Universal Hitchin moduli spaces

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Geometric Structures and Stability

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Joint work with Mario Garcia-Fernandez,
Oscar García-Prada and Samuel Trautwein

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Plan of the talk

1. Motivation and objectives
2. Flat connections
3. Coupled harmonic equations
4. Higgs bundles
5. Universal Hitchin moduli space

Motivation: dependence of moduli on complex structure

Fix a compact oriented surface Σ and a complex semisimple Lie group G.

Then the non-abelian Hodge correspondence on surfaces (Donaldson 1987, Hitchin 1987) gives, in particular, homeomorphisms

$$\mathcal{M}^{\mathrm{Flat}}(G) \cong \mathcal{M}^{\mathrm{Harm}}(G) \cong \mathcal{M}^{\mathrm{Hit}}(G) \cong \mathcal{M}^{\mathrm{Higgs}}(G)$$

Moduli space of reductive flat G-connections	Moduli space of harmonic G-bundles	Moduli space of solutions to Hitchin's equations	Moduli space of polystable Higgs bundles

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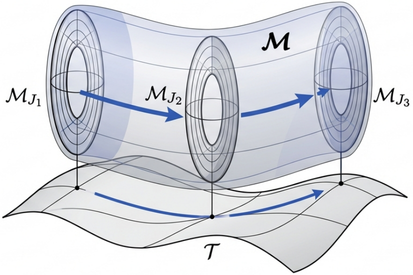

Objective: find universal symplectic constructions

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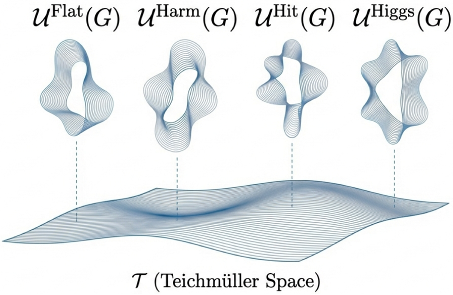

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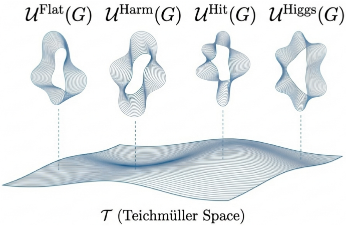

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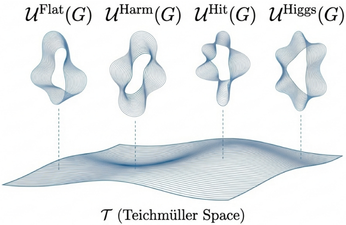

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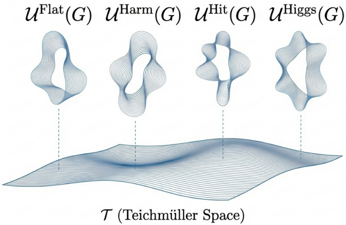

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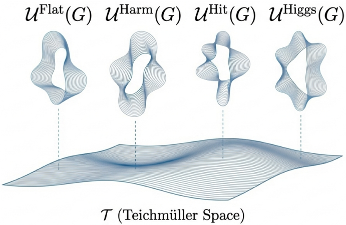

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Related algebraic approaches: Carlos Simpson (1994), V. Balaji, P. Barik & D. S. Nagaraj (2013), Sourav Das (2021), Oren Ben-Bassat, Sourav Das & Tony Pantev (2023), Ron Donagi & Andrés Fernandez Herrero (2024).

Origins of the project

Coupled equations for Kähler metrics and Yang-Mills connections

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The foundation (2009, 2013)

Based on the PhD thesis of Mario Garcia-Fernandez (UAM, Madrid). Moment map calculations for the extended gauge group acting on spaces of connections.

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Chapter 7

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Joint work with Samuel Trautwein (ETH Zürich). Moment map calculations for the extended gauge group acting on spaces of Higgs bundles.

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The revival (2025)

Revived by Richard Wentworth's lectures at ICMAT. Return to the universal moduli problem.

Basic set-up

We fix the following throughout this talk:

G complex semisimple connected Lie group with Lie algebra $\mathfrak{g}$
$K \subset G$ maximal compact Lie subgroup with Lie algebra $\mathfrak{k}$
$\tau\colon G \longrightarrow G$ Lie-group antiholomorphic involution with $K = G^\tau$
$\tau := \mathrm{d}\tau\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ Lie-algebra antiholomorphic involution
$\langle -, - \rangle = -\kappa$ with κ=Killing form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, so $\langle -, - \rangle|_{\mathfrak{k}} > 0$

