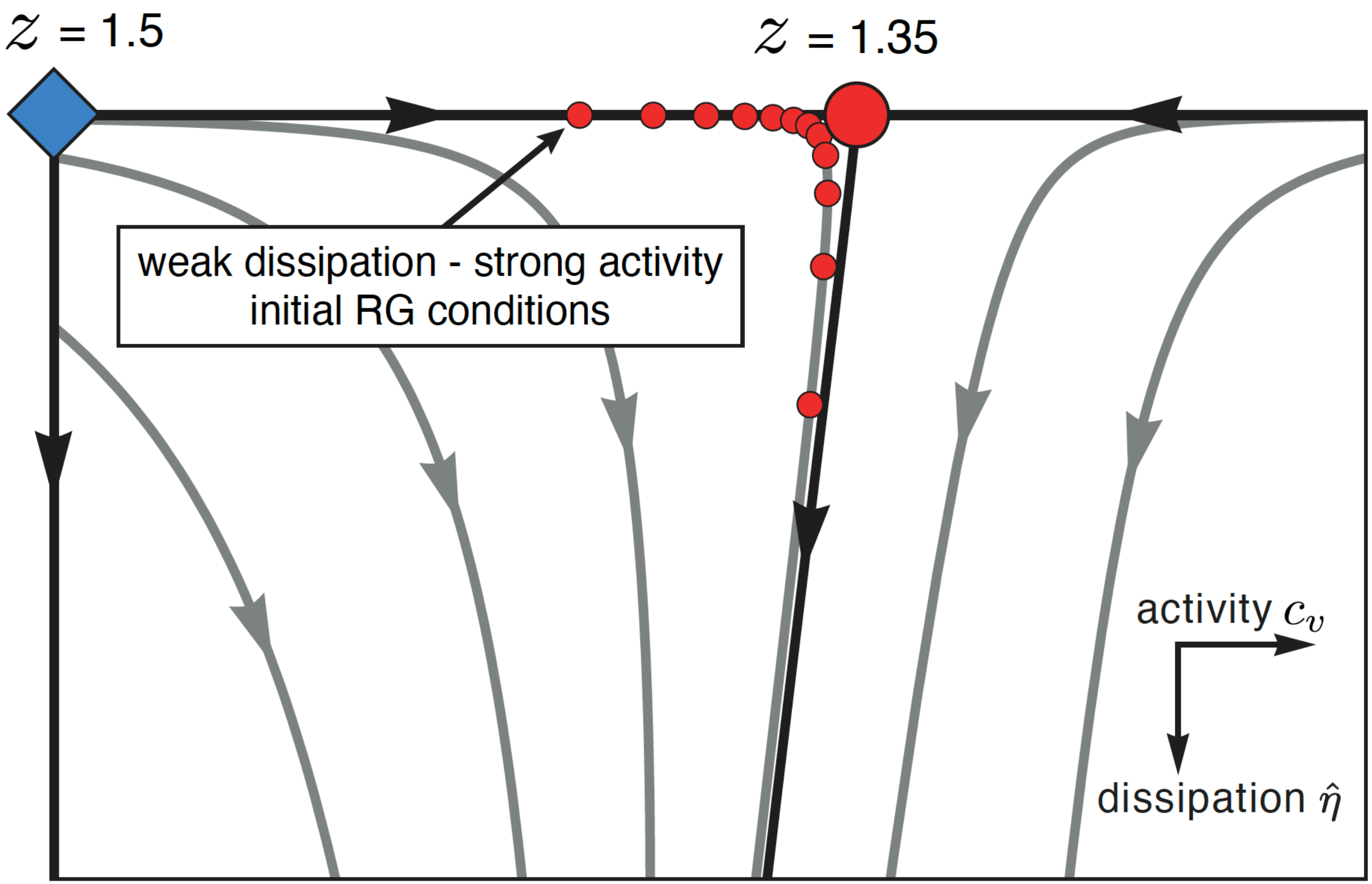
natural swarms - experiments



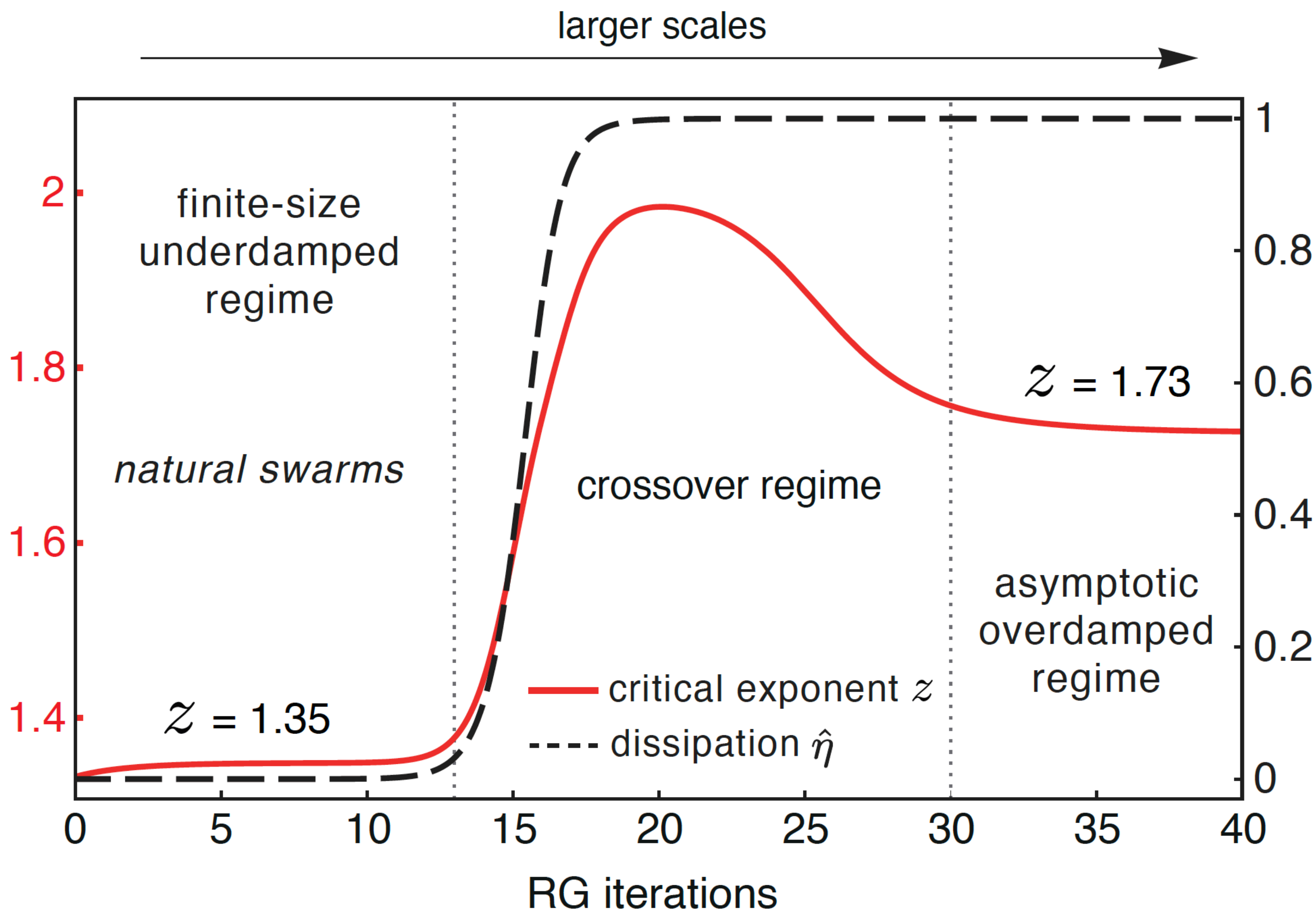
Vicsek swarms - simulations

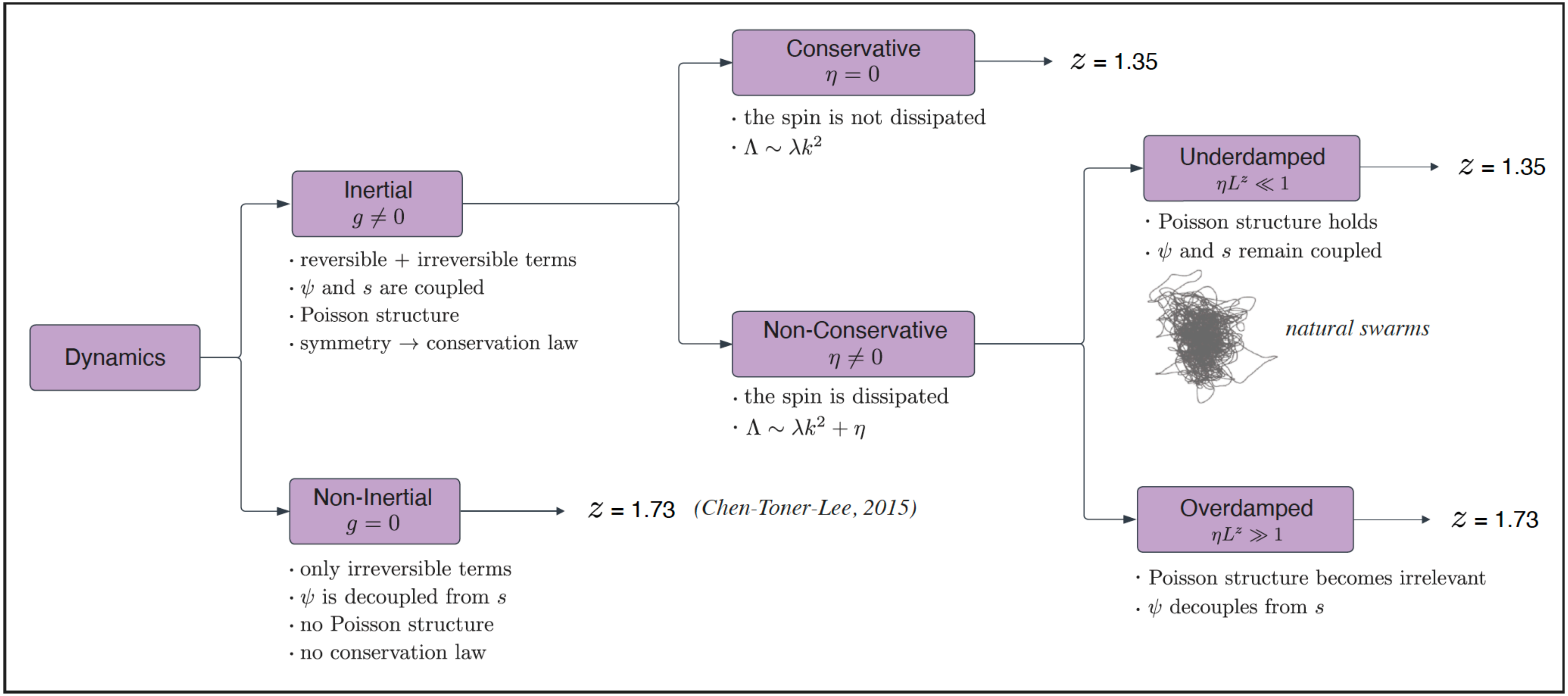


RG crossover



RG crossover





Dynamics
Inertial
$g \neq 0$
· reversible + irreversible terms
· ψ and s are coupled
· Poisson structure
· symmetry $\rightarrow$ conservation law
Non-Inertial
$g = 0$
· only irreversible terms
