LECTURE 3

Model Independent Approaches to Cosmological Tests of Gravity

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ICTS School on Physics of the Early Universe

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Beyond LCDM

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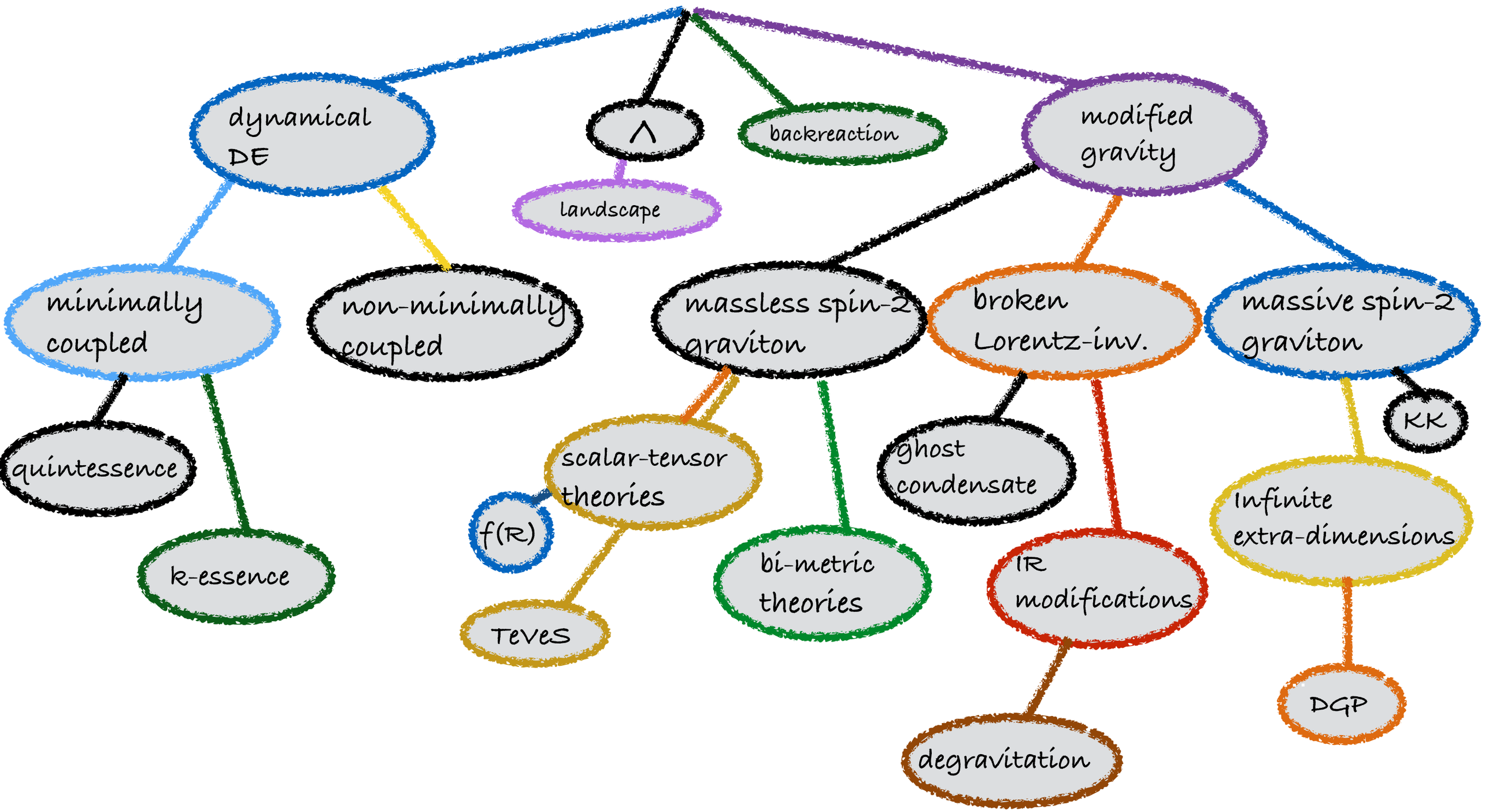
How can we optimally explore departures from the LCDM scenario?

The theoretical landscape



The theoretical landscape

PARAMETRIZED FRAMEWORKS

Parametrized Approaches

What to parametrize?

solutions of equations of motion → (μ, γ)
and equivalent choices

the action → EFT

Trigger parameters → $f = \dfrac{d \ln \delta}{d \ln a} \equiv \Omega_m(a)^\gamma$

Parametrized Approaches

What to parametrize?

solutions of equations of motion → (μ, γ) phenomenological

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Trigger parameters → $f = \dfrac{d\ln\delta}{d\ln a} \equiv \Omega_m(a)^\gamma$ observational

Trigger parameters

Trigger parameters

We could determine observationally some "trigger parameters" designed to detect a breakdown of the cosmological standard model. They are very specific combinations of observables that can be directly calculated from data; no need of evolving equations. They have a well known value in LCDM.

Any disagreement between the observed trigger parameter and its LCDM value would indicate some sort of modification of growth. Yet, their value does not necessarily have a physical meaning in any theory and they do not offer a consistent framework to make model-independent predictions for any observable

Trigger parameters: Growth Index

Linder's γ:

Linder & Cahn, Astropart.Phys.28 (2007)

Wang & Steinhardt, Astrophys. J. 508, (1998)

$$f = \frac{d\ln\delta}{d\ln a} \equiv \Omega_m(a)^\gamma$$

The idea behind this parametrization is that of capturing independently in Ω_m and γ the information from, respectively, the expansion and the growth history.

While for models of modified gravity and clustering dark energy in general γ will be a function of time and scale, in several cases for the regime of interest it can still be safely approximated by a constant, which can differ significantly from the ΛCDM value.

Trigger parameters: Growth Index

Working with the function $$G \equiv \frac{d\ln(\delta/a)}{d\ln a} = f - 1$$

The equation for the growth of structure can be written as a first order, non-linear diff. equation for the growth factor:

$$\frac{dG}{d\ln a} + \left(4 + \frac{1}{2}\frac{d\ln H^2}{d\ln a}\right) G + G^2 + 3 + \frac{1}{2}\frac{d\ln H^2}{d\ln a} - \frac{3}{2}\Omega_m(a) = 0$$

Assuming the quadratic term is negligible, we get the following solution by quadratures:

$$G(a) = -1 + \left(a^4 H(a)\right)^{-1} \int_0^a \frac{da'}{a'} a'^4 H(a') \cdot \left[1 + \frac{3}{2}\Omega_m(a')\right]$$

for a wCDM background $$\frac{H^2}{H_0^2} = E_m \left[1 + \frac{\Omega_w(a)}{\Omega_m(a)}\right]$$

we get $$G(a) \simeq -1 + \Omega_m^\gamma(a)$$ with $$\gamma \simeq \frac{6}{11} + 0.05(1 + w(z=1))$$

$$(\mu, \gamma)$$

The framework: (μ,γ)

$$ds^2 = -a^2(\tau)\left[(1+2\Psi(\tau,\vec{x}))\,d\tau^2 - (1-2\Phi(\tau,\vec{x}))\,d\vec{x}^2\right]$$

Continuity and Euler eqs.

$$\nabla_\mu T^{\mu\nu} = 0$$

$$\delta' + \frac{k}{aH}v - 3\Phi' = 0$$

$$v' + v - \frac{k}{aH}\Psi = 0$$

Einstein eqs.

Poisson:

$$k^2\Psi = -\mu(a,k)\frac{a^2}{2M_P^2}\rho\Delta$$

anisotropy:

$$\frac{\Phi}{\Psi} = \gamma(a,k)$$

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$$k^2\Psi = -\mu(a,k)\frac{a^2}{2M_P^2}\left\{\rho\Delta + 3\left(\rho+P\right)\sigma\right\}$$

anisotropy:

$$k^2\left[\Phi - \gamma(a,k)\Psi\right] = \mu(a,k)\frac{3a^2}{2M_P^2}\left(\rho+P\right)\sigma$$

Hojjati, Pogosian, Zhao, JCAP 1108:005 (2011)

The framework: (μ,γ)

* This is a consistent set of equations for the evolution of perturbations that can be incorporated into std Boltzmann codes, like CAMB.

* Solutions of linear cosmological perturbations in any particular theory can be expressed in terms of μ and γ; moreover, on sub-horizon scales they can have particularly simple forms.

* Everything that observations can tell us about the growth of structure can be stored as a measurement of μ and γ (and projected onto solutions of specific models if needed).

* They allow us to perform consistency tests of GR *as well as* exploring allowed parameter space of alternative models.

Alternative functions

lensing: $k^2\left(\Phi+\Psi\right)=-\Sigma(a,k)\frac{a^2}{M_P^2}\rho\Delta$ $\qquad (\mu,\gamma)\rightarrow(\mu,\Sigma)$

$$\mu \leftrightarrow G_{\rm matter}$$

$$\Sigma \leftrightarrow G_{\rm light}$$

$$\gamma \leftrightarrow \varpi = \frac{1}{\gamma}-1$$

$$\gamma \leftrightarrow \eta$$

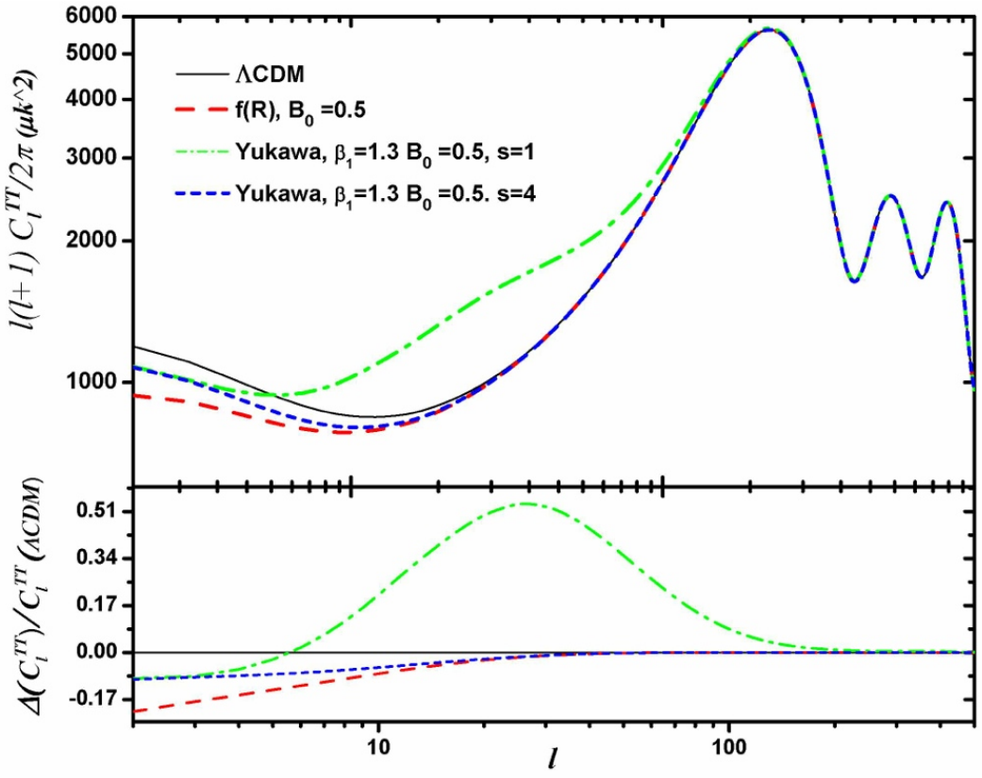
many references ...

Pogosian et al., Phys.Rev. D81 (2010)

MGCAMB

http://www.sfu.ca/~aha25/MGCAMB.html

Introduced in 2008 as a patch to the publicly available Boltzmann-Einstein solver CAMB to evolve linear scalar perturbations in a consistent parametrized framework and perform cosmological tests of gravity



'Searching for modified growth patterns with tomographic surveys'
Phys. Rev. D 79, 083513 (2009)
Zhao, Pogosian, Silvestri, Zylberberg

'Testing gravity with CAMB and CosmoMC'
JCAP 1108:005 (2011)
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Choices for (μ,γ)

What to do with μ and γ themselves?

- pick a specific functional form

$$\mu = \mu_0 \frac{\Omega_\Lambda(a)}{\Omega_\Lambda}$$

CFHTLenS:F. Simpson et al., arXiv: 1212.3339

$$\mu = \mu_0 + \frac{1-\mu_0}{2}\left(1 + \tanh\frac{z-z_s}{\Delta z}\right)$$

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  - QSA:

$$\mu = \frac{1+\beta_1\lambda_1^2 k^2 a^s}{1+\lambda_1^2 k^2 a^s}$$

Bertschinger & Zukin, Phys. Rev. D 78, 024015(2008)

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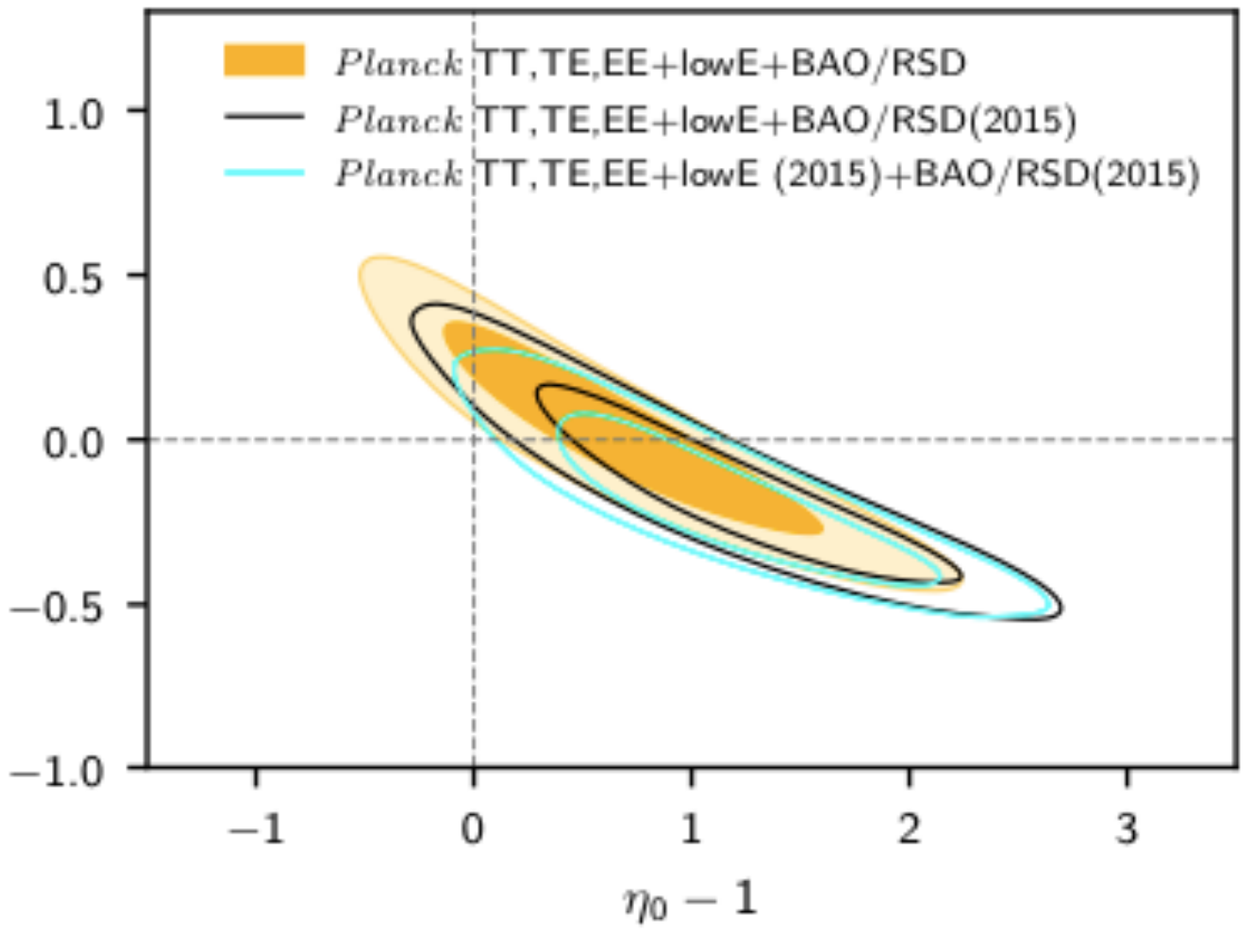
- bin them in time and space and constrain directly the resulting parameters or perform a 2D PCA (which is a very useful forecast tool). Or neglect the k-dependence and bin them in time, performing a 1D PCA.