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Then $$\mathcal{A} \times \Omega^1(\Sigma, E_K(\mathfrak{k})) \xrightarrow{\cong} \mathcal{D}, \quad (A, \psi) \longmapsto D = A + \mathrm{i}\psi$$

$$\forall J \in \mathcal{J}, \qquad \Omega^1(\Sigma, E_K(\mathfrak{k})) \xrightarrow{\cong} \Omega^{1,0}_J(E_G(\mathfrak{g}))$$
$$\psi \longmapsto \varphi = \psi^{1,0_J}$$

Flat connections

Moduli space of reductive flat connections

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$$\boldsymbol{J}(\dot{D}) = \mathrm{i}\dot{D}, \quad \boldsymbol{\Omega_J}(\dot{D}_1, \dot{D}_2) = \int_\Sigma \langle \dot{D}_1 \wedge \dot{D}_2 \rangle$$

for $D \in \mathcal{D}$, $\dot{D}_i \in T_D\mathcal{D} = \Omega^1(\Sigma, E_G(\mathfrak{g}))$.

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- The action of the (complex) gauge group $\mathcal{G}$ of E_G on $(\mathcal{D}, \boldsymbol{\Omega_J})$ is Hamiltonian. The zeros of the moment map $\mu_{\mathcal{G}} \colon \mathcal{D} \to (\operatorname{Lie}\mathcal{G})^*$ are the flat connections.

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- As usual with complex-symplectic quotients, to construct a Hausdorff moduli space, we can proceed in two steps:
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  (2) Take either **(a)** the GIT quotient $\mu_{\mathcal{G}}^{-1}(0)/\!/_{\mathrm{GIT}}\mathcal{G}$, or **(b)** the symplectic quotient $\mu_{\mathcal{K}}^{-1}(0)/\!/_{\mathrm{sym}}\mathcal{K}$ corresponding to the third symplectic form $\boldsymbol{\omega_J}$.

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  Choosing **(a)**, we restrict to the complex subspace $\mathcal{D}^{\mathrm{red}} \subset \mathcal{D}$ of reductive connections, obtaining the *moduli space of flat connections* as a 'GIT quotient':

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- The space $\mathcal{J}$ has a complex structure $\mathbb{J}_{\mathcal{J}}$ given for $\dot{J} \in T_J\mathcal{J}$ by

$$\mathbb{J}_{\mathcal{J}}\dot{J} = J\dot{J}.$$

Thus the projection map $\pi\colon \mathfrak{X} := \mathcal{J} \times \mathcal{D} \to \mathcal{J}$ is a (flat) holomorphic symplectic fibration (i.e., it is a holomorphic map whose fibres are holomorphic symplectic), for the product complex structure $\mathbb{J}$ on $\mathfrak{X} = (\mathcal{J}, \mathbb{J}_{\mathcal{J}}) \times (\mathcal{D}, \boldsymbol{J})$ and the pullback $\boldsymbol{\Omega}_{\mathbb{J}} := \pi^*\boldsymbol{\Omega_J}$.

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Complex gauge group: $\mathcal{G} := \left\{ \begin{array}{c} G\text{-equivariant automorphisms of } E_G \\ \text{covering the identity on } \Sigma \end{array} \right\}$

Group extension: $1 \to \mathcal{G} \longrightarrow \widetilde{\mathcal{G}} \longrightarrow \mathrm{Diff}_0(\Sigma) \to 1$

Moment map for $\widetilde{\mathcal{G}}$-action

The space $\mathfrak{X} = \mathcal{J} \times \mathcal{D}$ has a **canonical action** of the group $\widetilde{\mathcal{G}}$, preserving $\mathbb{J}$ and $\boldsymbol{\Omega}_{\mathbb{J}} := \pi^* \boldsymbol{\Omega}_{\boldsymbol{J}}$, and covering the $\mathrm{Diff}_0(\Sigma)$-action on $\mathcal{J}$.

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Proposition (AC, Garcia-Fernandez, Garcia-Prada, Trautwein 2025)

The $\widetilde{\mathcal{G}}$-action on $(\mathfrak{X}, \mathbf{\Omega}_{\mathbb{J}})$ is Hamiltonian, with $\widetilde{\mathcal{G}}$-equivariant moment map $\mu_{\widetilde{\mathcal{G}}} \colon \mathfrak{X} \to (\mathrm{Lie}\, \widetilde{\mathcal{G}})^*$ whose zeros are the pairs $(J, D) \in \mathfrak{X}$ such that $F_D = 0$.

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$$\langle \mu_{\widetilde{\mathcal{G}}}(J, D), \zeta \rangle = \int_\Sigma \langle F_D, \theta_D(\zeta) \rangle,$$

where $\theta_D(\zeta) \in \Gamma(\Sigma, E_G(\mathfrak{g}))$ is the vertical part of the G-equivariant vector field ζ on the total space of E_G with respect to the connection D.

