Outliers in weakly confined Coulomb-type systems

Topics in High Dimensional Probability

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joint work with

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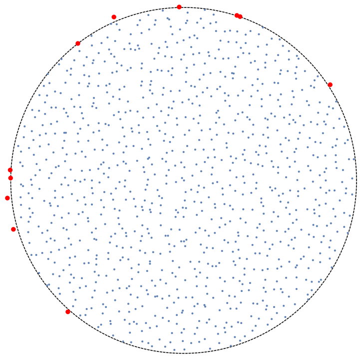
General picture

Outliers in planar particle systems

- Particle system ($\equiv$point process) in the plane.
- Number of particles goes to infinity.
- Most 'stay' on the droplet, some are 'outliers'



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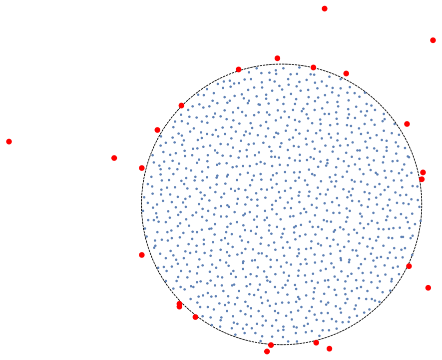

Plan of the talk

- Describe two models of weakly confined particle systems
  - Determinantal Coulomb gas with ‘weak’ external field
  - Zeros of random linear combinations of orthogonal polynomials
- Limiting point process of the outliers
  - Bergman process - conformal invariance
  - More complicated for domains which are multiply-connected
- Some ideas from the proof

Coulomb gas (or Jellium)

Planar Coulomb gas (or Jellium)

- System of N particles with charge -1 in a background 'electric' field $\kappa_N V$ of positive charge.
- (2D Coulombic) repulsion between electrons at z_1, z_2:

$$-\log |z_1 - z_2|$$

- Force of electric field at z:

$$\kappa_N V(z)$$

- Energy/Hamiltonian for N particles at $z_1, \ldots, z_N$:

$$H_N(z_1, \ldots, z_N) = \sum_{j<\ell} -\log |z_j - z_\ell| + \kappa_N \sum_{\ell} V(z_\ell)$$

Planar Coulomb gas (or Jellium) - cont.

Definition

Coulomb gas with N particles, with field V at (inverse) temperature $\beta > 0$ is a random vector in $\mathbb{C}^N$ whose density is proportional to

$$\exp\left(-\beta H_N\left(z_1, \ldots, z_N\right)\right)$$

with respect to area measure $\mathrm{d}\lambda_{\mathbb{C}^N}$.

- That is, density is

$$\frac{1}{Z_N} \prod_{j<\ell} \left|z_j - z_\ell\right|^\beta e^{-\beta \kappa_N \sum_\ell V(z_\ell)} \mathrm{d}\lambda\left(z_1\right) \ldots \mathrm{d}\lambda\left(z_N\right),$$

  where $Z_N = Z_N\left(\beta, V, \kappa_N\right)$ is a normalizing constant.

The 'Jellium'

- A canonical example is the Ginibre ensemble with

$$\kappa_N = N, \qquad V(z) = |z|^2, \qquad \beta = 2$$

corresponding to eigenvalues of random matrices with independent complex Gaussian coefficients.

  - Here we have 'strong' confinement

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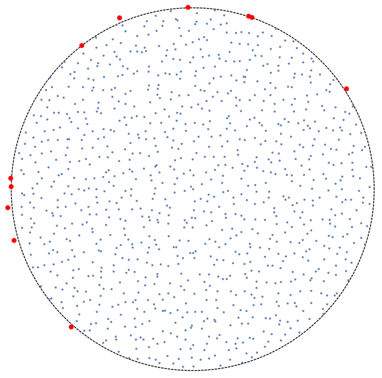

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- A canonical example is the Ginibre ensemble with

$$\kappa_N = N, \qquad V(z) = |z|^2, \qquad \beta = 2$$

  corresponding to eigenvalues of random matrices with independent complex Gaussian coefficients.
  - Here we have 'strong' confinement
- However, we will choose a 'weak' field V forcing us to take $\kappa_N > N$ (small charge imbalance) to make the system mathematically well-defined.

