Uniform Stability of High-Rank Lattices

Alex Lubotzky
Weizmann Institute and Hebrew University

Joint work with:
Lev Glebsky, Nicolas Monod and Bharat Rangarajan

§1. Rigidity and Stability

- G - (semi) simple Lie group over a local field K.
- $\Gamma \leq G$ - a lattice (discrete subgroup of finite covolume; uniform (=cocompact) or non-uniform).
- $rk(G) = rank_K(G)$
- "usually" a difference between $rk = 1$ and $rk \geq 2$ (= high rank),

  e.g., local rigidity, strong rigidity, super-rigidity, congruence subgroup problem.

$$SL_2(\mathbb{Z}) \text{ versus } SL_n(\mathbb{Z}), \ \geq 3$$

This talk is about Ulam Stability which is another such property.

A typical rigidity result: There is a clear, easy to understand, family of representations of Γ. The theorem says that if something is "similar" it is already in the family.

Ulam stability is such a statement for "**almost representations**"

More formally:

Let Γ be a group and $\mathfrak{g} = (G_n, d_n)$-family of groups with

$d_n =$ bi invariant metric.

Def: Γ is uniform/Ulam stable w.r.t. $\mathfrak{g}$, if $\forall \varepsilon > 0, \exists \delta > 0$, such that

$\forall n,\ \forall$map $\varphi : \Gamma \to G_n$ with

$$d_n\big(\varphi(gh), \varphi(g)\varphi(h)\big) \leq \delta,\ \forall g, h \in \Gamma \tag{$*$}$$

$\exists$ a homomorphism $\psi : \Gamma \to G_n$

s.t.

$$d_n\big(\varphi(g), \psi(g)\big) \leq \varepsilon,\ \forall g \in \Gamma \tag{$**$}$$

i.e., every "almost representation" is just a small deformation of a true representation.

Warning: Do not confuse with (ordinary) **stability** which is equivalent to:

If $\varphi_n : \Gamma \to G_n$ s.t. $\forall g, h \in \Gamma \;\; d_n(\varphi_n(gh), \varphi_n(g)\varphi_n(h)) \underset{n\to\infty}{\to} 0$

then $\exists \psi_n : \Gamma \to G_n$ homomorphisms with

$$d_n(\varphi_n(g),\ \psi_n(g)) \underset{n\to\infty}{\to} 0 \quad \forall g \in \Gamma$$

Uniform Stability is equivalent to

If $\varphi_n : \Gamma \to G_n$ s.t. $\sup_{g,h\in\Gamma} d_n(\varphi_n(g,h), \varphi_n(g)\varphi_n(h)) \underset{n\to\infty}{\to} 0$

then $\exists \psi_n : \Gamma \to G_n$ homomorphisms with

$$\sup_g d_n(\varphi_n(g), \psi_n(g)) \underset{n\to\infty}{\to} 0$$

Today we will work only with $G_n = U(n)$ and d_n- metric induced by a **submultiplicative** norm $\|\cdot\|$ on $M_n(\mathbb{C})$, i.e. $\|AB\| \leq \|A\| \ \|B\|$ and $d_n(A, B) = \|A - B\|$.

Ex: **(a)** the operator norm $\|\cdot\|_\infty = \|\cdot\|_{op}$

(b) The Frobenius norm $= L^2$-norm

(c) The p-Schatten norm $(1 \leq p < \infty)$ $\|A\|_p = (tr|A|^p)^{1/p}$ when $|A| = \sqrt{A * A}$ (So **(b)** is the case $p = 2$ of **(c)**)

Non-example: The Hilbert-Schmidt norm

$$\|A\|_{HS} = (tr\frac{1}{n}|A|^2)^{1/2} = \frac{1}{\sqrt{n}}\|A\|_2$$

Main Results

Theorem (Glebsky-Lubotzky-Monod-Rangarajan)

$\mathfrak{g} = \big(U(n), d_n\big)$, d_n-submultiplicative, Γ- a lattice, G - a high rank simple Lie group over a local field K. Then Γ is Ulam-stable provided G satisfies condition $G(Q_1, Q_2)$.