Proof. Apply methods of Garcia-Fernandez' thesis. □

Universal moduli space of flat G-connections

As in the absolute case, we restrict to the complex subspace $\mathcal{D}^{\mathrm{red}} \subset \mathcal{D}$ of reductive connections, so we consider the fibration

$$\pi \colon \mathfrak{X}^{\mathrm{red}} := \mathcal{J} \times \mathcal{D}^{\mathrm{red}} \longrightarrow \mathcal{J}$$

and define the **universal moduli space of flat G-connections** on E_G as the quotient

$$\mathcal{U}^{\mathrm{Flat}}(G) := \{(J, D) \in \mathfrak{X}^{\mathrm{red}} \text{ with } F_D = 0\}/\widetilde{\mathcal{G}} = \mu_{\widetilde{\mathcal{G}}}|_{\mathfrak{X}^{\mathrm{red}}}^{-1}(0)/\widetilde{\mathcal{G}}.$$

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By construction, $\mathcal{U}^{\mathrm{Flat}}(G)$ fibres over the Teichmüller space:

$$\mathcal{U}^{\mathrm{Flat}}(G) \longrightarrow \mathcal{T} := \mathcal{J} / \operatorname{Diff}_0(\Sigma).$$

This fibration is naturally holomorphic, and $\mathbf{\Omega}_{\mathbb{J}}$ induces a structure of holomorphic symplectic fibration (with flat Ehresmann connection).

Universal moduli space for the coupled harmonic equations

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A **Kähler fibration** is a holomorphic fibre bundle

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whose typical fibre (X, ω_X) is a Kähler manifold, such that the local transition functions take values in the group $\mathrm{Aut}(X, \omega)$ of Kähler isometries.

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Then ω_X induces a smoothly varying Kähler structure on the fibres: $\hat{\omega} \in \Gamma(\mathcal{B}, \wedge^2 V^*\mathfrak{X})$, where $V\mathfrak{X} = \ker(\mathrm{d}\pi) \subset T\mathfrak{X}$ is the vertical bundle.

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Coupling form: Suppose $\exists$ a closed real (1,1)-form σ on $\mathfrak{X}$ that restricts to $\hat{\omega}$ on the fibres. Then it determines a connection on the fibration, with horizontal distribution

$$H^\sigma := \{v \in T\mathfrak{X} \mid (i_v\sigma)|_{V\mathfrak{X}} = 0\}$$

that is Kähler, i.e., parallel transport $\tau_\gamma\colon \mathfrak{X}_{\gamma(0)} \to \mathfrak{X}_{\gamma(1)}$ is Kähler isometry.

Kähler fibrations

A **Kähler fibration** is a holomorphic fibre bundle $\pi\colon \mathfrak{X} \to \mathcal{B}$ whose typical fibre (X, ω_X) is a Kähler manifold, such that the local transition functions take values in the group of Kähler isometries.
Then ω_X induces a smoothly varying Kähler structure on the fibres: $\hat{\omega} \in \Gamma(\mathcal{B}, \wedge^2 V^*\mathfrak{X})$, where $V\mathfrak{X} = \ker(\mathrm{d}\pi) \subset T\mathfrak{X}$ is the vertical bundle.

Coupling form: Suppose $\exists$ a closed real (1,1)-form σ on $\mathfrak{X}$ that restricts to $\hat{\omega}$ on the fibres. Then it determines a connection on the fibration, with horizontal distribution

$$H^\sigma := \{v \in T\mathfrak{X} \mid (i_v\sigma)|_{V\mathfrak{X}} = 0\}$$

that is Kähler, i.e., parallel transport $\tau_\gamma\colon \mathfrak{X}_{\gamma(0)} \to \mathfrak{X}_{\gamma(1)}$ is Kähler isometry.

Weak coupling: A fundamental question is whether there exists a Kähler metric $\omega_{\mathfrak{X}}$ on $\mathfrak{X}$ which restricts to $\hat{\omega}$ on the fibres. Given a coupling form σ on $(\mathfrak{X}, \hat{\omega}) \to \mathcal{B}$ and a Kähler form $\omega_{\mathcal{B}}$ on $\mathcal{B}$, a natural *candidate* is

$$\omega_\alpha := \pi^*\omega_{\mathcal{B}} + \alpha\sigma$$

for a small coupling constant $0 < \alpha \ll 1$.