The Jellium - empirical measure and the droplet

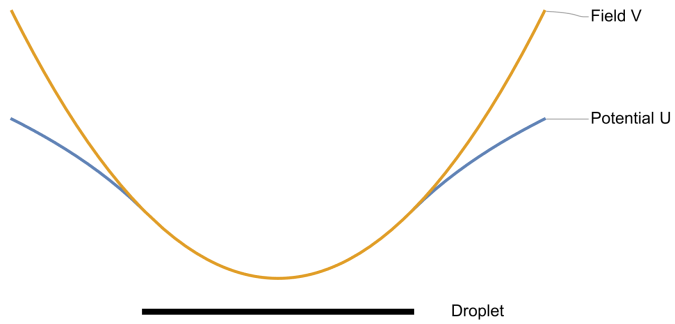
- Convergence to the limiting distribution:

$$\text{Empirical measure} \qquad \frac{1}{N}\sum_{\ell}\delta_{z_\ell} \xrightarrow[N\to\infty]{} \nu \qquad \text{limiting measure}$$

  - E.g. weak convergence in probability, under mild conditions

- Support of the limiting measure ν is called the droplet
  - A nice compact set under some mild conditions
  - On droplet $V - U^{\nu} \equiv$ const., where U^{ν} – log. potential

The Jellium - empirical measure and the droplet



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The Jellium - empirical measure and the droplet

- Support of the limiting measure ν is called the droplet
  - A nice compact set under some mild conditions
  - On droplet $V - U^{\nu} \equiv$ const., where U^{ν} – $\log.\ \text{potential}$

- Roughly speaking, weak confinement arises when the

$$\text{effective field} \qquad V - U^{\nu}$$

<u>does not</u> form a ‘well’ away from the droplet.

The Jellium in our model

- Determinantal $\equiv \beta = 2$
- To obtain weak confinement, our field

$$V(z) = U^{\mu}(z) = \int \log|z-w|\,\mathrm{d}\mu(w)$$

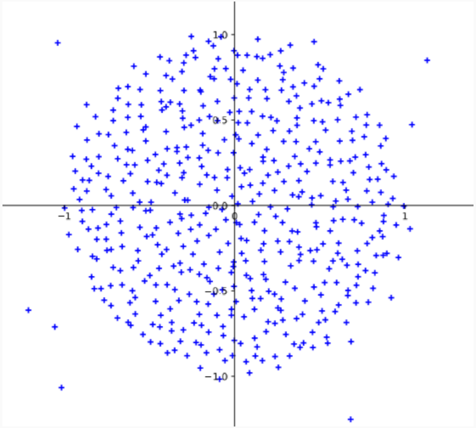
  is the logarithmic potential of a probability measure μ.
- In this case the limiting measure ν is equal to μ.
- In addition, κ_N is a sequence such that

$$N < \kappa_N \leq N+1 \qquad\qquad (\text{e.g. } \kappa_N = N + \alpha)$$

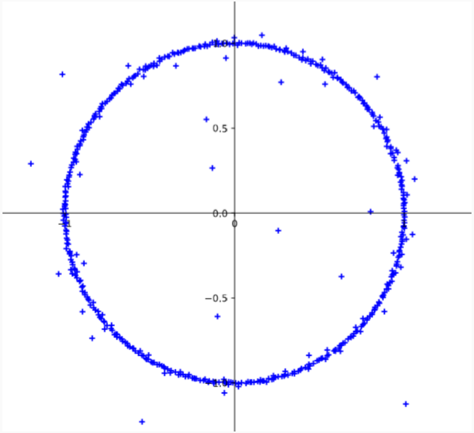
  - Larger κ_N: different limit process (García-Zelada '18) or strong confinement (Ameur '21, general β)

The Jellium - illustration

Figure 1: Jellium at $\beta = 2$



(a) μ - uniform on the disc



(b) μ - uniform on the circle

Outliers process

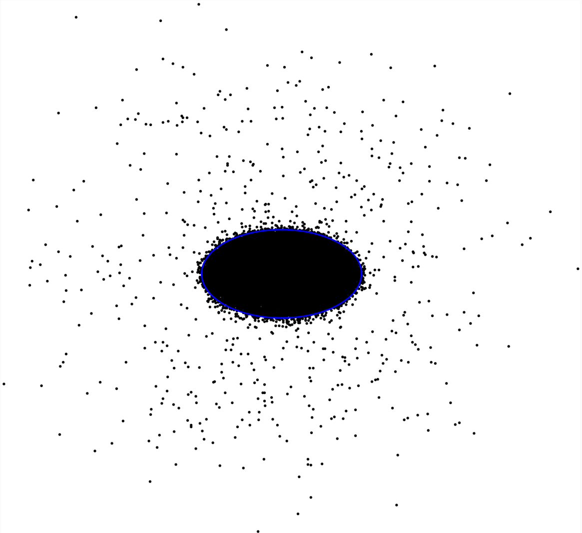
Questions about outliers

Weakly confined system with N particles:

1. Are there ‘many’ particles outside the droplet?
2. How many are there, depending on N?
3. Can we describe the outliers as a point process?
4. If we can, how does it depend on the measure μ?