Remark: $G(Q_1, Q_2)$ to be defined later, is satisfied by "most" simple groups. E.g., always if K is non-archimedian, **or** for $SL_d(\mathbb{R})$ if $d \geq 4$ and $SL_d(\mathbb{C})$, $d \geq 3$.

As of now: we do not know for $G = SL_3(\mathbb{R})$.

§. 2 History

Theorem (Kazhdan 1982)

Amenable groups are **strongly** Ulam stable w.r.t. the operator norm.

Strongly means even w.r.t. infinite dimensional Hilbert space.

Theorem (Burger-Ozawa-Thom 2013)

If Γ any discrete group containing a free non-abelian subgroup, then Γ is not strongly Ulam stable.

Pf. Free groups are **not** Ulam stable and induce the "almost rep". $\square$

Open problem: Does "strong Ulam stability" characterize amenability?

Theorem (Burger-Ozawa-Thom 2013)

If $H_b^2(\Gamma, \mathbb{R}) \to H^2(\Gamma, \mathbb{R})$ is not injective, then Γ does not have Ulam-Stability.

Cor: Lattices in rank one groups are not Ulam stable.

Theorem (Burger-Ozawa-Thom 2013)

$SL_d(\mathbb{Z})$ (or more generally $SL_d(\mathcal{O}_S)$) are Ulam stable for $d \geq 3$.

Remark: The proof used "bounded generation" of these groups.

According to a recent result of **Corvaja, Rapinchuk, Ren** and **Zannier**, cocompact lattices are never boundedly generated.

Recall: **(i)** $H_b^n(\Gamma, V) = 0 \; \forall n$ when Γ is amenable.

(ii) $H_b^n(\Gamma, V) = 0 \; \forall n$ when Γ is high-rank lattice and $V^\Gamma = \{0\}$.

- **Burger-Monod** for $n = 2$
- **Burger-Shalom** for $n = 2$, more general V, different pf
- **Monod** all n.

This led Monod already in his ICM talk (2000) to ask:

Is there a connection between Ulam stability and bounded cohomology?

Answer: Yes, but . . .

§3 Stability and 2nd cohomology

There has been quite a lot of progress on ordinary stability in recent years. A major technical tool to prove stability is the following ($\mathfrak{u} =$ ultrafilter on $\mathbb{N}$):

Theorem (de Chiffre, Glebsky, Lubotzky, Thom)

Let $\mathcal{L}_n = Lie\big(U(n)\big)$ with the norms as before and $\mathcal{L} = \prod_{\mathfrak{u}} \mathcal{L}_n$ the topological ultra product of $\mathcal{L}_n$. The "almost homomorphisms" φ_n (in the standard sense) define an action of Γ on $\mathcal{L}$. If $H^2(\Gamma, \mathcal{L}) = 0$ then $\{\varphi_n\}$ are close to homo's ψ_n. In particular, if we work for example with the Frobenius norm and $H^2(\Gamma, V) = 0$ for every Hilbert space, then Γ is Frobenius stable.

What does H^2 have to do with stability?
The maps $\varphi_n : \Gamma \to U(n)$ give rise to true homo: φ^*:

$$\begin{array}{ccc} & & \Pi U(n) \\ & \overset{\psi}{\nearrow} & \downarrow \\ \Gamma & \xrightarrow[\varphi^*]{} & \Pi U(n)/inf \end{array}$$

Γ is stable iff this φ^* can be lifted to $homomorphism\ \psi : \Gamma \to \Pi U(n)$.

The kernel K is a “very” non-commutative group, but if **the norm is submultiplicative** K can be approximated by abelian small steps.

Vanishing of H^2 gives small extensions and the limit gives ψ.