The Kähler fibration for harmonic G-connections

The projection $\pi\colon \mathfrak{X} := \mathcal{J} \times \mathcal{D} \to \mathcal{J}$ is a Kähler fibration, for the product complex structure $\mathbb{J}$ on $\mathfrak{X} = (\mathcal{J}, \mathbb{J}_{\mathcal{J}}) \times (\mathcal{D}, \boldsymbol{J})$ (already defined) and the **family of fibrewise Kähler structures** $\hat{\boldsymbol{\omega}}_{\boldsymbol{J}} = \boldsymbol{g}(\boldsymbol{J}(-), -)$, parametrized by $J \in \mathcal{J}$, with

$$\hat{\boldsymbol{\omega}}_{\boldsymbol{J}}(a_1 + \mathrm{i}\dot{\psi}_1, a_2 + \mathrm{i}\dot{\psi}_2) = \int_\Sigma \langle a_1 \wedge J\dot{\psi}_2 \rangle - \int_\Sigma \langle \dot{\psi}_1 \wedge Ja_2 \rangle$$

for $a_i + \mathrm{i}\dot{\psi}_i \in T_{(A,\mathrm{i}\psi)}(\mathcal{A} \times \mathrm{i}\Omega^1(\Sigma, E_K(\mathfrak{k}))) \cong T_D\mathcal{D}$.

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To perform symplectic reduction on $\mathfrak{X}$, we first search for a coupling form

$$\boldsymbol{\sigma}^{\mathbb{J}} \in \Omega^{1,1}(\mathfrak{X}, \mathbb{R}).$$

The coupling form for G-connections

Proposition (AC-GF-GP-T 2025; after Hitchin 1987)

- There is a 'coupling' closed (1,1)-form $\boldsymbol{\sigma}^{\mathbb{J}}$ on $\mathfrak{X}$ whose restriction to the fibres $\mathfrak{X}_J \cong \mathcal{D}$ is $\widehat{\boldsymbol{\omega}}_{\boldsymbol{J}}$. It is given in terms of the potential

$$\nu \colon \mathfrak{X} \longrightarrow \mathbb{R}, \quad (J, D = A + \mathrm{i}\psi) \longmapsto \|\psi\|^2_{L^2,J} = \int_\Sigma \langle \psi \wedge J\psi \rangle$$

by the formula

$$\boldsymbol{\sigma}^{\mathbb{J}} = \mathrm{i}\partial_{\mathbb{J}}\bar{\partial}_{\mathbb{J}}\nu,$$

where $\mathbb{J}$ is the product complex structure on $\mathfrak{X} = (\mathcal{J}, \mathbb{J}_{\mathcal{J}}) \times (\mathcal{D}, \boldsymbol{J})$.

- The coupling form $\boldsymbol{\sigma}^{\mathbb{J}}$ determines a Kähler connection on this fibration with horizontal distribution

$$H^{\mathbb{J}} = \{(\dot{J}, \psi(\dot{J}-), (J\psi)(\dot{J}-)) \mid (J, A + \mathrm{i}\psi) \in \mathfrak{X}, \dot{J} \in T_J\mathcal{J}\}$$

and non-zero curvature $F_{\mathbb{J}} := -\Gamma[(-)^\Gamma, (-)^\Gamma] \in \Omega^2(\mathfrak{X}, V\mathfrak{X})$ given by

$$F_{\mathbb{J}}(v_1, v_2) = (0, \psi([\dot{J}_1, \dot{J}_2]), 0), \ \forall v_i = (\dot{J}_i, \dot{D}_i) \in T_{(J,A+\mathrm{i}\psi)}\mathfrak{X}.$$

Weak coupling

Fix a symplectic form ω on Σ (compatible with its orientation).

- We have constructed a 'coupling' closed (1,1)-form $\boldsymbol{\sigma}^{\mathbb{J}}$ on $\mathfrak{X}$ that restricts to $\widehat{\boldsymbol{\omega}}_{\boldsymbol{J}}$ on the fibres $\mathfrak{X}_J \cong \mathcal{D}$ of $\mathfrak{X} = \mathcal{J} \times \mathcal{D} \to \mathcal{J}$.
- The space $(\mathcal{J}, \mathbb{J}_{\mathcal{J}})$ has a Kähler form $\omega_{\mathcal{J}}$ given for $\dot{J}_i \in T_J\mathcal{J}$ by

$$\boldsymbol{\omega}_{\mathcal{J}}(\dot{J}_1, \dot{J}_2) = \frac{1}{2}\int_\Sigma \mathrm{Tr}(J\dot{J}_1\dot{J}_2)\omega.$$

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In the weak-coupling approach, one gets a closed 2-form on $\mathfrak{X}$ as follows:

- Fix a parameter $\varepsilon \in \{-1, 1\}$ and a real coupling constant $\alpha > 0$.
- The corresponding 'minimally coupled' closed $(1,1)$-form on $(\mathfrak{X}, \mathbb{J})$ is

$$\boldsymbol{\omega}^{\mathbb{J}}_{\alpha,\varepsilon} := \varepsilon\pi^*\boldsymbol{\omega}_{\mathcal{J}} + \alpha\boldsymbol{\sigma}^{\mathbb{J}}$$

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Signature and non-degeneracy properties of $\boldsymbol{\omega}^{\mathbb{J}}_{\alpha,\varepsilon}$:

- For $\varepsilon = -1$, the symmetric tensor $\boldsymbol{g}^{\mathbb{J}}_{\alpha,-1} = \boldsymbol{\sigma}^{\mathbb{J}}_{\alpha,-1}(-, \mathbb{J}-)$ is negative definite in the horizontal direction $H^{\mathbb{J}}$ (and positive definite in the vertical direction by construction), so $\boldsymbol{\omega}^{\mathbb{J}}_{\alpha,-1}$ is a (non-degenerate) symplectic form on $\mathfrak{X}$.
- For $\varepsilon = +1$ and $\psi \neq 0$, the symmetric tensor $\boldsymbol{g}^{\mathbb{J}}_{\alpha,+1} = \boldsymbol{\sigma}^{\mathbb{J}}_{\alpha,+1}(-, \mathbb{J}-)$ changes signature along the lines $\{(J, A + \mathrm{i}\lambda\psi) \mid \lambda \in \mathbb{R}\} \subset \mathfrak{X}$, becoming null at $H^{\mathbb{J}}_{(J,A+\mathrm{i}\lambda_0\psi)}$ for a specific value $\lambda_0 \in \mathbb{R}$.

The Hamiltonian extended gauge group

Group of hamiltonian symplectomorphisms:

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Gauge group: $\mathcal{K} := \left\{ \begin{array}{c} K\text{-equivariant automorphisms of } E_K \\ \text{covering the identity on } \Sigma \end{array} \right\}$

Group extension: $1 \to \mathcal{K} \longrightarrow \widetilde{\mathcal{K}} \longrightarrow \mathcal{H} \to 1$

Harmonicity and coupling to the scalar curvature

The space $\mathfrak{X} = \mathcal{J} \times \mathcal{D}$ has a **canonical action** of the group $\widetilde{\mathcal{K}}$ preserving $\mathbb{J}$ and $\boldsymbol{\omega}^{\mathbb{J}}_{\alpha,\varepsilon}$.

Proposition (AC, Garcia-Fernandez, Garcia-Prada, Trautwein 2025)

The $\widetilde{\mathcal{K}}$-action on $(\mathfrak{X}, \boldsymbol{\omega}^{\mathbb{J}}_{\alpha,\varepsilon})$ is Hamiltonian, with $\widetilde{\mathcal{K}}$-equivariant moment map $\mu^{\mathbb{J}}_{\widetilde{\mathcal{K}}} \colon \mathfrak{X} \to (\mathrm{Lie}\,\widetilde{\mathcal{K}})^*$ whose zero locus consists of the triples $(J, D = A + \mathrm{i}\psi) \in \mathfrak{X}$ that solve the coupled equations

$$
\begin{aligned}
\mathrm{d}_A^* \psi &= 0, \\
S_g - \varepsilon\alpha * \mathrm{d}\langle \Lambda_\omega F_A, *\psi \rangle &= \frac{2\pi}{V}\chi(\Sigma),
\end{aligned}
$$

where S_g is the scalar curvature of metric $g = \omega(-, J-)$ and $V = \int_\Sigma \omega$.

Proof. Apply methods of Garcia-Fernandez' thesis, including Donaldson–Fujiki's interpretation of S_g as a moment map for the $\mathcal{H}$-action on $\mathcal{J}$. □

The coupled harmonic equations

Fix $\varepsilon \in \{-1, 1\}$ and $\alpha > 0$.
Imposing the flatness condition $F_D = 0$ (coming from the vanishing of the moment map $\mu_{\widetilde{\mathcal{G}}} \colon \mathfrak{X} \to (\mathrm{Lie}\, \widetilde{\mathcal{G}})^*$ and the vanishing of the moment map $\mu^{\mathbb{J}}_{\widetilde{\mathcal{K}}} \colon \mathfrak{X} \to (\mathrm{Lie}\, \widetilde{\mathcal{K}})^*$, we get the following.

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Definition (AC-GF-GP-T 2025)

The *coupled harmonic equations* (CHE) for a triple $(J, A + \mathrm{i}\psi) \in \mathfrak{X}$ are

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F_A - \frac{1}{2}[\psi, \psi] &= 0 \\
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Ignoring singularities, it is presymplectic (symplectic for $\varepsilon = -1$).
It fibres over the Teichmüller space $\mathcal{T}$ via the obvious map

$$\mathcal{U}^{\mathrm{Harm}}_{\alpha,\varepsilon}(G) \longrightarrow \mathcal{U}^{\mathrm{Flat}}(G) \longrightarrow \mathcal{T}.$$

Since the symmetric tensor $\boldsymbol{g}^{\mathbb{J}}_{\alpha,\varepsilon} = \boldsymbol{\sigma}^{\mathbb{J}}_{\alpha,\varepsilon}(-, \mathbb{J}-)$ is not positive definite, it is not clear *a priori* whether $\mathbb{J}$ induces a complex structure on $\mathcal{U}^{\mathrm{Harm}}_{\alpha,\varepsilon}(G)$.