  - QSA: fix their scale-dependence, according to general arguments of locality and then perform a 1D PCA on the time-dependence.

What have data told us so far about/via (μ,γ)?

Planck 2018

$$\mu(a) - 1 = (\mu_0 - 1)\Omega_{\rm DE}(a)$$



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Planck 2018 Results. VI Cosmological Parameters. arXiv:1807.06209

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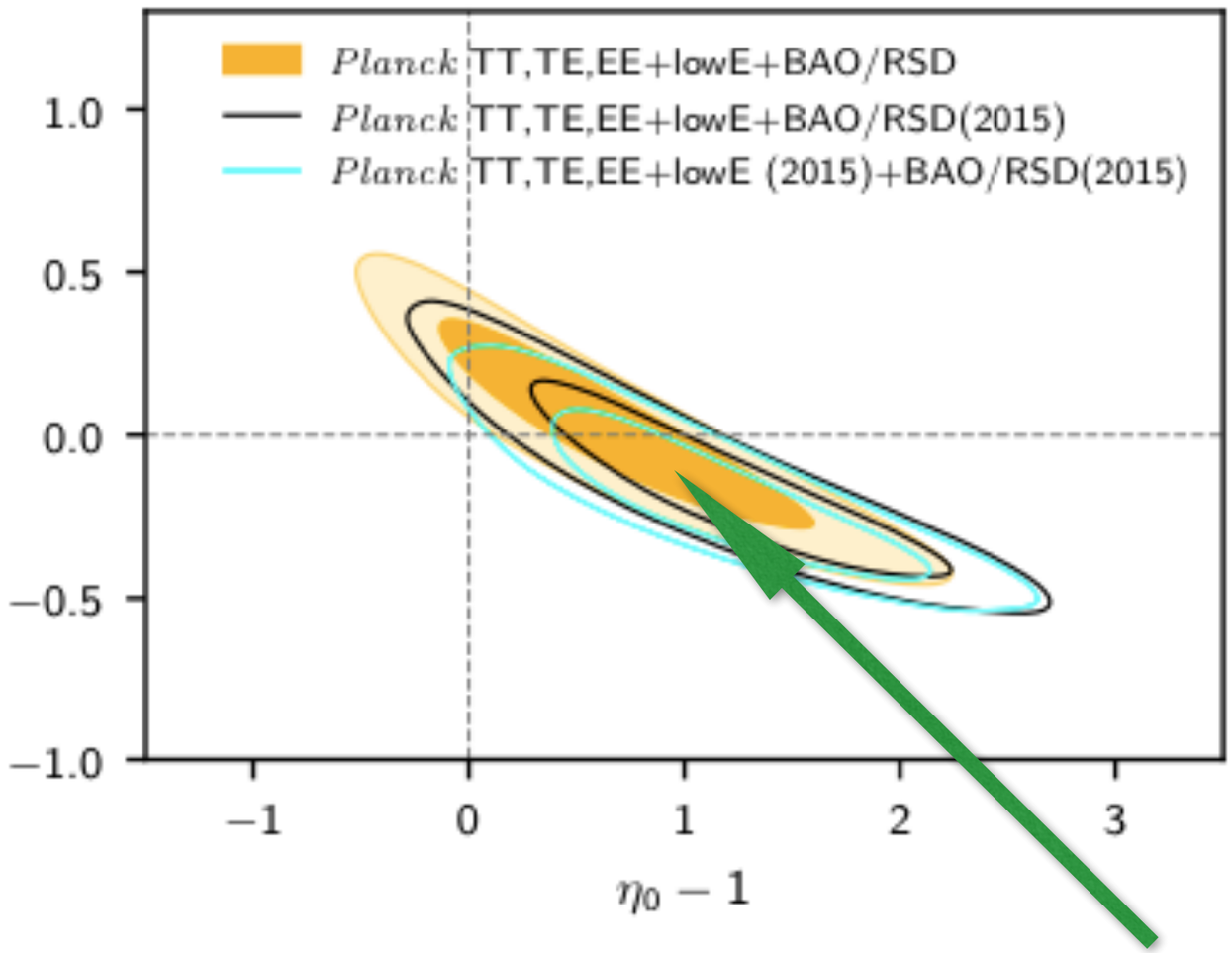

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which theory is it?

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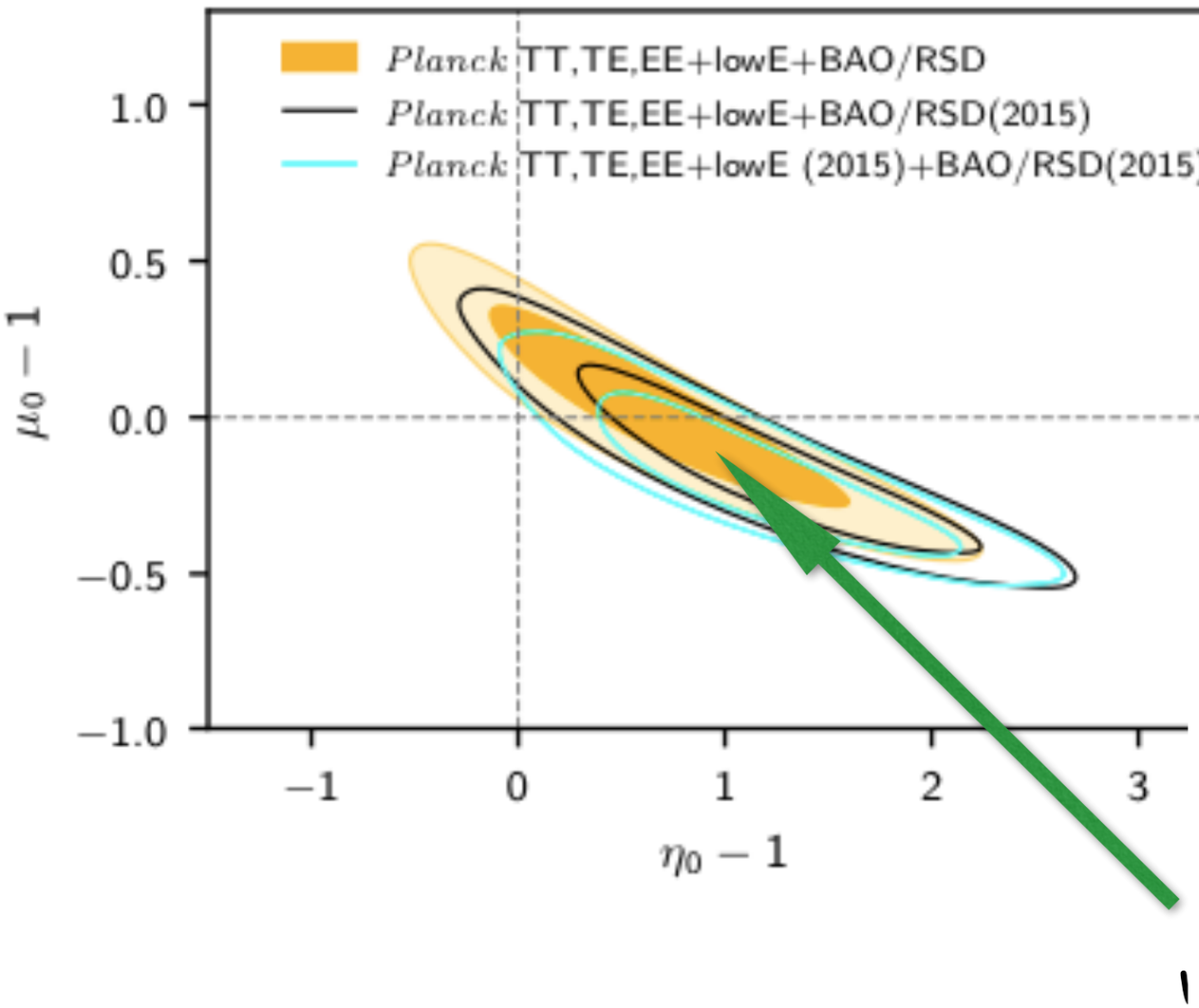

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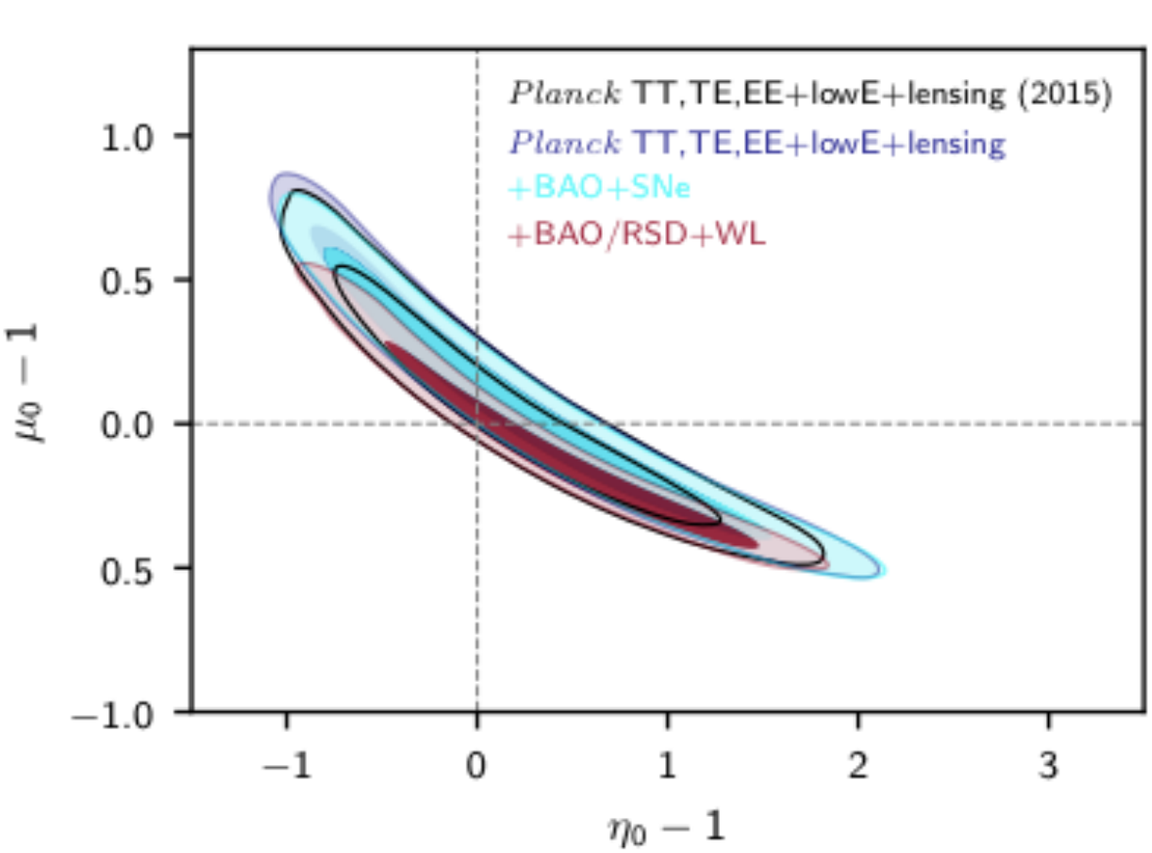