Superposition of 100 simulations with $N = 200$

Figure 2: Probability measure: uniform on ellipse



Determinantal point processes (DPP)

- A simple point process $\mathcal{X}$ is determinantal with kernel K with respect to background measure m, if the density of the ℓ-th correlation function w.r.t. m is of the form

$$\rho_\ell(z_1, \ldots, z_\ell) = \det\left(K(z_i, z_j)_{i,j=1}^{\ell}\right)$$

- The kernel K and the measure m determine the distribution.
- The distribution of the number of points in a given set can be expressed as a sum of independent Bernoulli (0/1) random variables.

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- The kernel K and the measure m determine the distribution.
- The distribution of the number of points in a given set can be expressed as a sum of independent Bernoulli (0/1) random variables.
  - However, the parameters of the random variables are difficult to compute except in the case of a radial field/set.

Determinantal point processes - convergence

- **Convergence of point processes**: weak convergence of Radon measures - or the distribution of the number of points in compact sets.
- Generally speaking:

$$\text{Convergence of kernels} \implies \text{Convergence of DPPs}$$

- $\{\mathcal{X}_N\}$ is a sequences of DPPs with nice kernels $\{K_N\}$ with respect to the same background measure m.
- If the kernels K_N converge uniformly on compact sets to K_∞, then there exists a DPP $\mathcal{X}_\infty$ with that kernel such that

$$\mathcal{X}_N \xrightarrow[N\to\infty]{} \mathcal{X}_\infty \qquad \text{in distribution.}$$

Determinantal Jellium ($\beta = 2$)

- For N particles, and field $\kappa_N V$, the kernel has the following form

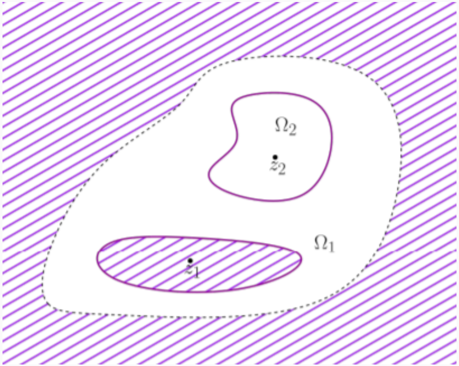
$$K_N(z, w) = \sum_{j=0}^{N-1} P_{k,N}(z) \overline{P_{k,N}(w)} e^{-\kappa_N (V(z)+V(w))}$$

  where $P_{k,N}$ are the orthogonal polynomials (of degree k) with respect to the inner product in the space $L^2\left(e^{-\kappa_N V}\right)$.
- The kernel is not unique, e.g. one can multiply by a phase $e^{i\widetilde{V}}$.
- We will denote the points of the Jellium by $\mathcal{X}_N$.

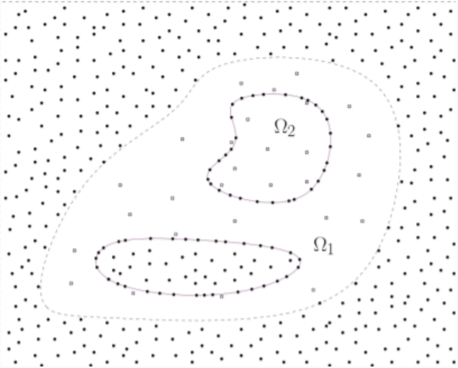
Nice measures

- Recall that we consider fields $V = U^\mu$ (μ prob. measure)

Figure 3: Admissible measures and the Jellium



(a) (Part of) admissible measure μ



(b) Illustration of the Jellium

Limiting process of outliers - simpler case

- Ω – A simply connected component of $\widehat{\mathbb{C}}\setminus\mathrm{supp}(\mu)$.

Theorem
μ – nice probability measure and $\kappa_N = N + 1$, then

$$\mathcal{X}_N \cap \Omega \xrightarrow[N\to\infty]{} \mathcal{B}_\Omega \qquad \textit{in distribution.}$$

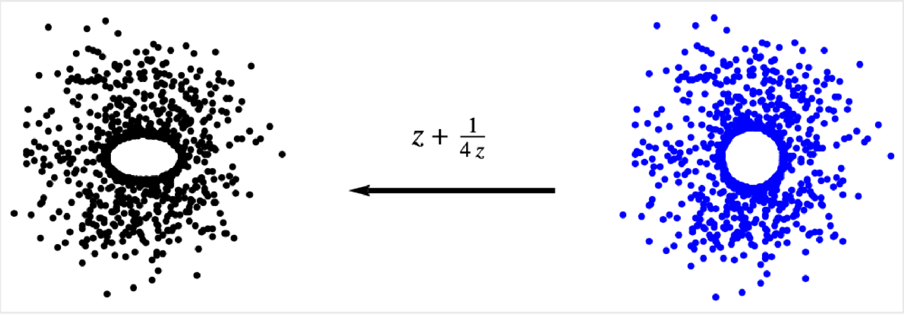
- $\mathcal{B}_\Omega$ – the (unweighted) Bergman point process on Ω.
  - DPP associated with the Bergman kernel of Ω (also denoted by $\mathcal{B}_\Omega$).
- Radial case with no regularity assumptions was proved by Butez and García-Zelada.