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Theorem (AC-GF-GP-T 2025)

Fix volume $V > 0$, $\varepsilon \in \{-1, 1\}$ and $(J, D = A + \mathrm{i}\psi) \in \mathfrak{X}$, with D flat and **irreducible**. Suppose genus $g(\Sigma) \geq 2$.

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Furthermore,

- the composite $\mathcal{U}_* \hookrightarrow \mathcal{U}^{\mathrm{Harm}}_{\alpha,\varepsilon}(G) \to \mathcal{U}^{\mathrm{Flat}}(G) \to \mathcal{T}$ is holomorphic,
- for $\varepsilon = -1$, $\mathrm{g}^{\mathbb{J}}_{\alpha,\varepsilon}$ is **non-degenerate**, and defines a pseudo-Kähler structure on the moduli space,
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In either case, $\boldsymbol{\omega}^{\mathbb{J}}_{\alpha,\varepsilon}|_{\mathcal{U}_*}$ admits a Kähler potential, i.e., $\boldsymbol{\omega}^{\mathbb{J}}_{\alpha,\varepsilon}|_{\mathcal{U}_*} = \mathrm{i}\partial_{\mathbb{J}}\bar{\partial}_{\mathbb{J}}\Phi$, where $\Phi = \varepsilon\nu_{\mathcal{J}} + \alpha\|\psi\|^2_{L^2}$, with $\nu_{\mathcal{J}}$ induced by the Kähler potential on $\mathcal{J}$.

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Proof: involves appropriate choice of gauge fixing and application of results about **generalized ellipticity of multi-degree differential operators** (Douglis & Nirenberg) following Lockhart & McOwen (1985). □

Universal Higgs-bundle moduli space

The tautological complex structure for Higgs bundles

The fibration $\mathcal{Y} := \mathcal{J} \times \mathcal{A} \to \mathcal{J}$ has canonical complex structure given by

$$\mathbb{I}_{\mathcal{Y}}(\dot{J}, a) = (J\dot{J}, Ja), \quad \forall (\dot{J}, a) \in T_{(J,A)}\mathcal{Y},$$

where $Ja := -a \circ J$. Identifying $\mathfrak{X} = \mathcal{J} \times \mathcal{D} \cong \mathcal{J} \times T^*\mathcal{A}$ with the relative cotangent bundle $T^*_{\mathcal{J}}\mathcal{Y}$ of this fibration, the complex structure $\mathbb{I}_{\mathcal{Y}}$ induces another one $\mathbb{I}$ on $\mathfrak{X}$, given on $T_{(J,A,\psi)}\mathfrak{X}$ by

$$\mathbb{I}(\dot{J}, a, \dot{\psi}) = (J\dot{J}, Ja, J\dot{\psi} + \psi \circ \dot{J}).$$

This complex structure leaves invariant the tangent bundle to the space

$$\mathfrak{X}^{\mathrm{Higgs}} := \{(J, A, \psi) \in \mathfrak{X} \mid \bar{\partial}_{J,A}\varphi = 0\}$$

where J converts $\psi \in \Omega^1(\Sigma, E_K(\mathfrak{k}))$ into a *Higgs field*

$$\varphi := \psi^{1,0_J} \in \Omega^{1,0}(\Sigma, E_G(\mathfrak{g})).$$

Universal moduli space of G-Higgs bundles

Recall we have a group extension where $\mathcal{G}$ is the gauge group of E_G:

$$1 \to \mathcal{G} \longrightarrow \widetilde{\mathcal{G}} \longrightarrow \mathrm{Diff}_0(\Sigma) \to 1$$

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The identity-component extended gauge group $\widetilde{\mathcal{G}}$ acts holomorphically on $\mathfrak{X}$ preserving both the *universal space of Higgs fields* $\mathfrak{X}^{\mathrm{Higgs}}$ and its subset

$$\mathfrak{X}^{\mathrm{ps}} := \{(J, A, \psi) \in \mathfrak{X}^{\mathrm{Higgs}} \mid (\mathcal{E}, \varphi) \text{ is polystable over } (\Sigma, J)\}$$

where $(\mathcal{E}, \varphi)$ is the G-Higgs bundle determined by (J, A, ψ).