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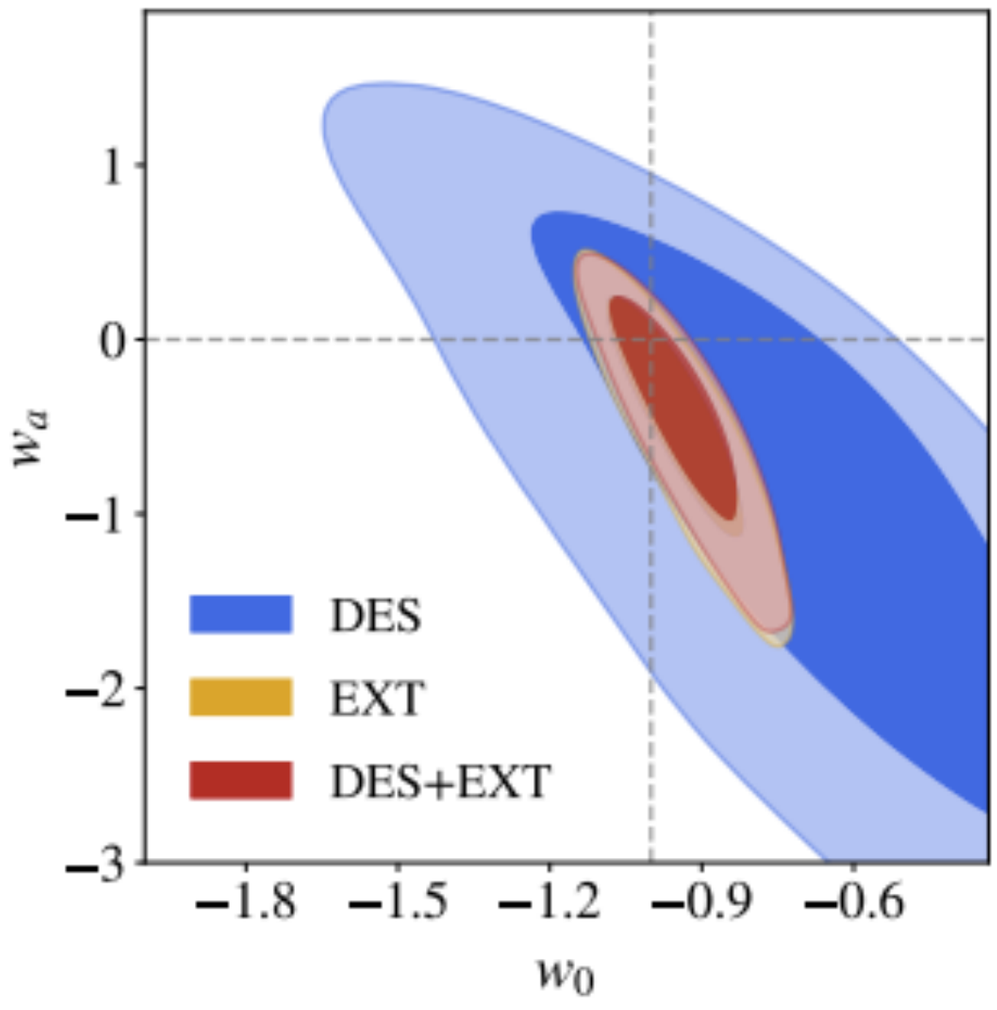
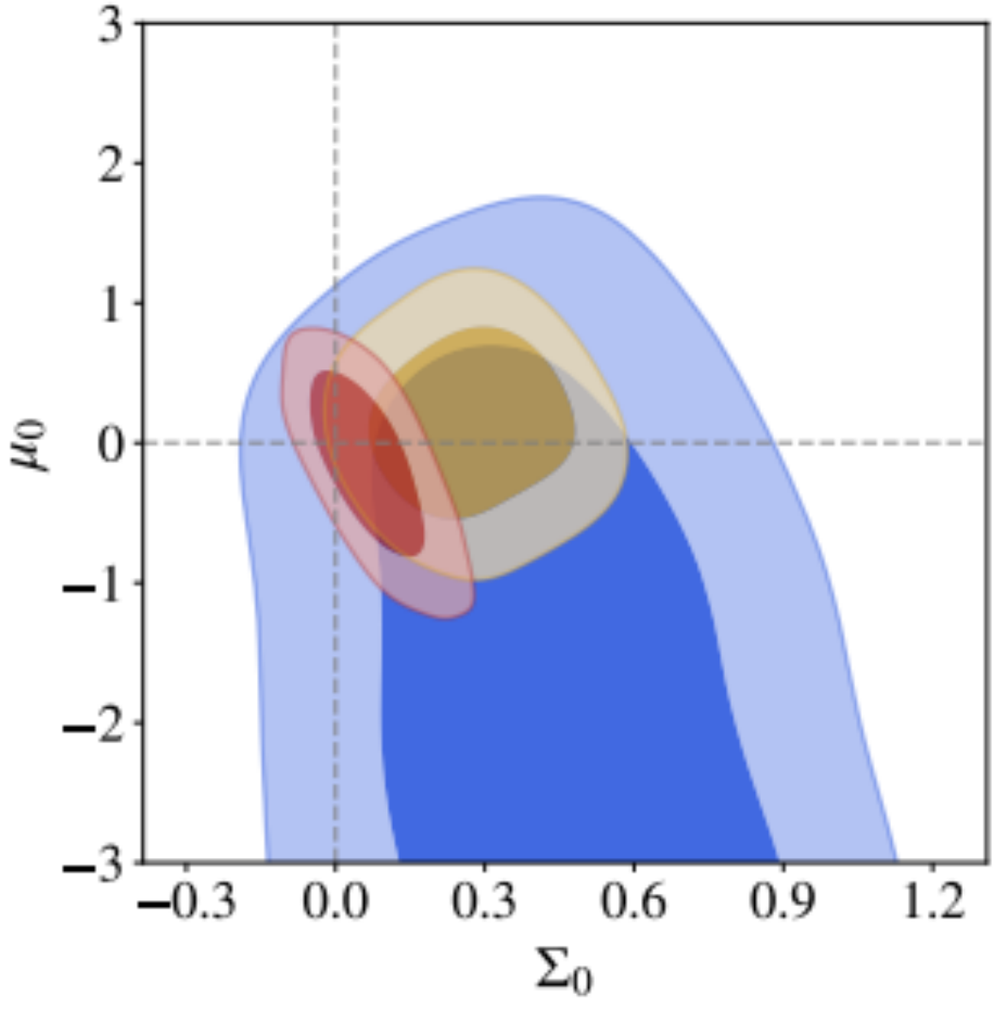
$$w_{\rm DE} = w_0 + w_a(1-a)$$



$$\Sigma(a) = \Sigma_0 \frac{\Omega_\Lambda(a)}{\Omega_\Lambda}$$



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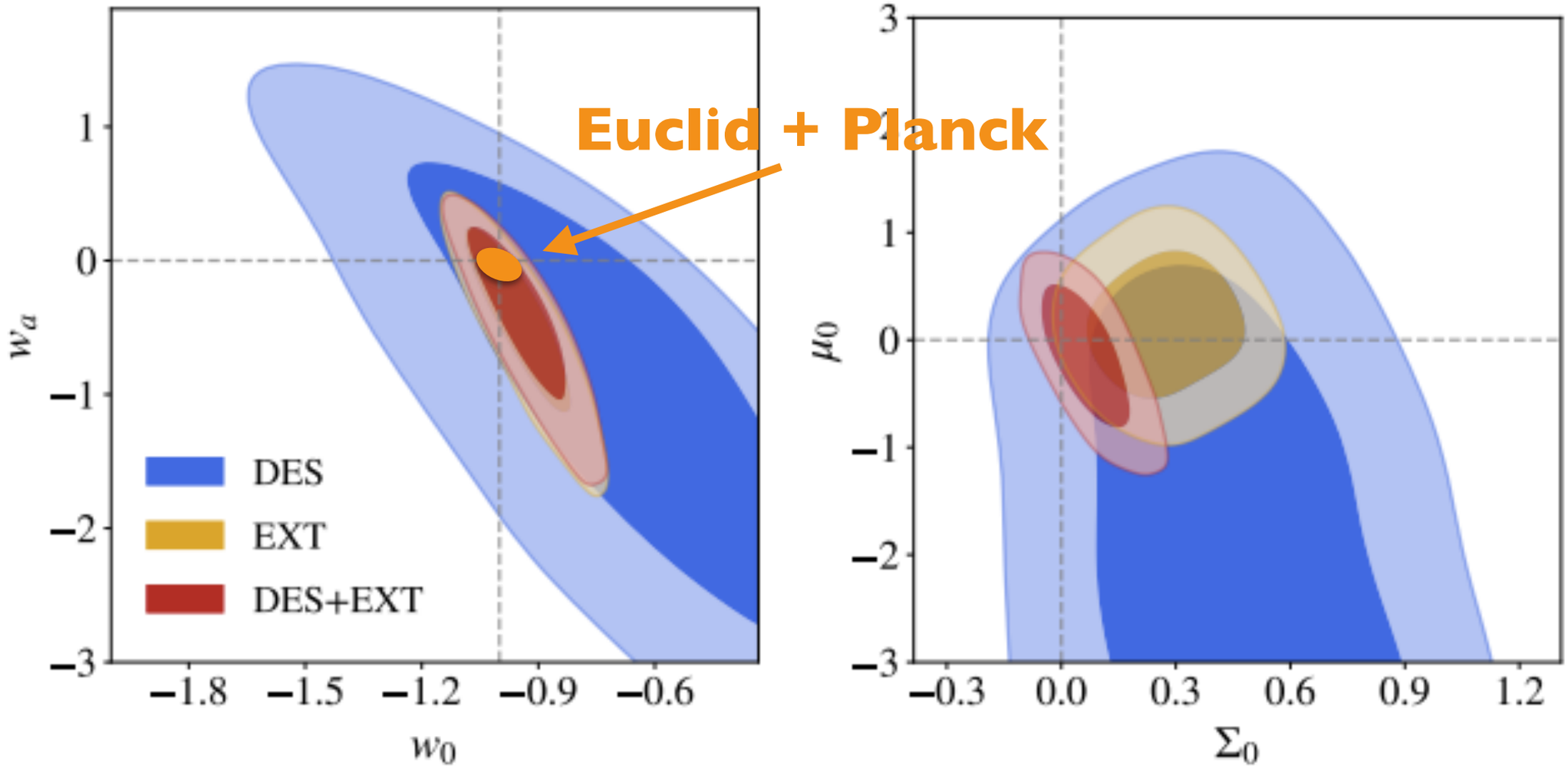

DES year 1

What about scale-dependence?

Is the k-dependence of (μ,γ) really completely general?

Shall we bother at all about the k-dependence of (μ,γ)?

k-dependence

The k-dependence of these functions should not be completely arbitrary if we wish to consider local covariant theories with equations of motion derived from a variational principle.

Starting from a 4d Lagrangian that contains an arbitrary function of geometric invariants $R, R_{\alpha\beta}R^{\alpha\beta}, R^{\alpha\beta}\nabla_\alpha\nabla_\beta R, \ldots$, as well as any number of scalar d.o.f. (including longitudinal modes of vectors and tensors), which can be non-minimally coupled to the metric and to each other, we have:

Poisson eq. $$\hat{a}\Phi + \hat{b}\Psi + \hat{c}^i\delta\phi_i = -\frac{a^2}{2M_P^2}\rho\Delta$$

anisotropy eq. $$\hat{d}\Phi + \hat{e}\Psi + \hat{f}^i\delta\phi_i = 0$$

scalar field eq. $$\hat{g}\Phi + \hat{h}\Psi + \hat{l}^i\delta\phi_i = 0$$

$$\hat{a} = \sum_{nm} A_{nm}(t)k^n\partial_0^m$$

A.S., Pogosian, Buniy, Phys.Rev. D87, 104015 (2013)

Baker, Ferreira , Skordis, Phys.Rev. D87, 024015 (2013)

k-dependence

Quasi-static approximation: (QSA)

- sub-horizon: $k \gg aH$

- all time dependences are negligible

$$\hat{a} \approx \sum_n A_{n0}(t) k^n$$

For models with one scalar obeying 2nd order EOMs, (this includes non-minimal coupling and theories with functions of Lovelock invariants), in the QUASI-STATIC regime μ and γ reduce to:

$$\gamma = \frac{p_1(t) + p_2(t)k^2}{1 + p_3(t)k^2}$$

$$\mu = \frac{1 + p_3(t)k^2}{p_4(t) + p_5(t)k^2}$$

AS, L. Pogosian, R. Buniy, Phys.Rev. D87 (2013)

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f(R), f(R,G), quintessence, k-essence, coupled quintessence, covariant galileon, scalar-tensor, ...

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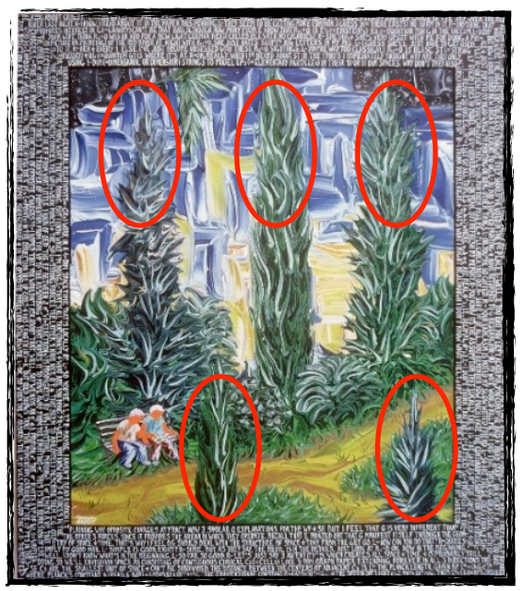

AS, L. Pogosian, R. Buniy, Phys.Rev. D87 (2013)

Some shortcomings

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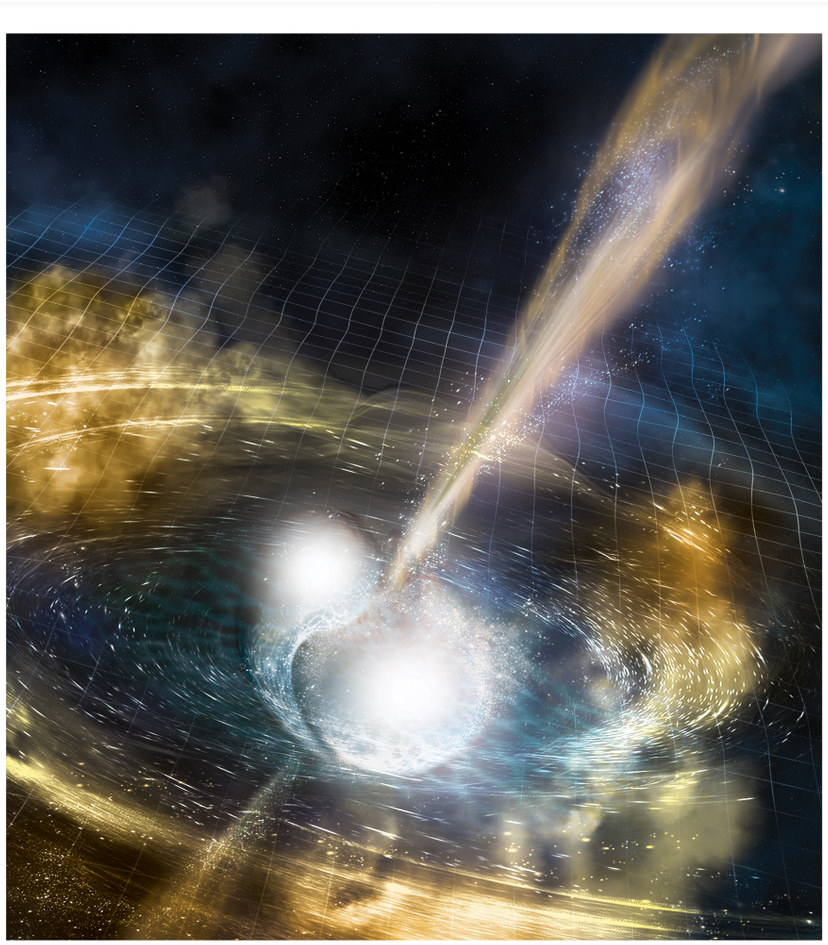