(unweighted) Bergman process

- Bergman kernel is the reproducing kernel for $A^2(\Omega)$ - Hilbert space of $L^2(\Omega)$ analytic functions.
- In general, if $\{\phi_\ell\}$ is any orthonormal basis for $A^2(\Omega)$ then
$$\mathcal{B}_\Omega(z,w) = \sum_\ell \phi_\ell(z)\overline{\phi_\ell(w)}, \qquad z,w \in \Omega$$
- For the unit disc $\mathbb{D}$ the kernel is explicit (z^ℓ are orthogonal)
$$\mathcal{B}_\mathbb{D}(z,w) = \frac{1}{\pi}(1 - z\overline{w})^{-2}$$
- If $\varphi : \Omega \to \mathbb{D}$ is a conformal map, then
$$\mathcal{B}_\Omega(z,w) = \mathcal{B}_\mathbb{D}(\varphi(z), \varphi(w))\,\varphi'(z)\overline{\varphi'(w)}$$
- This implies the conformal invariance of the process
$$\varphi(\mathcal{B}_\Omega) = \mathcal{B}_\mathbb{D} \qquad \text{in distribution.}$$

(unweighted) Bergman process

- Bergman kernel is the reproducing kernel for



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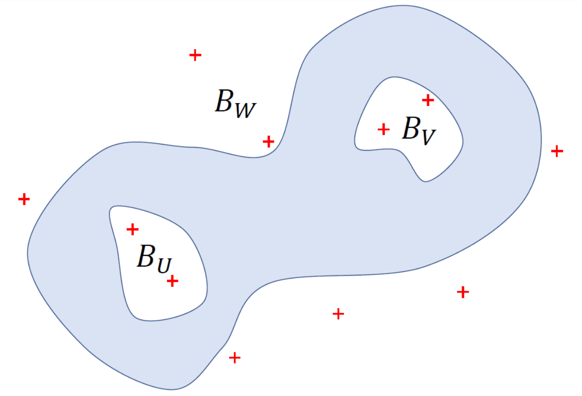
(unweighted) Bergman process - cont.



- The process $\mathcal{B}_\Omega$ also appears in zeros of Gaussian analytic functions (Peres and Virág '05)

Independence of limiting processes

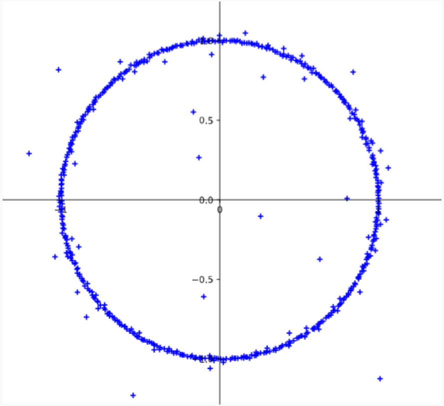
Theorem
μ – an admissible probability measure, $\Omega_1 \neq \Omega_2$ two simply connected components of $\mathbb{C}\backslash\mathrm{supp}\,(\mu)$ and $\mathcal{X}_N^1$, $\mathcal{X}_N^2$ the associated outlier processes. Then

$$\left(\mathcal{X}_N^1, \mathcal{X}_N^2\right) \xrightarrow[N\to\infty]{} \left(\mathcal{B}_{\Omega_1}, \mathcal{B}_{\Omega_2}\right) \qquad \textit{in distribution}$$

where $\mathcal{B}_{\Omega_1}, \mathcal{B}_{\Omega_2}$ are independent.

- The result holds also for multiply-connected Ω_1, Ω_2.
- This is the ‘screening’ phenomenon of Coulomb gas.

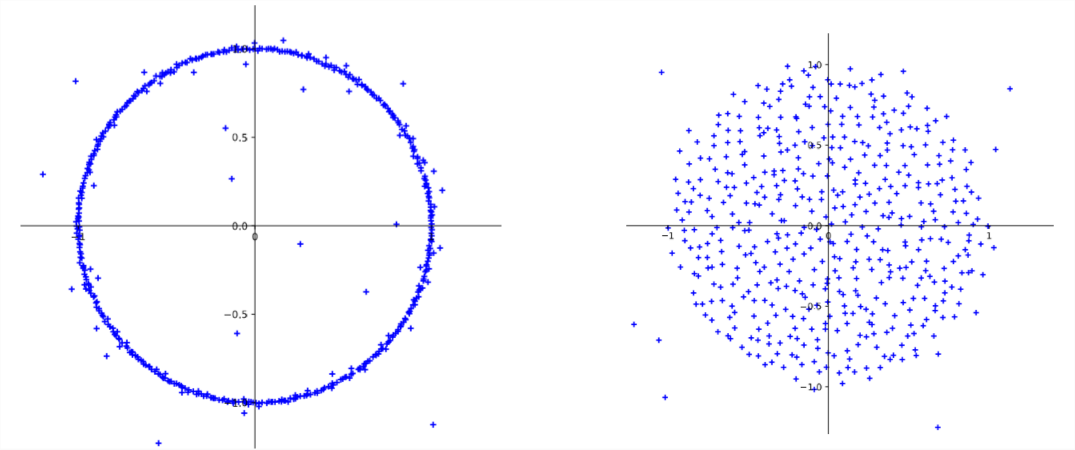
Example: Bergman process in the unit disk