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where we define the *universal moduli space of G-Higgs bundles* as the 'GIT quotient'

$$\mathcal{U}^{\mathrm{Higgs}}(G) := \mathfrak{X}^{\mathrm{ps}} / \widetilde{\mathcal{G}}.$$

The coupled Hitchin equations and the universal Hitchin moduli space

The Kähler fibration for Higgs fields

- Kähler fibration $\pi : (\mathfrak{X}, \mathbb{I}) \to \mathcal{J}$ given by the **constant** fibrewise Kähler form $\hat{\omega}_{\mathbb{I}} = \pi^* \boldsymbol{\omega_I}$, where $\boldsymbol{\omega_I} = \mathrm{Re}\, \boldsymbol{\Omega_J}$ is standard 'I-Kähler form' on $\mathcal{D}$. In terms of $a_i + \mathrm{i}\dot{\psi}_i \in T_D\mathcal{D} = T_A\mathcal{A} \times \mathrm{i}\Omega^1(E_K(\mathfrak{k}))$,

$$\boldsymbol{\omega_I}(a_1 + \mathrm{i}\dot{\psi}_1, a_2 + \mathrm{i}\dot{\psi}_2) = \int_\Sigma \Big(\langle a_1 \wedge a_2 \rangle - \langle \dot{\psi}_1 \wedge \dot{\psi}_2 \rangle \Big)$$

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- Kähler fibration $\pi : (\mathfrak{X}, \mathbb{I}) \to \mathcal{J}$ given by the **constant** fibrewise Kähler form $\hat{\omega}_{\mathbb{I}} = \pi^* \boldsymbol{\omega_I}$, where $\boldsymbol{\omega_I} = \mathrm{Re}\, \boldsymbol{\Omega_J}$ is standard 'I-Kähler form' on $\mathcal{D}$. In terms of $a_i + \mathrm{i}\dot{\psi}_i \in T_D\mathcal{D} = T_A\mathcal{A} \times \mathrm{i}\Omega^1(E_K(\mathfrak{k}))$,

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- The following formula, for $v_i = (a_i, \dot{\psi}_i) \in T_{(J,A,\mathrm{i}\psi)}\mathfrak{X}$, defines a closed $(1,1)$-form on $\mathfrak{X}$ that is a coupling form for a Kähler Ehresmann connection Γ (so it preserves $\mathfrak{X}^{\mathrm{Higgs}} \subset \mathfrak{X}$) that is compatible with $\hat{\boldsymbol{\omega}}_{\mathbb{I}}$:

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- Γ has curvature $F^{\mathbb{I}}(v_1, v_2) = (0, 0, \frac{\psi}{4}([\dot{J}_1, \dot{J}_2]))$, for $\dot{J}_i := \mathrm{d}\pi(v_i) \in T_J\mathcal{J}$.
- The associated symmetric tensor $\boldsymbol{\sigma}^{\mathbb{I}}(-, \mathbb{I}-)$ is negative semi-definite along the horizontal direction of $\Gamma^{\mathbb{I}}$ (and positive definite in the vertical direction by construction).

Weak coupling

Fix a symplectic form ω on Σ (compatible with its orientation).

- We have constructed a 'coupling' closed (1,1)-form $\boldsymbol{\sigma}^{\mathbb{I}}$ on $\mathfrak{X}$ that restricts to $\widehat{\boldsymbol{\omega}}_{\boldsymbol{I}}$ on the fibres $\mathfrak{X}_J \cong \mathcal{D}$ of $\mathfrak{X} = \mathcal{J} \times \mathcal{D} \to \mathcal{J}$.
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In the weak-coupling approach, one gets a closed 2-form on $\mathfrak{X}$ as follows:

- Fix a parameter $\varepsilon \in \{-1, 1\}$ and a real coupling constant $\alpha > 0$.
- The corresponding 'minimally coupled' closed $(1,1)$-form on $(\mathfrak{X}, \mathbb{I})$ is

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Signature and non-degeneracy properties of $\boldsymbol{\omega}^{\mathbb{J}}_{\alpha,\varepsilon}$:

- For $\varepsilon = -1$, the symmetric tensor $\boldsymbol{g}^{\mathbb{I}}_{\alpha,-1} = \boldsymbol{\sigma}^{\mathbb{I}}_{\alpha,-1}(-, \mathbb{I}-)$ is negative definite in the horizontal direction $H^{\mathbb{I}}$ (and positive definite in the vertical direction by construction), so $\boldsymbol{\omega}^{\mathbb{I}}_{\alpha,-1}$ is a (non-degenerate) symplectic form on $\mathfrak{X}$.
- For $\varepsilon = +1$ and $\psi \neq 0$, the symmetric tensor $\boldsymbol{g}^{\mathbb{I}}_{\alpha,+1} = \boldsymbol{\sigma}^{\mathbb{I}}_{\alpha,+1}(-, \mathbb{I}-)$ changes sign along the lines $\{(J, A + \mathrm{i}\lambda\psi) \mid \lambda \in \mathbb{R}\} \subset \mathfrak{X}$, becoming null at $H^{\mathbb{J}}_{(J, A+\mathrm{i}\lambda_0\psi)}$ for a specific value $\lambda_0 \in \mathbb{R}$.