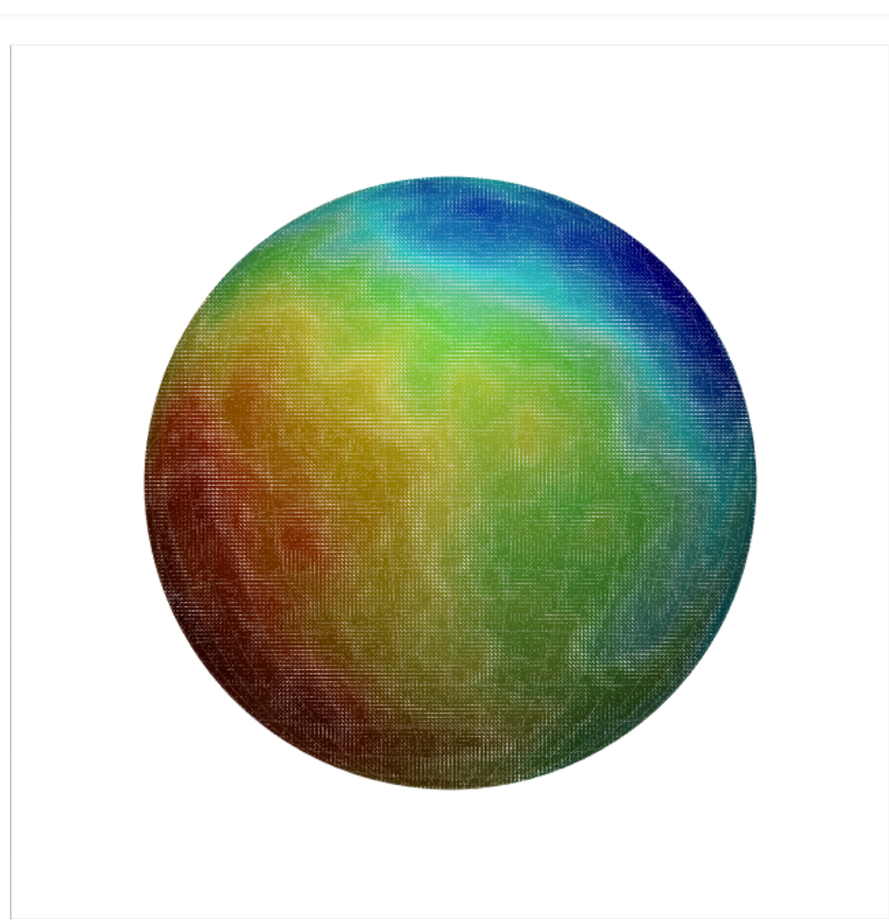

EFT of Dark Energy

A unified action

Jordan frame

$$S = \int d^4x \sqrt{-g} \left\{ \frac{m_0^2}{2} [1 \qquad\qquad] R + \right.$$

Gubitosi, Piazza, Vernizzi, JCAP 1302 (2013) 032

J. Gleyzes et al., Phys.Rev.Lett. 114 (2015)

Piazza, Vernizzi, Class.Quant.Grav. 30 (2013) 214007

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R. Kase, S. Tsujikawa, Int.J.Mod.Phys. D23 (2015)

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matter follows geodesic of the metric

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2nd order eoms: $\bar{M}_2^2 = -\bar{M}_3^2 = 2\hat{M}^2$

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IN UNITARY GAUGE

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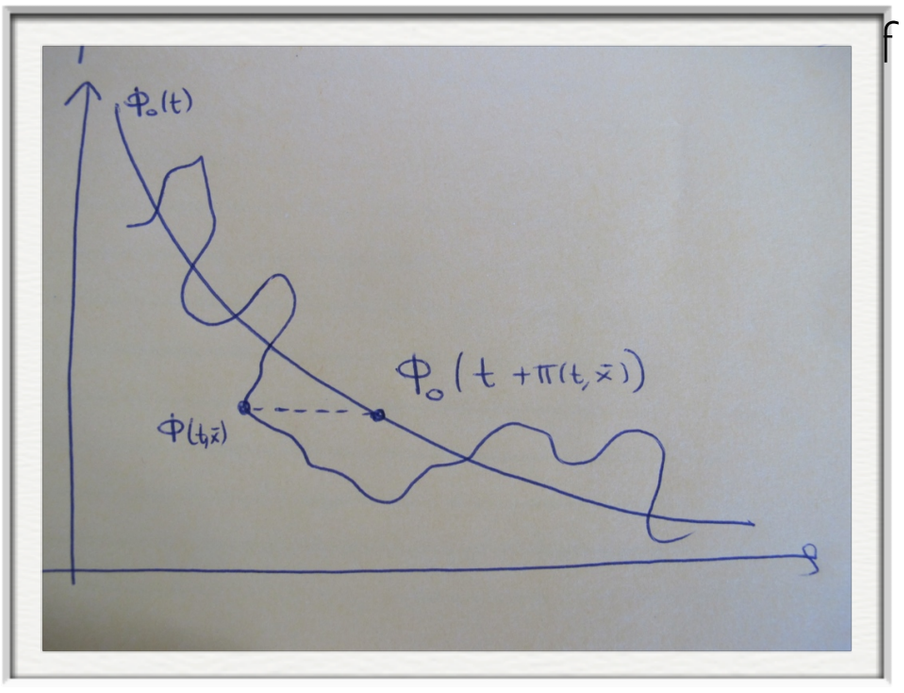
R. Kase, S. Tsujikawa, Int.J.Mod.Phys. D23 (2015)

A unified action

We seek a unified action to study linear perturbations around a FRW background in 'all' models of gravity with 1 tensor and 1 scalar d.o.f..
We strive to stay as model-independent as possible, hence we want to use general symmetry arguments to construct such action.

The presence of an extra dynamical scalar d.o.f. wrt the standard model of cosmology naturally singles out a preferred foliation of space-time, i.e. the one for which constant time hypersurfaces correspond to the uniform field hypersurfaces.



If we perform such a choice, we bring ourselves in the so-called unitary gauge, where all the d.o.f. are in the metric. If we write:

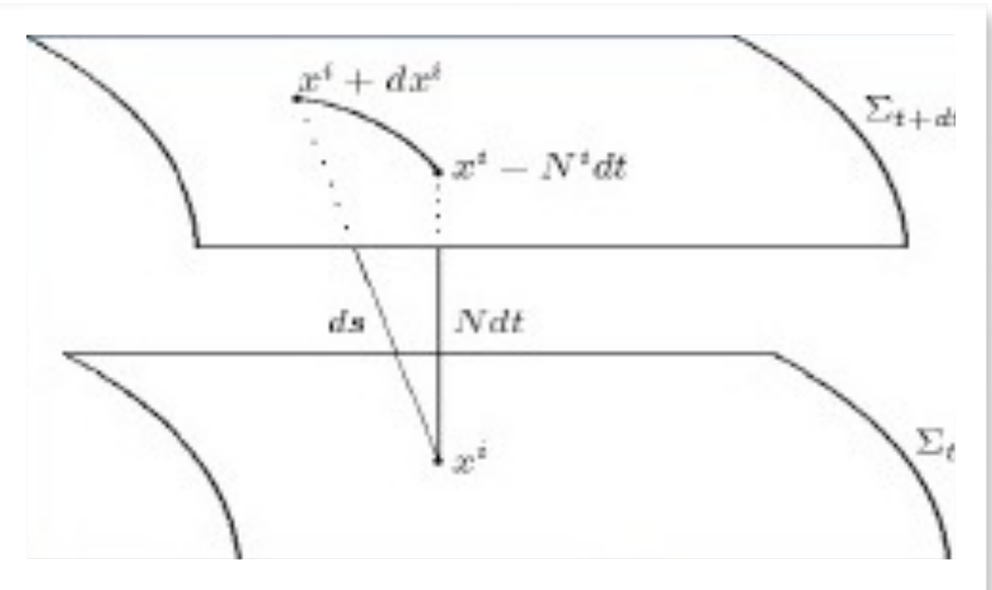
$$\phi(t,\bar{x}) = \phi_0(t+\pi(t,\bar{x}))$$

the unitary gauge corresponds to the preferred foliation identified by the orthonormal vector:

$$n_\mu \equiv \frac{\partial_\mu \phi}{\sqrt{(\partial_\mu \phi)^2}}$$

A unified action

It is then natural to employ the ADM (Arnowitt-Deser-Misner) decomposition of spacetime:



$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

lapse function — 3D metric — shift vector

If we define: $n_\mu \equiv N t_{;\mu}$ $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$

intrinsic curvature: $R^{(3)}_{\mu\nu}$

extrinsic curvature: $K_{\mu\nu} = h^\lambda_\mu n_{\nu;\mu}$ $K_{ij} = \frac{1}{2N}\left(\partial_t h_{ij} - \nabla_i N_j - \nabla_j N_i\right)$

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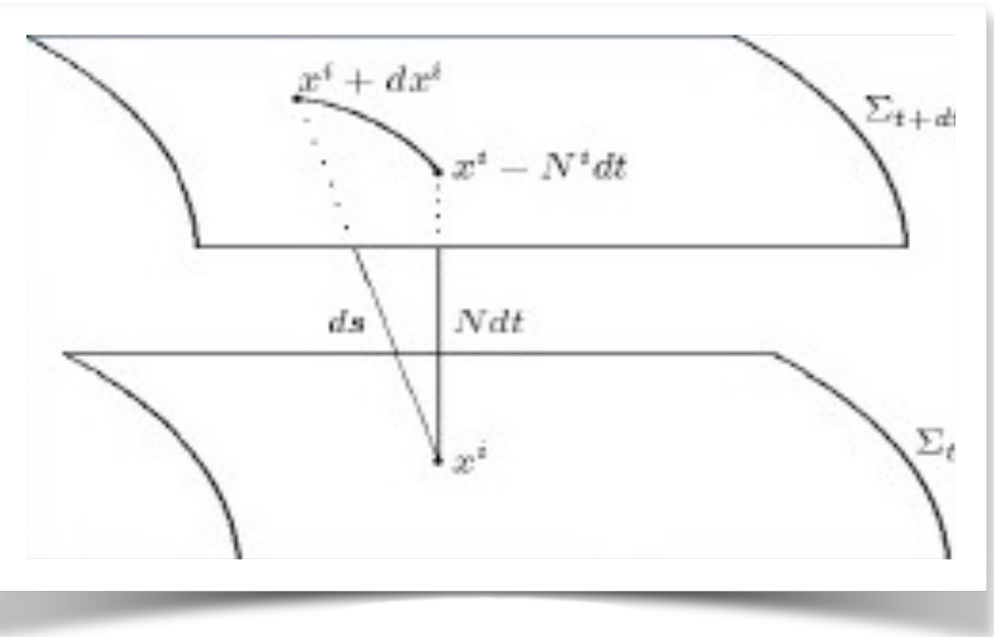

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UNITARY GAUGE → $n_\mu \equiv \frac{\partial_\mu \phi}{\sqrt{(\partial_\mu \phi)^2}}$

And the action is built out of all geometrical quantities that are invariant under the time-dependent 3D spatial diffeomorphisms.

A unified action

From:

$$n_\mu \equiv \frac{\partial_\mu \phi}{\sqrt{(\partial_\mu \phi)^2}} \qquad \gamma = \frac{1}{\sqrt{-X}}\,, \;\; K_{\mu\nu} = h^\sigma_\mu n_{\nu;\sigma}$$

$$\dot{n}_\mu = n^\nu n_{\mu;\nu} \qquad X \equiv \phi^{;\mu}\phi_{;\mu}$$

one gets:

$$\phi_{;\mu\nu} = -\gamma^{-1}\left(K_{\mu\nu} - n_\mu \dot{n}_\nu - n_\nu \dot{n}\mu\right) + \frac{\gamma^2}{2}\phi^{;\lambda} X_{;\lambda} n_\mu n_\nu$$

$$\Box\phi = -\gamma^{-1}K + \frac{1}{2}\phi^{;\mu} X_{;\mu}/X$$

Restoring 4D diffeomorphism invariance

Stückelberg trick

Let us restore the full diffeomorphism invariance, by making the extra scalar d.o.f. explicit, by means of the Stückelberg trick. This amounts to performing a time diffeomorphism transformation, taking us out of the unitary gauge and making the perturbation to the scalar field explicit:

$$\tau \to \tau + \pi(x^\mu)$$

Under this transformation:

time-dep. functions $\quad c(\tau) \to c(\tau+\pi) = c(\tau) + \dot{c}(\tau)\pi + \frac{1}{2}\ddot{c}(\tau)\pi^2 + \dots$

4D diffeomorphism invariant operators remain unchanged

other operators:

$$g^{00} \to g^{00} + 2g^{0\mu}\partial_\mu\pi + g^{\mu\nu}\partial_\mu\pi\partial_\nu\pi$$
$$g^{0i} \to g^{0i} + g^{\mu i}\partial_\mu\pi$$
$$\delta K_{ij} \to \delta K_{ij} - \dot{H}\pi h_{ij} - \partial_i\partial_j\pi$$

Restoring 4D diffeomorphism invariance

$$S=\int d^4x\sqrt{-g}\left\{\frac{m_0^2}{2}\left[1+\Omega(\tau+\pi)\right]R+\Lambda(\tau+\pi)-c(\tau+\pi)a^2\left[\delta g^{00}-2\frac{\dot\pi}{a^2}+2\mathcal{H}\pi\left(\delta g^{00}-\frac{1}{a^2}-2\frac{\dot\pi}{a^2}\right)\right.\right.$$
$$\left.\left.+2\dot\pi\delta g^{00}+2g^{0i}\partial_i\pi-\frac{\dot\pi^2}{a^2}+g^{ij}\partial_i\pi\partial_j\pi-\left(2\mathcal{H}^2+\dot{\mathcal{H}}\right)\frac{\pi^2}{a^2}\right]+\dots\right\}+S_m[g_{\mu\nu}]$$

Jordan frame, Stuckelberg field action

A unified action

It is an interesting framework that offers both a model-independent parametrization of alternatives to LCDM and a unifying language to analyze specific DE/MG models.

pure EFT:

$$\{\Omega(\tau), \Lambda(\tau), M_2^4(\tau), \bar{M}_1^3(\tau)\bar{M}_2^2(\tau)\}$$

mapping EFT:

f(R) $\quad \Omega = f_R; \quad \Lambda = \dfrac{m_0^2}{2}\left[f - R f_R\right]; \quad c = 0$

minimally coupled quintessence $\quad \Omega = 0; \quad c - \Lambda = V(\phi); \quad c = \dfrac{\dot{\phi}^2}{2}$

Gubitosi, Piazza, Vernizzi, JCAP 1302 (2013) 032

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Gubitosi, Piazza, Vernizzi, JCAP 1302 (2013) 032

A unified action

Operator	Ω	Λ	c	M_2^4	$\bar{M}_1^3$	$\bar{M}_2^2$	$\bar{M}_3^2$	$\hat{M}^2$	m_2^2
Model	R		δg^{00}	$(\delta g^{00})^2$	$\delta g^{00}\delta K^{\mu}{}_{\mu}$	$(\delta K^{\mu}{}_{\mu})^2$	$\delta K^{\mu}{}_{\nu} K^{\nu}{}_{\mu}$	$\delta g^{00}\delta R^{(3)}$	$\frac{\tilde{g}^{ij}}{a^2}\partial_i g^{00}\partial_j g^{00}$
ΛCDM	1	✓	0	-	-	-	-	-	-
Quintessence	1/✓	✓	✓	-	-	-	-	-	-
$F(R)$	✓	✓	0	-	-	-	-	-	-
k-essence	1/✓	✓	✓	✓	-	-	-	-	-
Galileon [39] Kinetic Braiding [40]	1/✓	✓	✓	✓	✓	-	-	-	-
DGP [41]	✓	✓†	✓†	✓†	✓	-	-	-	-
Ghost Condensate [42]	1/✓	✓	0	-	-	✓	✓	-	-
Horndeski [26]	✓	✓	✓	✓	✓	✓†	✓†	✓†	-

Bloomfield, Flanagan, Park, Watson JCAP 1308 (2013) 010

A unified action

Operator	Ω	Λ	c	M_2^4	$\bar{M}_1^3$	$\bar{M}_2^2$	$\bar{M}_3^2$	$\hat{M}^2$	m_2^2
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Galileon [39] Kinetic Braiding [40]	1/✓	✓	✓	✓	✓	-	-	-	-
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Bloomfield, Flanagan, Park, Watson JCAP 1308 (2013) 010

α parametrization

$$S^{(2)} = \int dtdx^3\, a^3 \frac{M_*^2}{2}\left\{\delta K^i_j \delta K^j_i - \delta K^2 + R\delta N + (1+\alpha_T)\delta_2\left(\sqrt{h}R/a^3\right) + \alpha_K H^2\delta N^2 + 4\alpha_B H\delta K\delta N\right\}$$

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α_M

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α_M

α_M Running of Planck's constant, generated by non-minimal coupling

α_T Deviation of speed of GWs from unity; non-zero whenever there is a non-linear derivative coupling of the scalar field to the metric. Same non-linearity is responsible for non-zero anisotropic stress.