- The outliers inside and outside the unit disk converge to <u>independent</u> Bergman point processes.

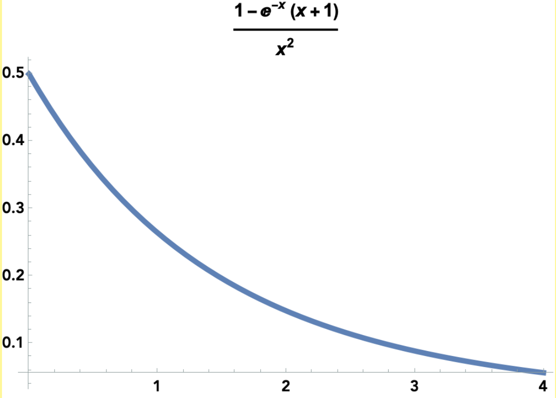
Some previous and related results

Figure 4: Jancovici '83



Some previous and related results

Jancovici was interested in profile of the boundary, for the circle he found:



Similar results near hard wall (Shirai '15, Seo '22)

Some previous and related results

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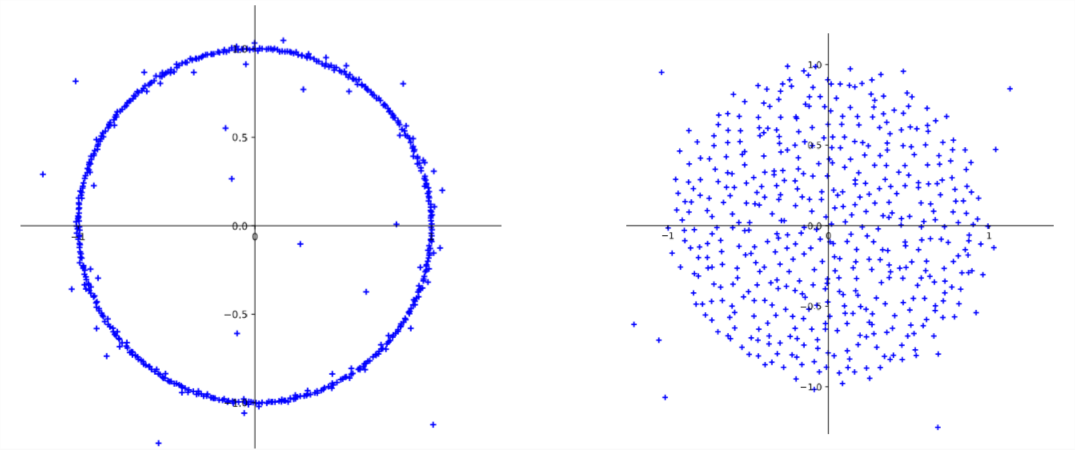
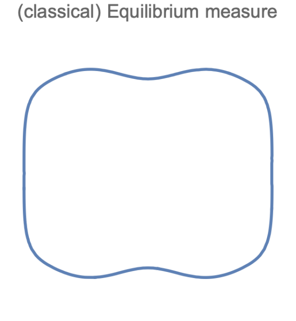


Some previous and related results - cont.

- Sinclair and Yattselev '12:
  - Equilibrium measure on a nice Jordan curve
  - Limiting Bergman outlier process
  - Limiting process near boundary

Figure 5: Sinclair and Yattselev '12



Expected number of outliers

- 'Most' of the points converge to the limiting measure μ, so the number of outliers in $\mathcal{X}_N$ is $o(N)$.
- Under some conditions we can estimate the expected number of outliers.

Theorem
μ – very nice probability measure, Ω – connected component of $\mathbb{C} \setminus \mathrm{supp}(\mu)$. Then

$$c\sqrt{N} \leq \mathbb{E}[\mathcal{X}_N \cap \Omega] \leq C\sqrt{N} \log N$$

with some numerical constants $c, C > 0$.

- A similar result also holds for strongly confining systems, but in that case all the particles are close to the droplet as $N \to \infty$ with high probability.