The same strategy can, in principle, work for the uniform stability. BUT

(I) The relevant cohomology here is the bounded cohomology!

(II) We need ψ to be **internal** i.e., $\psi = (\psi_n)_{n\in\mathbb{N}}$.

(In the ordinary stability we care about the values of ψ only on the generators; every ψ is internal).

Putting points **(I)** and **(II)** together gives:

Proposition

The group Γ is Ulam-$\mathfrak{g}$-stable if and only if every homomorphism φ **that has an internal lift**

$$
\begin{array}{ccc}
 & & \Pi U(n) \\
 & \overset{\psi}{\nearrow} & \downarrow \\
{}^*\Gamma & \xrightarrow[\varphi]{} & \Pi U(n)/inf
\end{array}
$$

has an **internal** lift homomorphism ψ.

What captures this lifting problem, when $\| \quad \|$ is **submultiplicative**, is a new cohomology theory, which we call the asymptotic cohomology $H_a^n(\Gamma, \mathcal{L})$ of Γ (really of ${}^*\Gamma$) which deals only with **internal** cochains.

Main technical theorem:

Theorem (Glebsky, Lubotzky, Monod, Ramanujan)

If $H^2_a(\Gamma,\mathcal{L})=0$ (w.r.t. to φ obtained from almost homo in the uniform sense - φ_n) then $\{\varphi_n\}$ are near true homo's $\{\psi_n\}$.

Remarks:

1) There is a canonical map

$$H^n_a(\Gamma,\mathcal{L}) \longrightarrow H^n_b(\Gamma,\mathcal{L})$$

but we do not know if this is injective(?) surjective(?)

2) $H^n_a(\Gamma,V)=0\ \forall n,\ \forall V$ if Γ is amenable, so we recover Kazhdan Thm.

Our goal: Prove $H_a^2(\Gamma, \mathcal{L}) = 0$ when Γ is high-rank lattice

Step one: Shapiro Induction

Theorem

If Γ high-rank lattice in G, then $\forall$ dual Banach space V with Γ action:

$$H_b^2(\Gamma, V) = H_b^2(G, W)$$

where $W = Ind_\Gamma^G(V)$.

By **a lot** of work, we can imitate an Eckman-Shapiro approach to get

$$H_a^2(\Gamma, \mathcal{L}) = H_a^2(G, \mathcal{W})$$

for $\mathcal{W} = Ind_\Gamma^G(\mathcal{L})$.

Why a lot of work? Even a standard result like Shapiro Lemma requires for ${}^*\Gamma$ in ${}^*G$ (e.g. what is $L^\infty({}^*\Gamma \setminus {}^*G)$? what is $L^2({}^*\Gamma \setminus {}^*G)$?).

Step two: Reducing to trivial module

In the Monod-Shalom proof, they obtain

Theorem

For dual separable Banach space W with G action:

$$H_b^2(G, W) = H_b^2(G, W^G)$$

In particular, if $V^\Gamma = W^G = 0$, then $H_b^2(G, W) = 0$.

A similar conclusion holds in our setting too:

Theorem

Suppose $\mathcal{W}$ has no fixed point "upto infinitesimals". Then

$$H_a^2(G, \mathcal{W}) = 0$$

Step three: Dealing with Fixed Points

We can do it for all lattices Γ is a simple Lie group G if G satisfies $G(Q_1, Q_2)$.

$G(Q_1, Q_2)$ **means:** G has two parabolic subgroups Q_1 and Q_2 satisfying:

(i) $Q_1 \cap Q_2$ contains a minimal parabolic P, and $\langle Q_1, Q_2 \rangle = G$

(ii) $H_b^2(Q_i, \mathbb{R}) = 0$ and
$H_b^3(Q_i, \mathbb{R})$ is Hausdorff.

(Note: conditions on the bounded cohomology with trivial coefficients).

"Most" simple groups satisfy $G(Q_1, Q_2)$, so our result, as of now, is quite general but still not complete.