α_K Quantifies the independent dynamics of the scalar-field

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Horndeski in EFT: $\{\Omega, \Lambda, M_2^4, \bar{M}_1^3, \bar{M}_2^2\}$ or $\{H, \Omega, M_2^4, \bar{M}_1^3, \bar{M}_2^2\}$

designer

DHOST

In unitary gauge:

$$S^{(2)} = \int dx^3\, dt\, a^3 \frac{M_*^2}{2} \left\{ \delta K_{ij}\delta K^{ij} - \left(1+\frac{2}{3}\alpha_L\right)\delta K^2 + (1+\alpha_T)\delta_2\left(\sqrt{h}R/a^3\right) + \alpha_K H^2\delta N^2 + 4\alpha_B H\delta K\delta N \right.$$
$$\left. +\alpha_H R\delta N + 4\beta_1\delta K\delta\dot{N} + \beta_2\delta\dot{N}^2 + \frac{\beta_3}{a^2}(\partial_i\delta N)^2 \right\}$$

* with some degeneracy conditions on the α_L and β_i parameters

Crisostomi et al., JCAP(2019)

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$$\left. + \alpha_H R\delta N + 4\beta_1\delta K\delta\dot{N} + \beta_2\delta\dot{N}^2 + \frac{\beta_3}{a^2}(\partial_i\delta N)^2 \right\}$$

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Dark-Energy Instabilities induced by Gravitational Waves,
Creminelli et al. arXiv:1910.14035 [gr-qc]

Resonant Decay of Gravitational Waves into Dark Energy,
Creminelli et al. JCAP(2018), JCAP(2019)

Crisostomi et al., JCAP(2019)

Linear scalar perturbations

At this point, in the mapping case we have all is needed to determine all EFT functions, also those affecting only perturbations. In the pure EFT case, we need to make a choice for the EFT functions multiplying the higher order operators (i.e. those that affect only perturbations). After this choice is made, we can move on and study the dynamics of perturbations.

energy-momentum equations: standard ones since we are in the Jordan frame

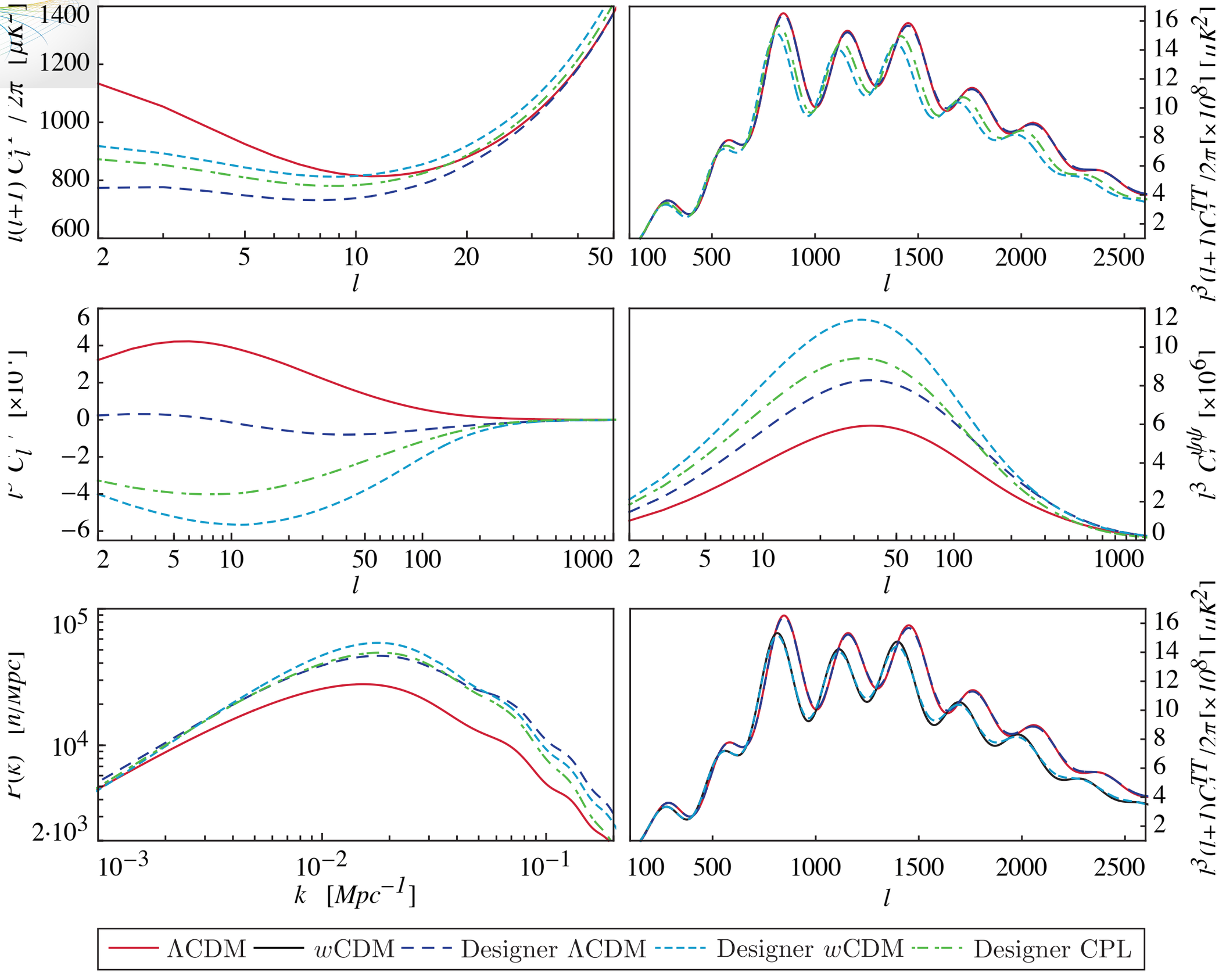
Einstein equations: messy equations involving contributions from 'all' EFT functions

π field equation:

$$A\ddot{\pi} + B\dot{\pi} + \left(C + k^2 D\right)\pi + E = 0$$

$$A = A[c, \Lambda, \Omega, \dots](\tau, k)$$

EffectiveFieldTheory
CAMB



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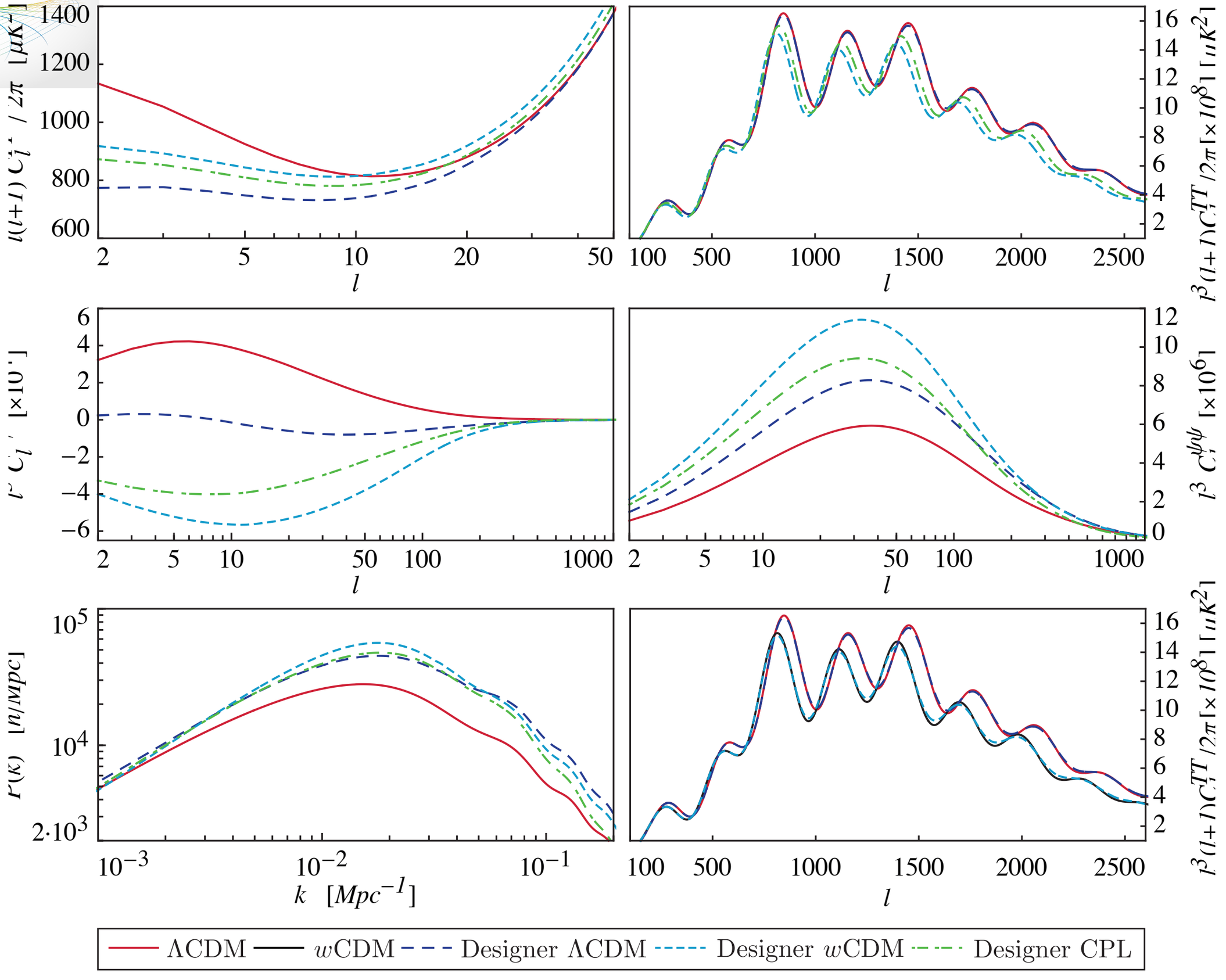
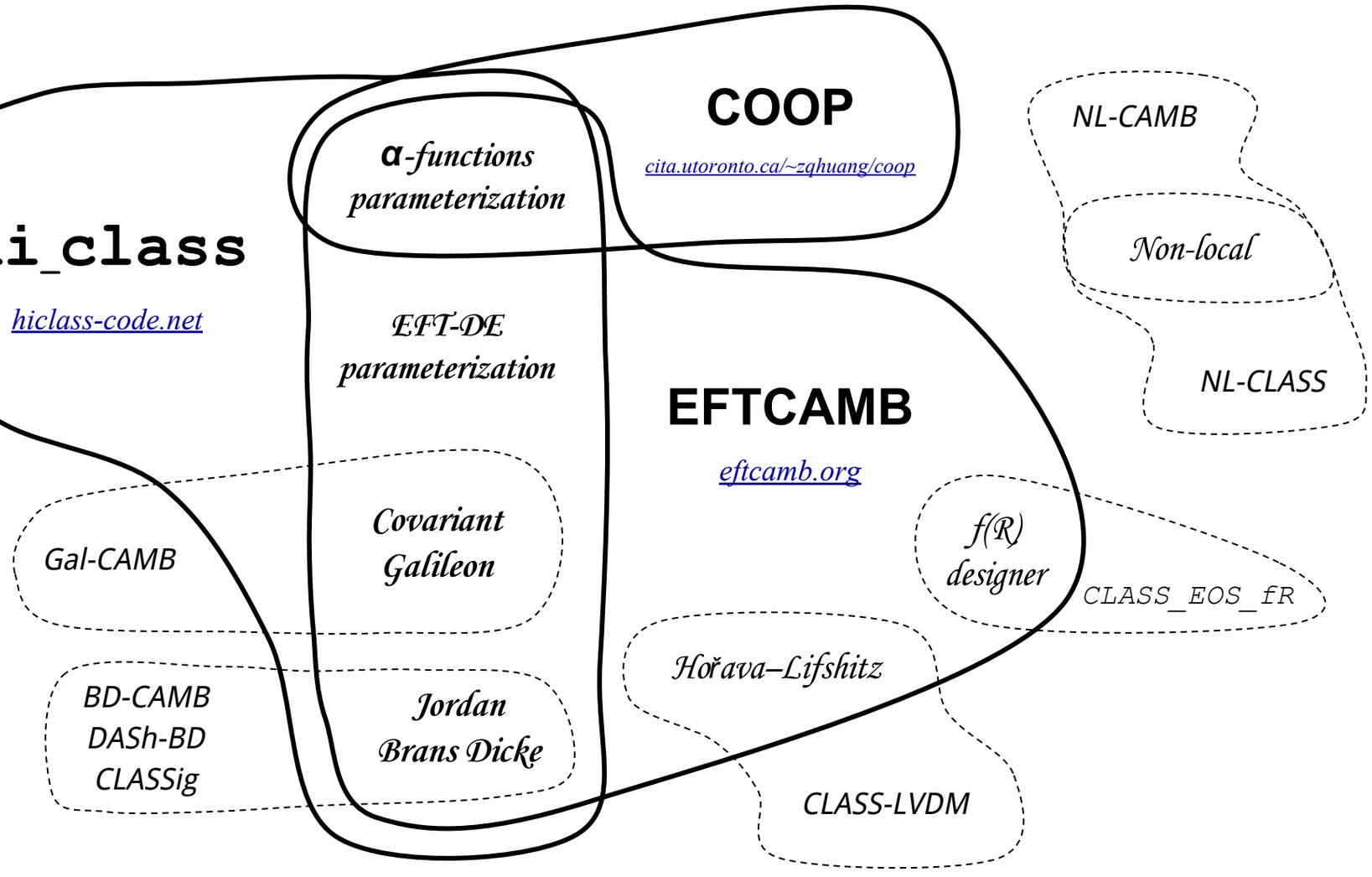


From Theory to Observables



A comparison of Einstein-Boltzmann solvers for testing General Relativity

E. Bellini,[1] A. Barreira,[2] N. Frusciante,[3] B. Hu,[4] S. Peirone,[5] M. Raveri,[6] M. Zumalacárregui,[7,8] A. Avilez-Lopez,[9] M. Ballardini,[10,11,12,13] R. A. Battye,[14] B. Bolliet,[15] E. Calabrese,[1,16] Y. Dirian,[17] P. G. Ferreira,[1] F. Finelli,[12,13] Z. Huang,[18] M. M. Ivanov,[19,20] J. Lesgourgues,[21] B. Li,[22] N. A. Lima,[23] F. Pace,[14] D. Paoletti,[12,13] I. Sawicki,[24] A. Silvestri,[5] C. Skordis,[24,25] C. Umiltà,[26,27,28] and F. Vernizzi[29]

Stability priors

Since we start from an action, we can identify general, model-independent viability conditions that are well motivated theoretically and ensure also numerical stability.