Zeros of random polynomials

Model of random polynomials

- We consider a model of random polynomials introduced by Zeitouni and Zelditch.
- μ is again a probability measure. We define an inner product

$$\langle f, g \rangle_N = \int_{\mathbb{C}} f(z) \overline{g(z)} e^{-2N \cdot U^{\mu}(z)} \mathrm{d}\mu(z)$$

- Let $\{Q_{\ell,N}\}_{\ell=0}^{N}$ be the orthonormal polynomials of degrees $0, \ldots, N$, and define

$$P_N(z) = \sum_{\ell} \xi_\ell Q_{\ell,N}(z)$$

  - $\{\xi_\ell\}$ i.i.d. standard complex Gaussians.

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E.g. with the choice μ - uniform probability measure on circle $\{|z|=1\}$, we get the Kac polynomials:

$$P_N(z) = \sum_{\ell=0}^{N} \xi_\ell z^\ell$$

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Model of random polynomials - cont.

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- The distribution of P_N depends only on the inner product.
- Denote by Ψ_N the zero process of P_N, it is known that it converges after normalization to the measure μ.

Outliers of random polynomials

- Ψ_N – zeros of the random polynomial

Theorem
μ – probability measure of the form

$$\mathrm{d}\mu = \rho\,\mathrm{d}\sigma_\Gamma$$

where Γ is a simple closed analytic curve, σ_Γ is the arc length measure, and $\rho > 0$ is a real-analytic density.

Ω_1, Ω_2 – the two components of $\mathbb{C} \backslash \Gamma$, then

$$(\Psi_N \cap \Omega_1, \Psi_N \cap \Omega_2) \xrightarrow[N\to\infty]{} (\mathcal{B}_{\Omega_1}, \mathcal{B}_{\Omega_2}) \qquad \textit{in distribution}$$

where the Bergman point processes $\mathcal{B}_{\Omega_1}, \mathcal{B}_{\Omega_2}$ are independent.

Outliers of random polynomials - cont.

- We have a result under a restrictive condition.
- Comparing with results in the radial case by Butez and García-Zelada it is likely that a similar result holds for more general measures.
- Note that Ψ_N is not a determinantal point process.

Weighted Bergman kernels

Result for multiply-connected domains

Theorem

μ – nice probability measure, Ω – connected component of $\widehat{\mathbb{C}}\setminus\mathrm{supp}\,(\mu)$ with ℓ holes. Fix arbitrary points $w_1, \dots, w_\ell$ in each of the holes, and let $q_1, \dots, q_\ell$ be the μ-masses of the closed holes.

Result for multiply-connected domains

Theorem

μ – nice probability measure, Ω – connected component of $\widehat{\mathbb{C}}\backslash\mathrm{supp}\,(\mu)$ with ℓ holes. Fix arbitrary points $w_1, \ldots, w_\ell$ in each of the holes, and let $q_1, \ldots, q_\ell$ be the μ-masses of the closed holes. Assume that along a subsequence $\mathcal{A} \subset \mathbb{N}$,

$$\left(e^{2\pi i \kappa_N q_1}, \ldots, e^{2\pi i \kappa_N q_\ell}\right) \xrightarrow[N\in\mathcal{A},\, N\to\infty]{} \left(e^{2\pi i Q_1}, \ldots, e^{2\pi i Q_\ell}\right)$$

for some $\mathbf{Q} = (Q_1, \ldots, Q_\ell) \in (\mathbb{R}\backslash\mathbb{Z})^\ell$

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for some $\mathbf{Q} = (Q_1, \ldots, Q_\ell) \in (\mathbb{R}\backslash\mathbb{Z})^\ell$, then

$$\mathcal{X}_N \cap \Omega \xrightarrow[N\in\mathcal{A},\, N\to\infty]{} \mathcal{B}_{\Omega,\mathbf{Q}} \qquad \text{in distribution.}$$

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$$\mathcal{X}_N \cap \Omega \xrightarrow[N \in \mathcal{A},\, N \to \infty]{} \mathcal{B}_{\Omega, \mathbf{Q}} \qquad \textit{in distribution.}$$

Here $\mathcal{B}_{\Omega, \mathbf{Q}}$ is the Bergman point process on Ω with weight

$$\omega_{\mathbf{Q}}(z) = \prod_{j=1}^{\ell} |z - w_j|^{-2Q_j}$$

Multiply-connected domains

- In general, the process of outliers does not converge on multiply-connected domains, but we can identify the subsequential limits.
- Weighted Bergman space $A^2(\mathcal{D}, \rho)$ – $\mathcal{D} \subset \mathbb{C}$ open set, $\rho > 0$ continuous weight function – consists of all analytic functions in $L^2(\mathcal{D}, \rho)$.
- It is a reproducing kernel Hilbert space. The kernel can be written as

$$K_{\mathcal{D},\rho}(z, w) = \sum_{\ell} \varphi_\ell(z)\overline{\varphi_\ell(w)} \qquad \{\varphi_\ell\} \text{ an orthonormal basis}$$