· ψ is decoupled from s
· no Poisson structure
· no conservation law
z = 1.73 (Chen-Toner-Lee, 2015)
Conservative
$\eta = 0$
· the spin is not dissipated
· $\Lambda \sim \lambda k^2$
z = 1.35
Non-Conservative
$\eta \neq 0$
· the spin is dissipated
· $\Lambda \sim \lambda k^2 + \eta$
Underdamped
$\eta L^z \ll 1$
· Poisson structure holds
· ψ and s remain coupled
z = 1.35
natural swarms
Overdamped
$\eta L^z \gg 1$
· Poisson structure becomes irrelevant
· ψ decouples from s
z = 1.73

the 3.99 group

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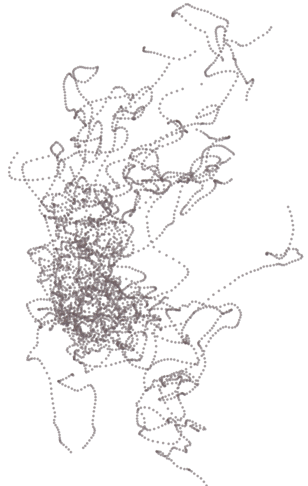
Göttingen

Mattia Scandolo
Rome



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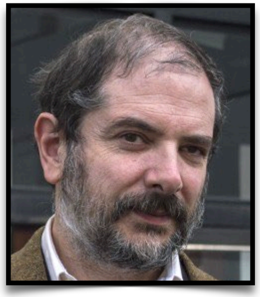

Tomas Grigera
La Plata

"Natural Swarms in 3.99 Dimensions"
Nature Physics, 2023