This can significantly restrict the parameter space, and represents a powerful tool for the advocated open-minded approach to cosmological tests of GR.

The world is beautiful outside
only if it is stable inside.

Stability priors

Let us use an ADM decomposition of the metric:

$$ds^2 = -N^2 dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right)$$

And let us focus on scalar and tensor perturbations:

$$\delta N\,, \quad N^i = \partial^i \psi\,, \quad h_{ij} = a^2(t) e^{2\xi}\left(\delta_{ij} + h^T_{ij}\right)\,.$$

Expanding the given action up to second order in the perturbations, and removing spurious DOFs, we can inspect the dynamics of perturbations; in this case one scalar field, i.e. ξ, and the tensor:

$$S^{(2)}_{\xi,h^T} = \frac{1}{(2\pi)^3}\int d^3k\, dt\, a^3 \left\{\left[\mathcal{L}_{\dot\xi\dot\xi}\dot\xi^2 - k^2 G\xi^2\right] + \frac{A_T}{8}\left[\left(\dot h^T_{ij}\right)^2 - \frac{c_T^2}{a^2}\left(h^T_{ij}\right)^2\right]\right\}$$

Stability priors

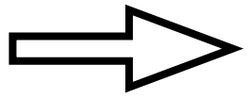
E.g., avoidance of ghost and gradient instabilities translate into the following set of conditions:

$$\mathcal{L}_{\dot{\xi}\dot{\xi}} > 0$$

$$c_s^2 \equiv \frac{G}{\mathcal{L}_{\dot{\xi}\dot{\xi}}} > 0 \quad *$$

$$A_T > 0$$

$$c_T^2 > 0$$



very general conditions in terms of the EFT functions, that constitute a stability check to be run at the very beginning!

In terms of EFT functions:

$$A_T(t) = m_0^2\,(1+\Omega) - \bar{M}_3^2$$

$$c_T^2(t) = \frac{m_0^2\,(1+\Omega)}{m_0^2\,(1+\Omega) - \bar{M}_3^2}$$

$*$ including the matter sector these will change

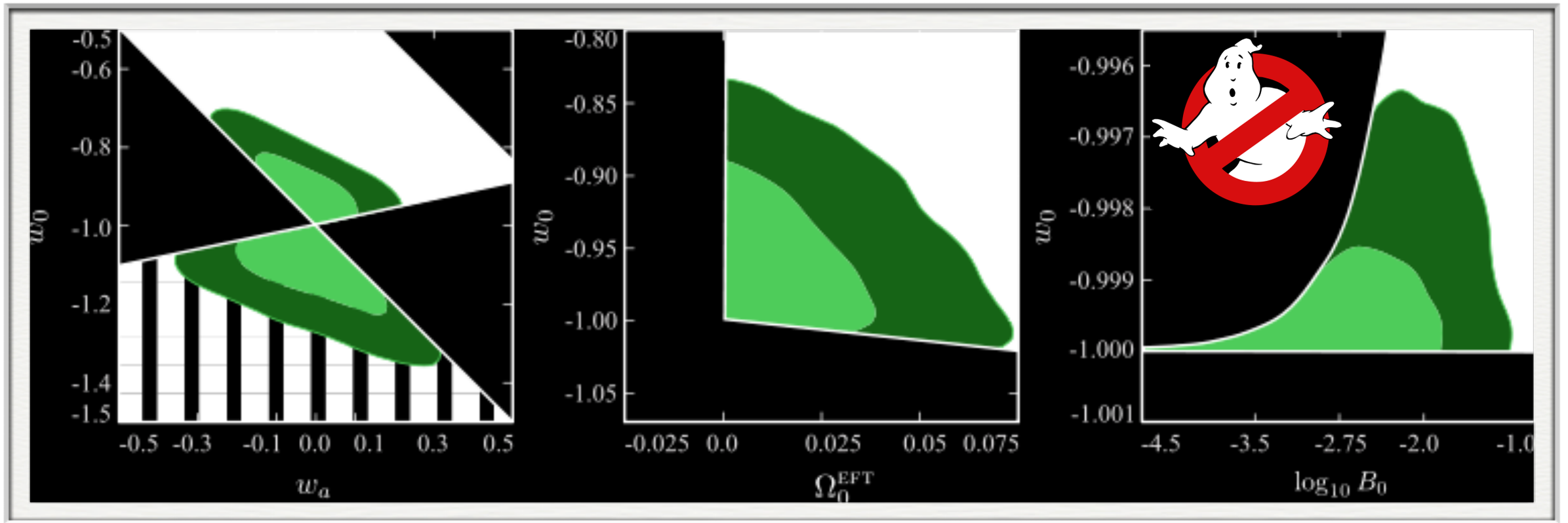
… more ?!

Frusciante, Papadomanolakis, AS JCAP 1607 (2016).

De Felice et al., JCAP 1703 (2017); J. Gleyzes et al., Int.J.Mod.Phys. D23 (2015); R. Kase, S. Tsujikawa, Int.J.Mod.Phys. D23 (2015)

Stability priors

Physics remains the guiding principle in the agnostic exploration of dark energy!



Marginalized joint likelihoods for parameters of interest in three different dark energy models: quintessence, non-minimally coupled quintessence and f(R) gravity.

Raveri, Hu, Frusciante, A

Stability priors

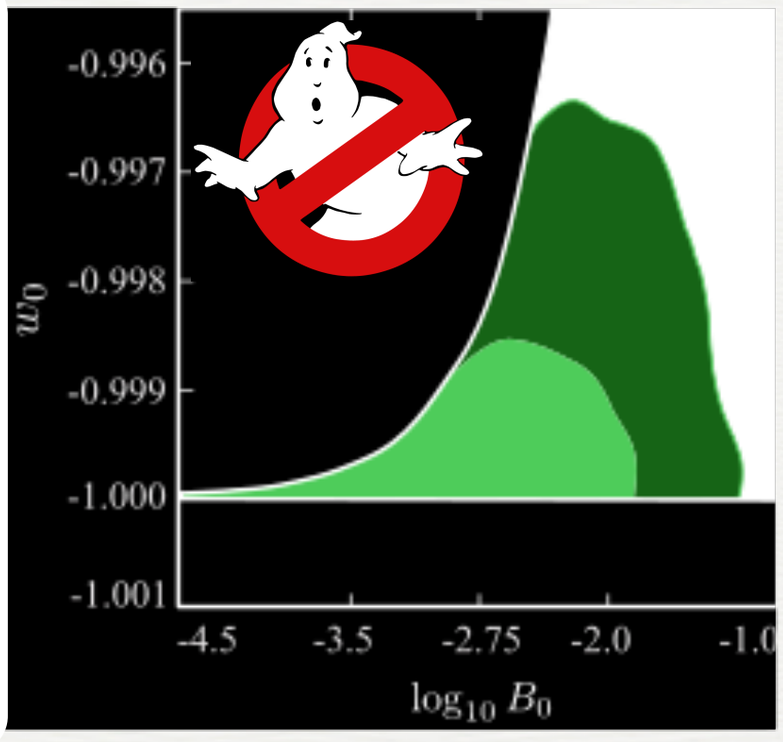
Physics remains the guiding principle in the agnostic exploration of dark energy!

designer f(R) on wCDM background:

$$w_0 \ \epsilon \ (-1, -0.9997) \ \ (95\%\mathrm{C.L.})$$

with Planck, lensing, WP, BAO data

B_0 and w_0 are strongly correlated via a theoretical prior



Marginalized joint likelihoods for parameters of interest in three different dark energy models: quintessence, non-minimally coupled quintessence and f(R) gravity.

Raveri, Hu, Frusciante, A

Horndeski theoretical priors …

HORNDESKI gravity → $\{\Omega(a), \Lambda(a), M_2^4(a), \bar{M}_1^3(a), \bar{M}_2^2(a)\}$

$$f(a) = \sum_{n=0}^{N} \frac{\alpha_n}{n!}(a - a_0)^n$$

$$f(a) = \sum_{n=0}^{N} \alpha_n (a - a_0)^n \qquad a_0 = 0, 1$$

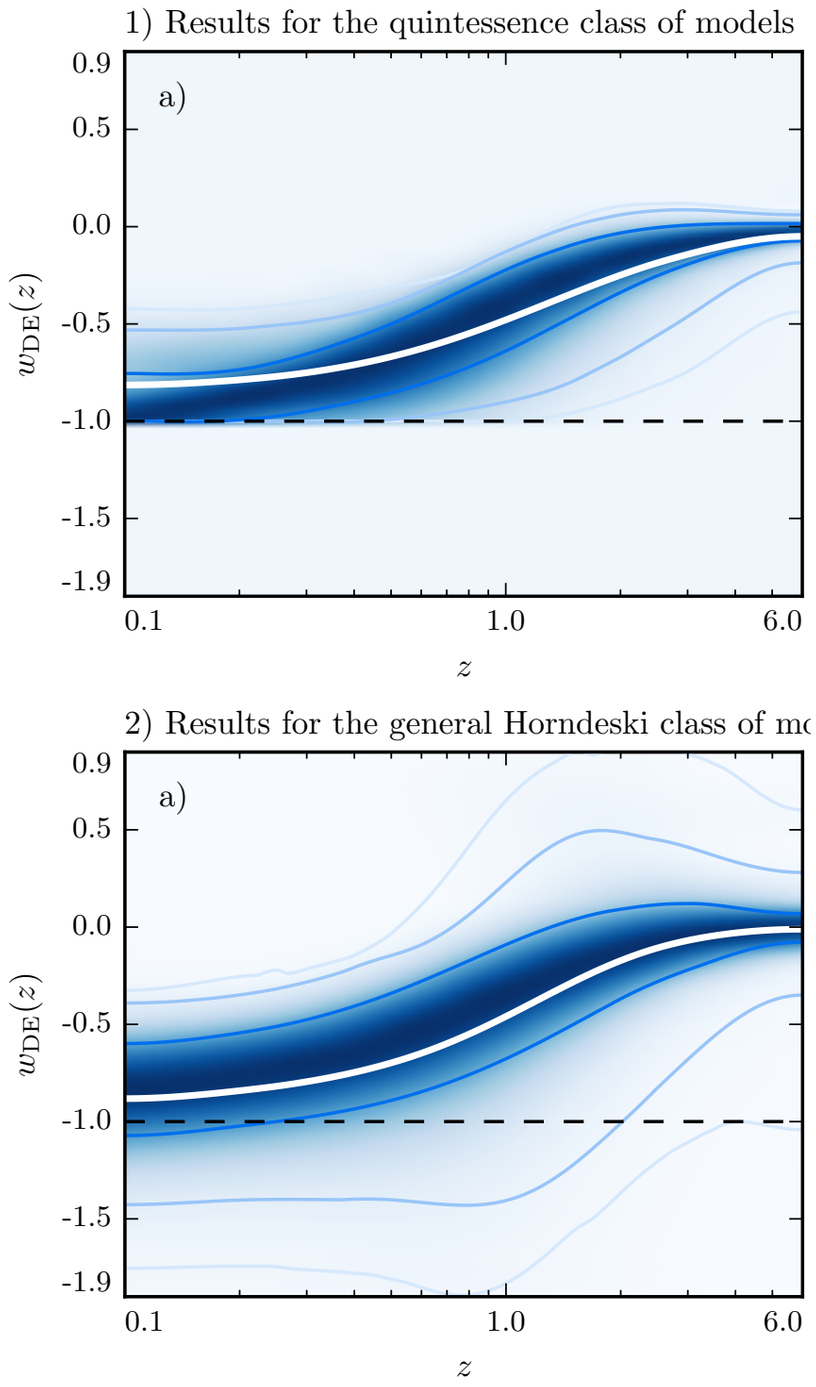
$$f(a) = \frac{\sum_{n=0}^{N} \alpha_n (a - a_0)^n}{1 + \sum_{m=1}^{M} \beta_m (a - a_0)^m} \qquad \alpha_n, \beta_m \, \epsilon [-1, 1] \qquad M, N = 9$$

$O(10^8)$ **models!**

Raveri, Bull, AS, Pogosian, arXiv:1703.05297

… on DE equation of state

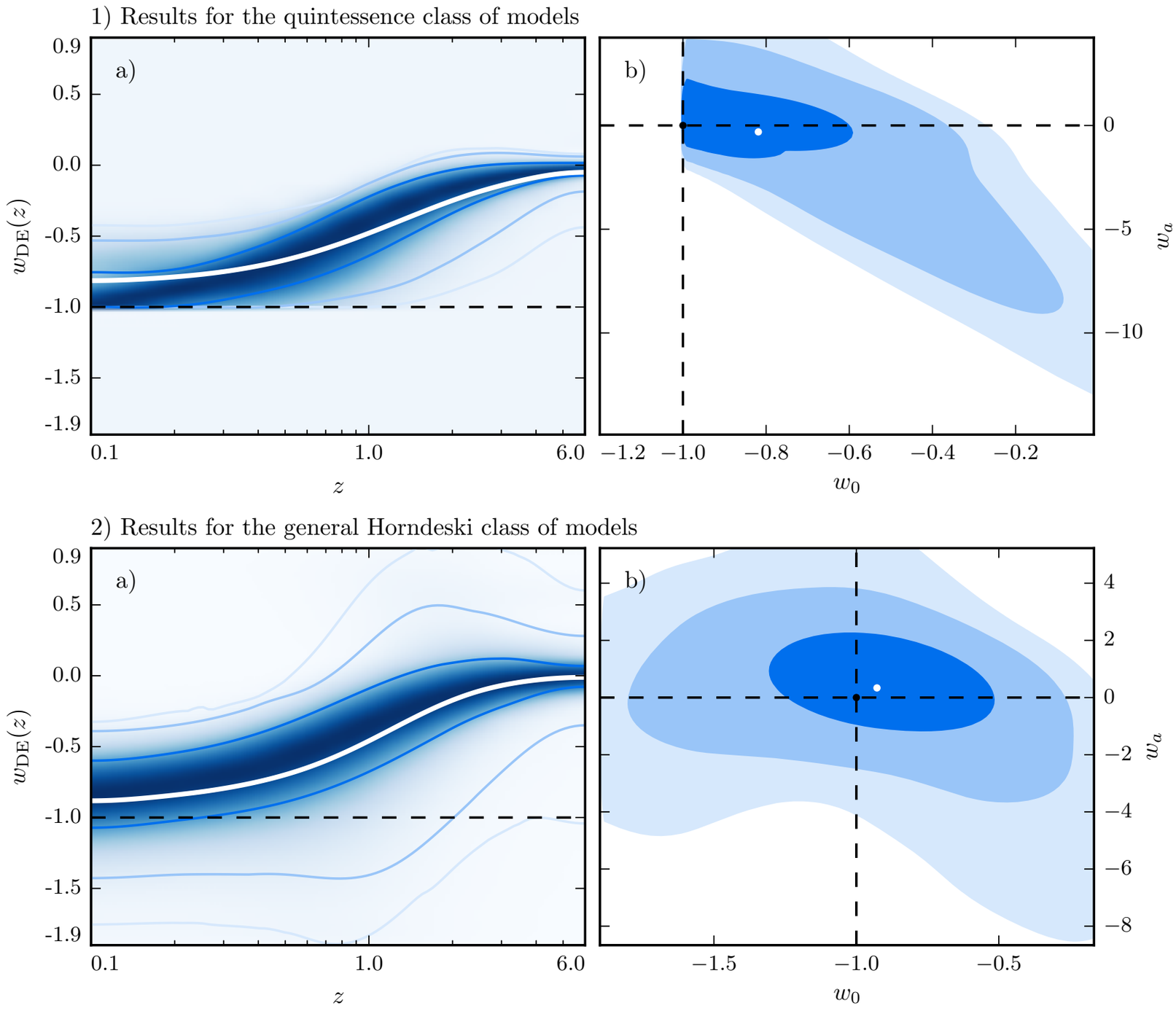
$$\mathcal{H}(a) \to w(a) = \frac{\rho_{\rm DE}}{p_{\rm DE}}$$