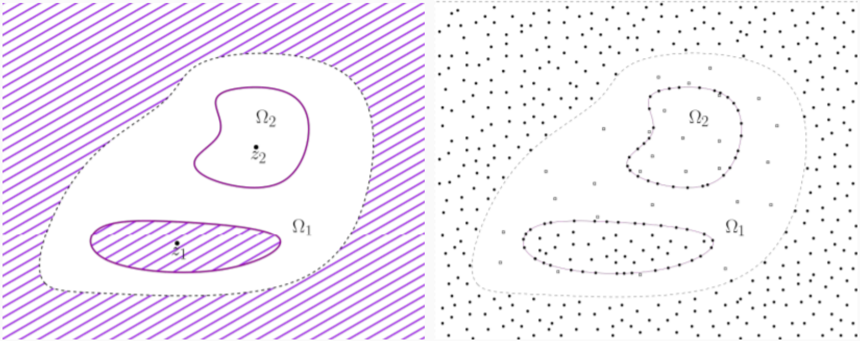
- The correlation kernel is given by

$$\mathcal{K}_{\mathcal{D},\rho}(z, w) = K_{\mathcal{D},\rho}(z, w)\sqrt{\rho(z)\rho(w)}$$

- $\mathcal{B}_{\Omega,\mathbf{Q}}$ is the DPP associated with the kernel $\mathcal{K}_{\Omega,\omega_{\mathbf{Q}}}$.

Multiply-connected domains - illustration

Figure 6: Admissible measures and the Jellium



(a) (Part of) admissible measure μ **(b)** Illustration of the Jellium

- Ω_2 is simply connected.
- Ω_1 is 2-connected, we fix a point in each hole.

Kernel convergence

Kernel convergence

Zeros of random polynomials

Convergence of the kernels - overview

- Want to show local uniform convergence of the kernels $\mathcal{K}_N(z,w)$ to $\mathcal{B}_\Omega(z,w)$.
- In one direction we have the crucial inequality

$$\mathcal{K}_N(z,z) \leq \mathcal{B}_{\Omega,\mathbf{Q}}(z,z) \qquad \forall z \in \Omega$$

- Together with analyticity of the kernels this gives precompactness of the family $\{\mathcal{K}_N\}$ and we have to identify the possible limits.
- For a lower bound we construct special orthonormal polynomials that approximate a suitable orthonormal basis of the Bergman space $A^2(\Omega, \omega_{\mathbf{Q}})$.

More on the Bergman point process

- Consider an open set $\mathcal{D} \subset \mathbb{C}$ and a weight function $\rho = e^{-2V}$.
- The Bergman point process $\mathcal{B}_{\mathcal{D},V}$ is the DPP associated with the kernel

$$\mathcal{K}_{\mathcal{D},e^{-2V}}(z,w) = K_{\mathcal{D},e^{-2V}}(z,w)\, e^{-V(z)-V(w)}$$

- If V is harmonic on $\mathcal{D}$ and $f \neq 0$ is analytic on $\mathcal{D}$, then

$$\mathcal{B}_{\mathcal{D},V} = \mathcal{B}_{\mathcal{D},V-\log|f|} \qquad \text{in distribution.}$$

- This is also true in the other direction.
- We show that weights $\omega_{\mathbf{Q}}(z) = \prod_{j=1}^{\ell} |z - w_j|^{-2Q_j}$ correspond to different processes (for different $\mathbf{Q}$ values in $(\mathbb{R}\backslash\mathbb{Z})^{\ell}$).

Upper bound for the kernels

- Consider the unweighted case $\mathbf{Q} = \mathbf{0}$, Ω simply connected.
- Can modify the kernel $\mathcal{K}_N$ to make it analytic in z and $\overline{w}$.
  - This is also true for $\mathcal{B}_\Omega$.
  - Here we use the fact U^μ is harmonic on Ω.
- Reproducing kernel property together with Montel's theorem (in several complex variables) show that the bound

$$\mathcal{K}_N(z, z) \leq \mathcal{B}_\Omega(z, z) \qquad z \in \Omega$$

  gives that $\{\mathcal{K}_N\}$ is precompact.
- Upper bound is proved by comparing the extremal characterization of the kernel.

Complications of the multiply-connected case

- Want to modify the kernel $\mathcal{K}_N$ to make it analytic in z and $\overline{w}$.
- The potential U^μ is still harmonic on Ω.
- The *logarithmic conjugation theorem* tells us that there is an analytic function V_0 in Ω such that

$$U^\mu(z) = \mathbf{q} \cdot \mathrm{Log}(z) + \mathrm{Re}\, V_0(z)$$

where

$$\mathrm{Log}(\cdot) = (\log|\cdot - w_1|, \ldots, \log|\cdot - w_\ell|)$$

- This leads to comparisons with the weighted kernels $\mathcal{B}_{\Omega,\{\kappa_N \mathbf{q}\}}$.