The coupled Hitchin equations

Recall we have a group extension of the group $\mathcal{H}$ of Hamiltonian symplectomorphisms on (Σ, ω) by the gauge group $\mathcal{K}$ of E_K.

$$1 \to \mathcal{K} \longrightarrow \widetilde{\mathcal{K}} \longrightarrow \mathcal{H} \to 1$$

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The *Hamiltonian extended gauge group* $\widetilde{\mathcal{G}}$ has an action on $\mathfrak{X}$ that preserves $\mathbb{I}$ and $\boldsymbol{\omega}^{\mathbb{I}}_{\alpha,\varepsilon}$, and the universal space of Higgs fields $\mathfrak{X}^{\mathrm{Higgs}} \subset \mathfrak{X}$.

Proposition/Definition (AC-GF-GP-T 2025)

The $\widetilde{\mathcal{K}}$-action on $(\mathfrak{X}^{\mathrm{Higgs}}, \boldsymbol{\omega}^{\mathbb{I}}_{\alpha,\varepsilon})$ is Hamiltonian, with $\widetilde{\mathcal{K}}$-equivariant moment map $\mu^{\mathbb{I}}_{\widetilde{\mathcal{K}}} \colon \mathfrak{X} \to (\operatorname{Lie} \widetilde{\mathcal{K}})^*$ whose zero locus consists of the triples $(J, D = A + \mathrm{i}\psi) \in \mathfrak{X}$ that solve the **coupled Hitchin equations**

$$F_A - [\varphi, \tau(\varphi)] = 0, \quad \bar{\partial}_{J,A}\varphi = 0,$$
$$S_g = \frac{2\pi\chi(\Sigma)}{V},$$

where $\varphi := \psi^{1,0_J} \in \Omega^{1,0}(\Sigma, E_G(\mathfrak{g}))$ is the Higgs field determined by (J, A, ψ), S_g is the scalar curvature of $g = \omega(-, J-)$ and $V = \int_\Sigma \omega$.

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Remark. The $\widetilde{\mathcal{K}}$-action on $(\mathfrak{X}, \boldsymbol{\omega}^{\mathbb{I}}_{\alpha,\varepsilon})$ is also Hamiltonian, but in this case we loose holomorphicity of φ in the vanishing locus of the moment map.

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The **universal Hitchin moduli space** is the pre-symplectic quotient

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Ignoring singularities, it is presymplectic (symplectic for $\varepsilon = -1$).
It fibres over the Teichmüller space $\mathcal{T}$ via the obvious map

$$\mathcal{U}^{\mathrm{Hit}}_{\alpha,\varepsilon}(G) \longrightarrow \mathcal{U}^{\mathrm{Higgs}}(G) \longrightarrow \mathcal{T}.$$

The complex structure of universal Hitchin moduli space

As for $\mathcal{U}^{\mathrm{Harm}}_{\alpha,\varepsilon}(G)$, the symmetric tensor $\boldsymbol{g}^{\mathbb{I}}_{\alpha,\varepsilon} = \boldsymbol{\sigma}^{\mathbb{I}}_{\alpha,\varepsilon}(-, \mathbb{I}-)$ is not positive definite, so it is not clear *a priori* whether $\mathbb{I}$ induces a complex structure on $\mathcal{U}^{\mathrm{Hit}}_{\alpha,\varepsilon}(G)$.

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Theorem (AC-GF-GP-T 2025)

Fix volume $V > 0$, $\varepsilon \in \{-1, 1\}$ and $(J, D = A + \mathrm{i}\psi) \in \mathfrak{X}^{\mathrm{Higgs}}$ corresponding to a **stable** Higgs bundle. Suppose genus $g(\Sigma) \geq 2$.

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Then $\exists\, \alpha_0$ such that $\forall\; 0 < \alpha < \alpha_0$, the $\widetilde{\mathcal{G}}$-orbit of $(J, D = A + \mathrm{i}\psi)$ in $\mathfrak{X}^{\mathrm{Higgs}}$ contains a $\widetilde{\mathcal{K}}$-orbit $\mathcal{O}$ of solutions to the coupled Hitchin equations,

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Furthermore,

- the composite $\mathcal{U}_* \hookrightarrow \mathcal{U}^{\mathrm{Hit}}_{\alpha,\varepsilon}(G) \to \mathcal{U}^{\mathrm{Higgs}}(G) \to \mathcal{T}$ is holomorphic,
- for $\varepsilon = -1$, $\mathrm{g}^{\mathbb{I}}_{\alpha,\varepsilon}$ is non-degenerate, and defines a pseudo-Kähler structure on the moduli space,
- for $\varepsilon = 1$, $\mathrm{g}^{\mathbb{I}}_{\alpha,\varepsilon}$ is possibly degenerate.

Thank you!