Raveri, Bull, AS, Pogosian, PRD (2017)

… on DE equation of state

$$\mathcal{H}(a) \to w(a) = \frac{\rho_{\rm DE}}{p_{\rm DE}} \qquad w(a) = w_0 + w_a(1-a)$$



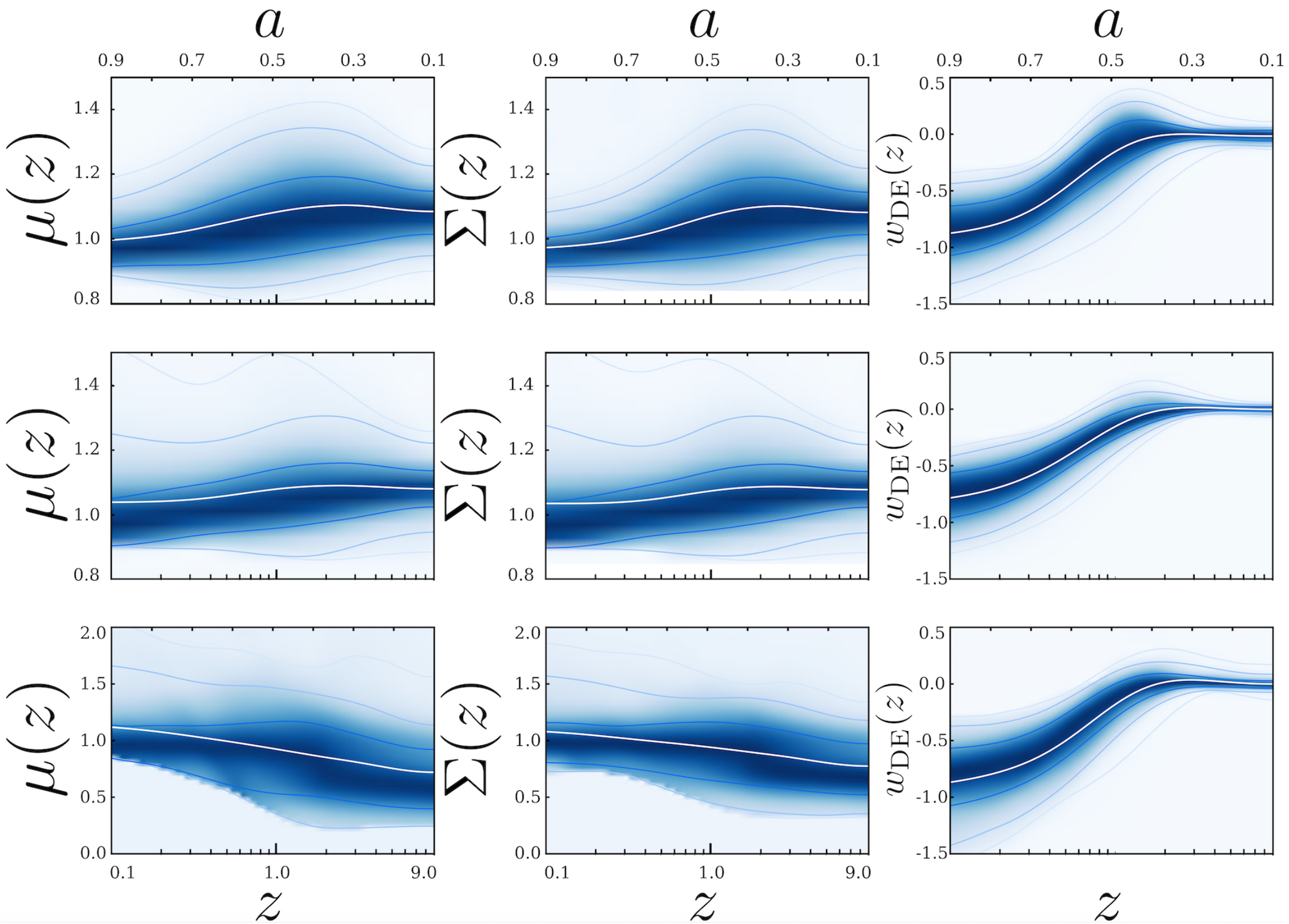
Raveri, Bull, AS, Pogosian, PRD (2017)

and on LSS Phenomenology!

GBD

H_S

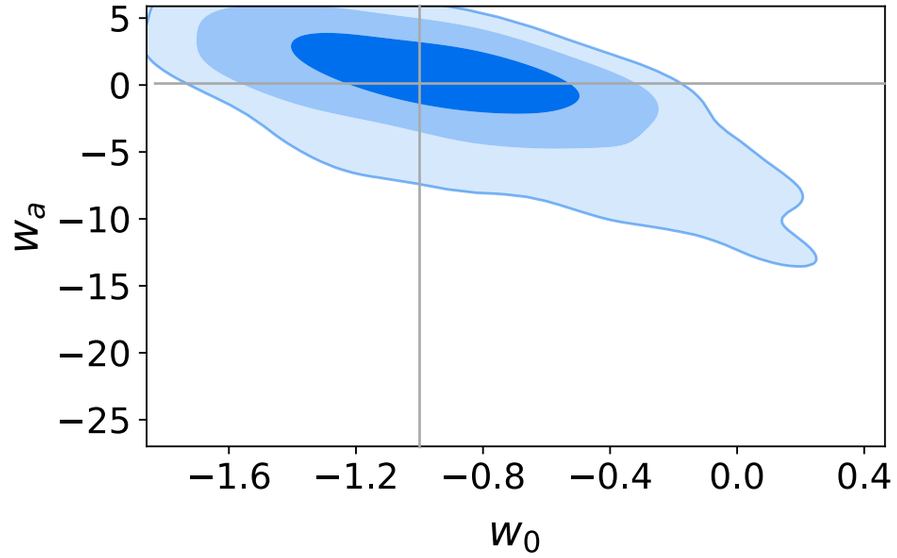
HOR

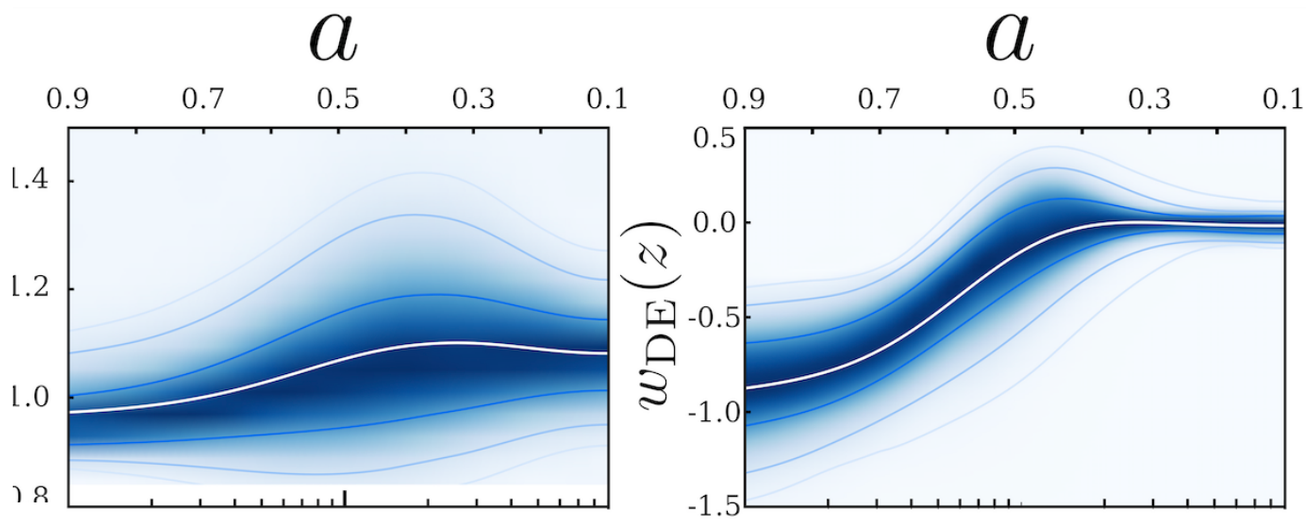


Espejo et al., PRD 2019

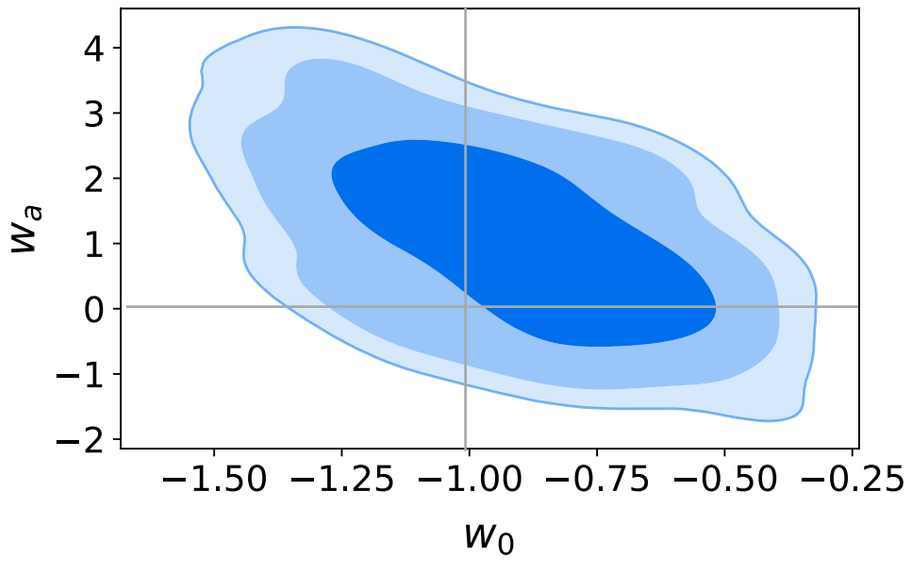
and on LSS Phenomenology!