Kernel convergence

- Consider unweighted case $\mathbf{Q} = \mathbf{0}$, Ω simply connected.
- Idea: find suitable ONB $\{\psi_\ell\}$ for $A^2(\Omega)$ and orthonormal set of polynomials $\{P_{\ell,N}\}$ in $L^2\left(\mathbb{C}, e^{-2\kappa_N U^\mu}\right)$ of degrees $\leq N-1$ such that

$$|P_{\ell,N}|^2 e^{2\kappa_N U^\mu} \xrightarrow[N\to\infty]{} |\psi_\ell|, \qquad \ell \leq \ell_0$$

- Then

$$\sum_{\ell=0}^{\ell_0} |\psi_\ell(z)|^2 \leq \liminf_{N\to\infty} \sum_{\ell=0}^{N-1} |P_{\ell,N}(z)|^2 e^{2\kappa_N U^\mu(z)} = \liminf_{N\to\infty} \mathcal{K}_N(z,z)$$

- Polynomials are <u>not</u> necessarily the OP w.r.t. weight function.

Construction of the polynomials

- We use a method based on $\overline{\partial}$ Hörmander estimates, developed by Hedenmalm and Wennman. In the following steps:

1. Take a suitable ONB $\{\psi_\ell\}$ for $A^2(\Omega)$ (more complicated in weighted case).
2. In a domain $\Omega' \supset \Omega$, define

$$F_{\ell,N}(z) = \chi(z)\,\psi_\ell(z)\,e^{\kappa_N V_0(z)},$$

   where χ is a cutoff function, equal to 1 on Ω, 0 outside Ω'.
3. $F_{\ell,N}$ are almost orthogonal in $L^2\left(e^{-2\kappa_N U^\mu}\right)$, but not polynomials, in fact not even analytic.
4. Use $\overline{\partial}$ method to construct analytic $G_{\ell,N}$ close to $F_{\ell,N}$ in L^2 norm, conclude that it is a polynomial (Liouville's theorem).

$\overline{\partial}$ Hörmander estimates

- We use the following version of the Hörmander estimate:

Theorem
Let ϕ be a nice subharmonic function on $\mathbb{C}$, and $\mathcal{D} \subset \mathbb{C}$ is a compact set where ϕ is strictly subharmonic. For any bounded f supported on $\mathcal{D}$, there is a solution $g : \mathbb{C} \to \mathbb{C}$ to the equation

$$\overline{\partial} g = f$$

that satisfies

$$\int_{\mathbb{C}} |g|^2 e^{-\phi} \leq 2 \int_{\mathcal{D}} |f|^2 \frac{e^{-\phi}}{\Delta \phi}$$

- We apply this theorem with $\phi = 2\kappa_N U^{\mu}$ and $f = \overline{\partial} F_{\ell,N}$.
- The use of this method requires some strong regularity assumptions on the measure μ.

Zeros of random polynomials

Kernel convergence

Zeros of random polynomials

Covariance kernel

- Zeros of random polynomials not a DPP – use another approach.
- Work directly with the covariance kernel of the polynomial, show that it converges to the correct limit (Szegő kernel).
- Since $P_N(z) = \sum_\ell \xi_\ell Q_{\ell,N}(z)$, the covariance kernel is given by

$$C_N(z,w) = \mathbb{E}\left[P_N(z)\overline{P_N(w)}\right] = \sum_\ell Q_{\ell,N}(z)\overline{Q_{\ell,N}(w)}$$

- May multiply it by any non-vanishing function – this does not change the zero set of the Gaussian process.
- To prove convergence we consider the normalized kernel

$$\widehat{C}_N(z,w) = C_N(z,w)\, e^{-N\left(V_0(z)+\overline{V_0(w)}\right)}$$

Szegő kernel

- Peres and Virág – if covariance kernel of Gaussian analytic function is given by Szegő kernel, then zero process of that function is the Bergman point process.
- E.g., for the unit disk $\mathbb{D}$ the Szegő kernel is
$$\frac{1}{2\pi}(1 - z\overline{w})^{-1} \qquad \{z^\ell\} \text{ ONB for Hardy space } H^2(\mathbb{D})$$
- Our case: consider Szegő kernels corresponding to weighted Hardy spaces.
- This is essentially why we can prove our results only in the case the measure μ is supported on a curve.

Summary - The end

- Weakly confined particle systems
  - with many outliers far from the droplet
- Limiting point process of the outliers exists
  - Universal – determinantal Bergman point process
  - Conformal invariant

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Thank you for listening!