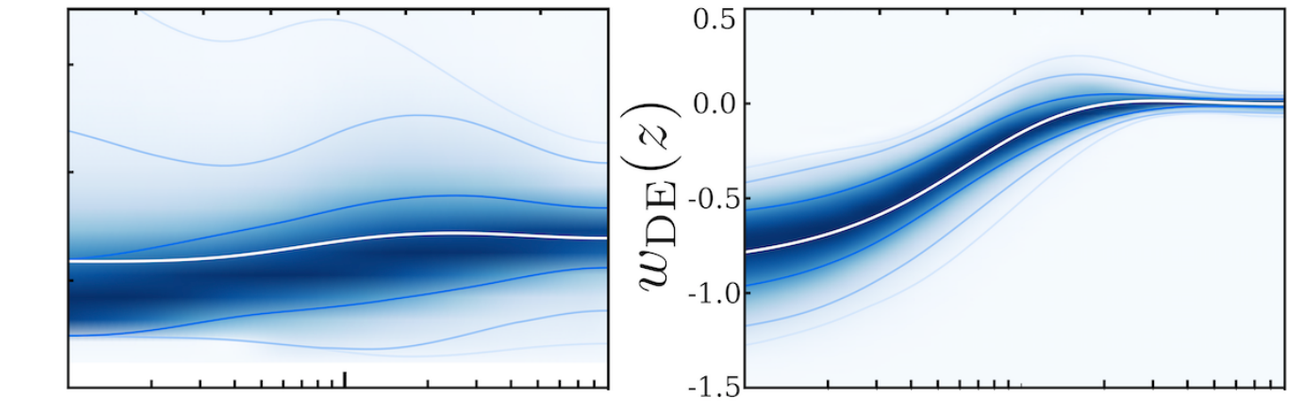
GBD



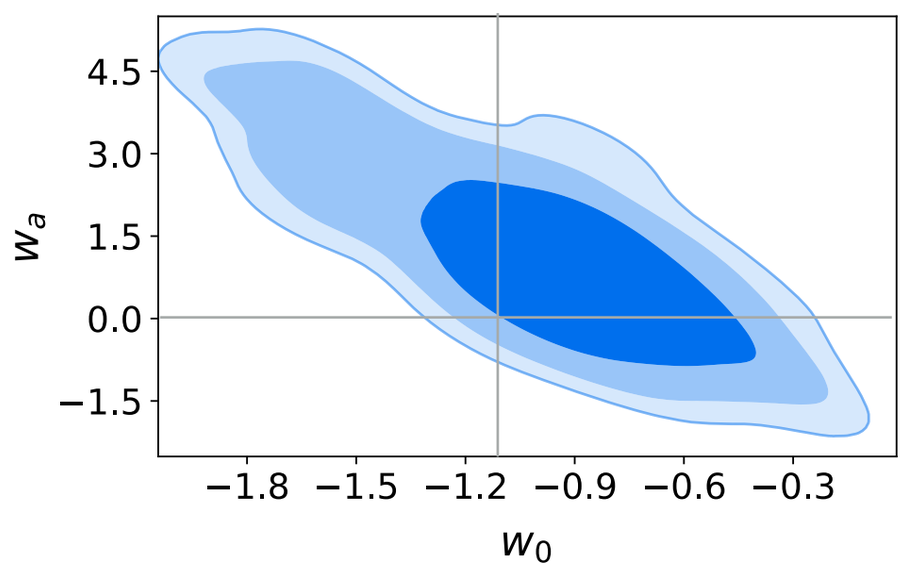


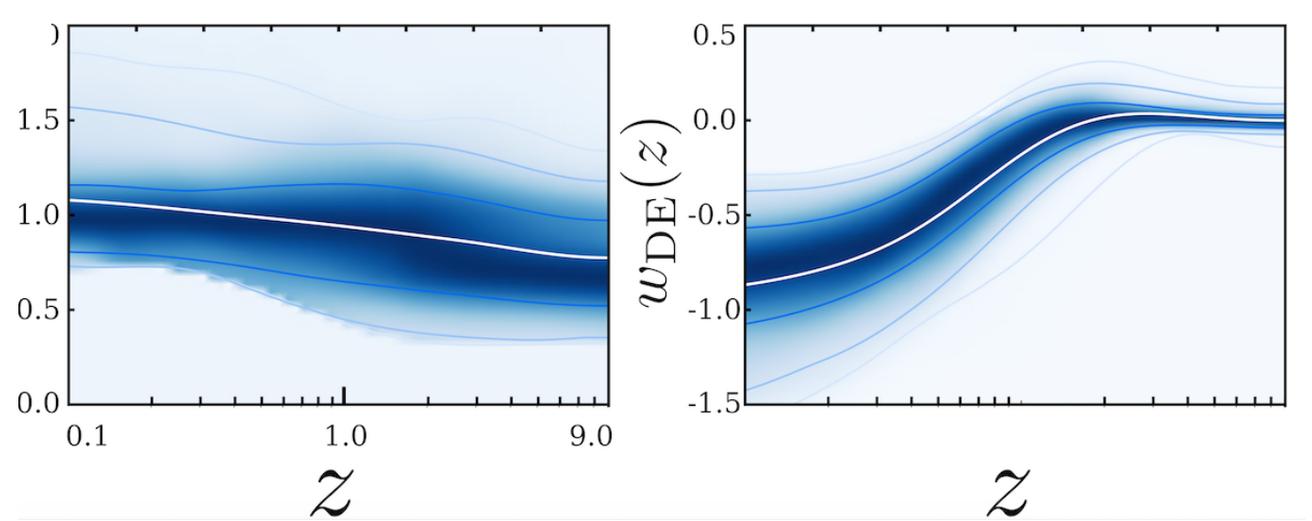
H_S





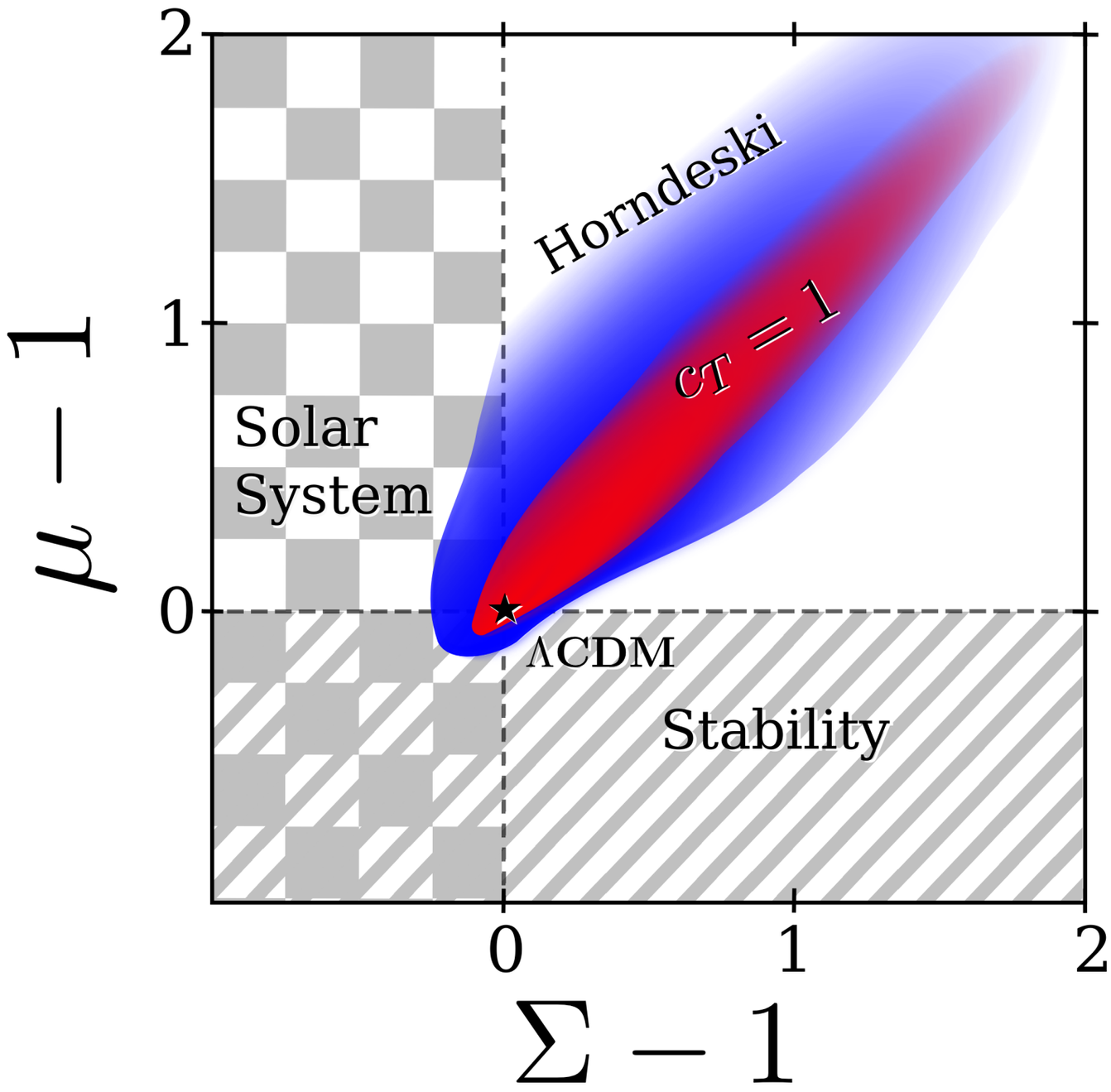
HOR





Espejo et al., PRD 2019

LSS Phenomenology



Peirone, Raveri, Pogosian, AS, Koyama, PRD 2018

Gravitational Waves

So, we are left with:

$$\ddot{h}_{ij} + (3 + \alpha_M)\, H\dot{h}_{ij} + (1 + \cancel{\alpha_T})\, k^2 h_{ij} = 0$$

Amendola et al., JCAP 08 (2018) 030; Belgacem et al., Phys.Rev.D 98 (2018)

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then, inside the horizon

$$h \propto \frac{1}{\tilde{a}} \qquad\qquad \frac{\dot{\tilde{a}}}{\tilde{a}} = \frac{3 + \alpha_M}{2} H$$

$$d_L^{\mathrm{GW}}(z) = \frac{a(z)}{\tilde{a}(z)} d_L^{\mathrm{EM}}(z)$$

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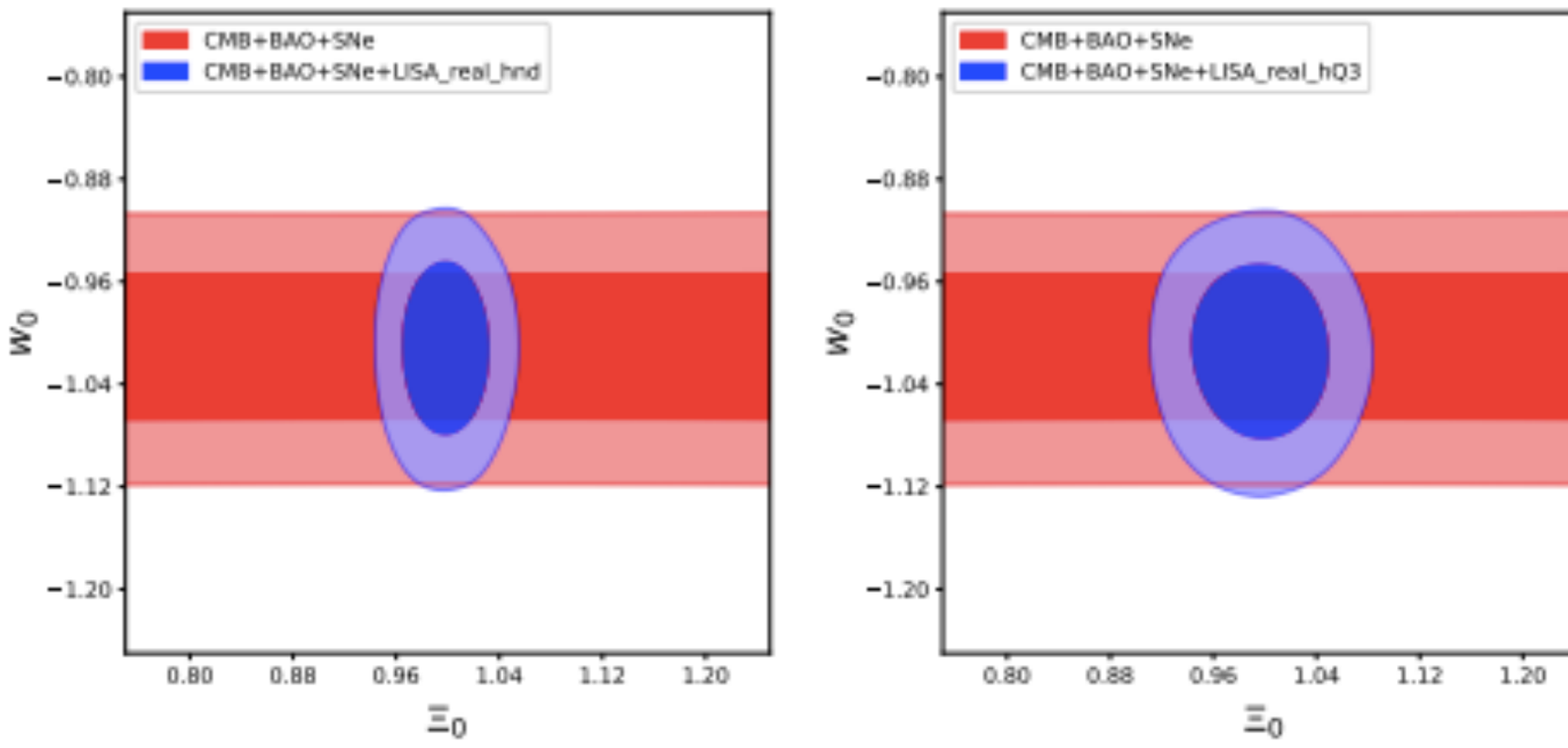
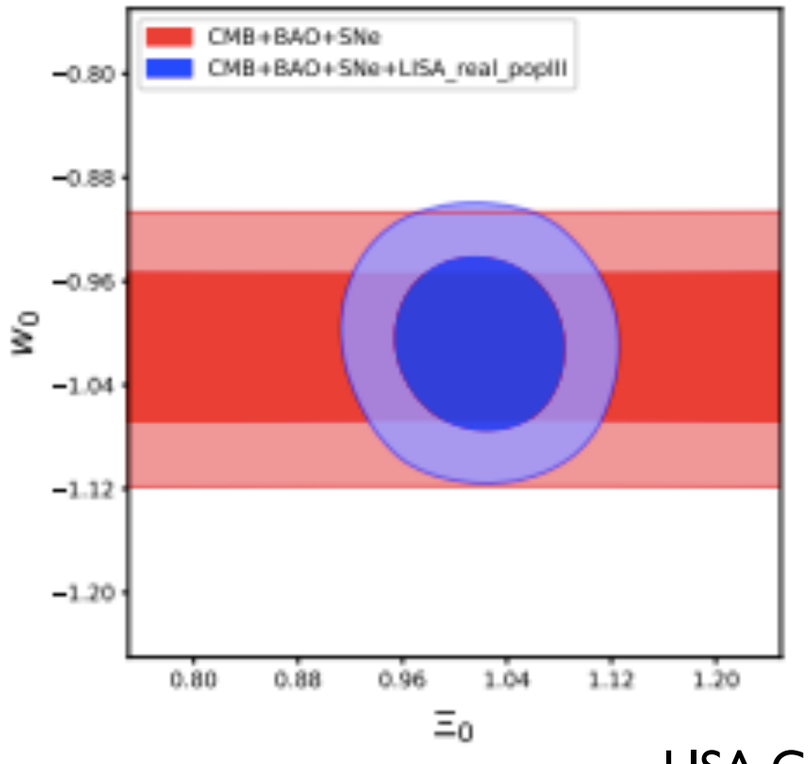


$$\Xi(z) \equiv \frac{d_L^{\mathrm{GW}}(z)}{d_L^{\mathrm{EM}}(z)}$$



LISA Cosmology Working Group, JCAP 07 (2019)

Gravitational Waves

We pick up additional, theory-dependent corrections from inhomogeneities along the line of sight:

$$\frac{\Delta d_L^{\rm SN}}{\bar{d}_L^{\rm SN}} = -\kappa - (\phi+\psi) + \frac{1}{\chi}\int_0^\chi d\tilde{\chi}\,(\phi+\psi) + \phi\left(\frac{1}{\chi\mathcal{H}}\right) + v_\parallel\left(1-\frac{1}{\mathcal{H}\chi}\right)$$
$$-\left(1-\frac{1}{\chi\mathcal{H}}\right)\int_0^\chi d\tilde{\chi}\,(\phi'+\psi')$$

A. Garoffolo et al., JCAP 11 (2020)

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$$-\left(1-\frac{1}{\chi\mathcal{H}}+\frac{M_P'}{\mathcal{H}M_P}\right)\int_0^\chi d\tilde{\chi}\,(\phi'+\psi') + \frac{M_{P,\varphi}}{M_P}\delta\varphi + \frac{M_{P,X}}{M_P}\delta X$$

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$$\Delta_\varphi(\hat{\theta},z) \equiv \frac{\Delta d_L^{\rm SN}(\hat{\theta},z)}{\bar{d}_L^{\rm SN}} - \frac{\Delta d_L^{\rm GW}(\hat{\theta},z)}{\bar{d}_L^{\rm GW}} = \frac{M_P'}{\mathcal{H}M_P}\left(\phi - v_\parallel + \int_0^\chi d\tilde{\chi}\,(\phi'+\psi')\right) - \frac{M_{P,\varphi}}{M_P}\delta\varphi - \frac{M_{P,X}}{M_P}\delta X$$

A.Garoffolo et al., JCAP 11 (2020)

APPENDIX

Super-horizon

Very general arguments based on the conservation of the energy-momentum tensor, imply that adiabatic linear scalar perturbations on super-horizon scales (k/(aH) ≪ 1) obey:

$$\zeta' \equiv (\delta - 3\Phi)' = \mathcal{O}(p^2) \qquad p \equiv \frac{k}{aH}$$

where ζ is the curvature perturbation on hypersurfaces of uniform density. This can be easily derived via the parallel universes approach.

The above condition can be rewritten in terms of an equation for the metric potentials.

$$\Phi'' + \Psi' - \frac{H''}{H'}\Phi' + \left(\frac{H'}{H} - \frac{H''}{H'}\right)\Psi = \mathcal{O}(p^2)$$

Which is basically a consistency relation, telling you that on super-horizon scales you really have the freedom of introducing one function, parametrizing the ratio between the two metric potentials.

Wands et al., Phys. Rev. D62, 043527 (2000)

Super-horizon

Is the parametrization (μ,γ) respecting the super-horizon consistency condition?

Combining the energy-momentum equations for matter with the Poisson equation we get:

$$\Psi'' + \left(2\frac{\gamma'}{\gamma} - \frac{H''}{H'} + \frac{1}{\gamma}\right)\Psi' + \left[\frac{\gamma''}{\gamma} - \frac{H''}{H'}\frac{\gamma'}{\gamma} + \left(\frac{H'}{H} - \frac{H''}{H'}\right)\frac{1}{\gamma}\right]\Psi = \mathcal{O}\left(\frac{p^2}{\mu\gamma}\right)$$

Yes, as long as μγ does not approach zero faster than k/aH, which should be the case. Hence the (μ,γ) parametrization is safe to use on all linear scales, from super- to sub-horizon. It will of course stop being valid in the non-linear regime.