High-order homogenization in optimal control by the Bloch wave method

MATHLEC-2021

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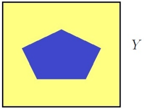
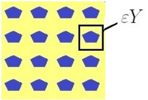
Classical homgenization problem

Unit cell $Y = [0,1]^n$



Periodic medium



Heterogeneous equation

$$-\nabla \cdot (A^\varepsilon(x)\nabla u_\varepsilon(x)) + u_\varepsilon(x) = f(x) \text{ in } \mathbb{R}^n$$

with $f \in L^2(\mathbb{R}^n)$ and $A^\varepsilon(x) := a_Y\left(\frac{x}{\varepsilon}\right)$.

- $a_Y \in L^\infty(\mathbb{R}^n, \mathbb{R}^{n\times n})$ is Y-periodic and elliptic

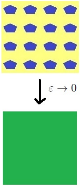
Classical question: What is the effective behavior of u^ε as $\varepsilon \to 0$?

Classical homogenization theorem

The weak limit u with $u_\varepsilon \rightharpoonup u$ in $H^1(\mathbb{R}^n)$ solves

$$-\nabla \cdot (A^*\nabla u(x)) + u(x) = f(x) \text{ in } \mathbb{R}^n$$

with constant **effective coefficient** $A^* \in \mathbb{R}^{n\times n}$.



A classical corrector result

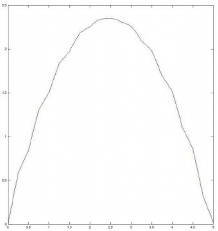
A^* is given in terms of a_Y and periodic **first order correctors** $\chi_j^{(1)}$.

Formula for A^*

$$A^*_{ij} = \int_Y [a_Y(y)(e_j + \nabla\chi_j^{(1)}(y))] \cdot e_i \, dy$$

Cell problem for the corrector

$$\nabla \cdot \Big(a_Y(y)(e_j + \nabla\chi_j^{(1)}(y)) \Big) = 0 \text{ in } Y$$



One can not expect strong convergence in H^1 or convergence rates in L^2 better than ε^α with $\alpha < 1$!

Corrector result

$$\|u_\varepsilon - (u + \varepsilon\chi^{(1)}\left(\tfrac{\cdot}{\varepsilon}\right) \cdot \nabla u)\|_{L^2(\mathbb{R}^n)} \le C\varepsilon^2$$

Problem: Adding a second order corrector $\varepsilon^2\chi^{(2)}(\frac{x}{\varepsilon}) \cdot \nabla^2 u(x)$ will not improve the L^2-rate!

Higher order estimates

Classical two-scale expansion: $u_\varepsilon(x) \sim \sum_{n\geq 0} \varepsilon^n u_n(x, \frac{x}{\varepsilon})$

Bakhvalov, Panasenko '89

1) $u_0(x,y) = u(x)$ with homogenized solution u.
2) Define $\tilde{u}_k(x) := \int_Y u_k(x,y)\,dy$. Then
$$-\nabla\cdot(A^*\nabla\tilde{u}_n) = \mathbb{B}^*_{n+2}\nabla^{n+2}u + \sum_{k=1}^{n-1} \mathbb{B}^*_{n+2-k}\nabla^{n+2-k}\tilde{u}_k$$
3) Define $U_\varepsilon(x) := u(x) + \sum_{n\geq 2}\varepsilon^n \tilde{u}_n(x)$. Then
$$u_\varepsilon(x) \sim \sum_{n\geq 0} \varepsilon^n \chi^{(n)}(\tfrac{x}{\varepsilon}) \cdot \nabla^n U_\varepsilon(x)$$

Separation of

- Macroscopic profile U_ε
- Highly oscillatory correctors $\chi^{(n)}$

Higher order estimates: Truncate the series in 3) at $n = N$ to obtain L^2-error of order ε^{N+1}.

Optimal control

General form

Energy: min $J^\varepsilon(\theta_\varepsilon, u_\varepsilon)$

Admissible set: $\theta_\varepsilon \in U_{ad} \subset L^2(\mathbb{R}^n)$

Constraint: $-\nabla \cdot (A^\varepsilon(x)\nabla u_\varepsilon(x)) + u_\varepsilon(x) = \theta_\varepsilon(x)$ in $\mathbb{R}^n$

- J^ε convex and continuous
- $U_{ad} \subset L^2(\mathbb{R}^d)$ convex and closed
- θ_ε^* optimal control with solution u_ε^*

Homogenization result. Weak convergens methods provide

1) $\theta_\varepsilon^* \rightharpoonup \theta^* \in U_{ad}$ and $u_\varepsilon^* \rightharpoonup u^*$ in $H^1(\mathbb{R}^n)$.
2) (θ^*, u^*) solves effective optimal control problem.

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Homogenization result. Weak convergens methods provide

1) $\theta_\varepsilon^* \rightharpoonup \theta^* \in U_{ad}$ and $u_\varepsilon^* \rightharpoonup u^*$ in $H^1(\mathbb{R}^n)$.
2) (θ^*, u^*) solves effective optimal control problem.

Aim: Find (θ_M^*, u_M^*) with $\|\theta_\varepsilon^* - \theta_M^*\|_{L^2\mathbb{R}^n} + \|u_\varepsilon^* - u_M^*\|_{L^2(\mathbb{R}^n)} \leq C\varepsilon^{2M}$.

Problem: Truncating the asymptotic expansion of θ_ε^* and u_ε^* is a bad idea.

Example: $U_{ad} := \{\theta_\varepsilon \in L^2(\mathbb{R}^n) \,|\, \|u_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq L\}$

$$u_M^* := u^* + \sum_{n=1}^{2M-1} u_n(x, \tfrac{x}{\varepsilon}) \;\not\Rightarrow\; \|u_M^*\|_{L^2(\mathbb{R}^n)} \leq L$$

Bloch expansions in a nutshell

1.) Fourier transform of $u : \mathbb{R}^n \to \mathbb{C}$:

$$u(x) = \int_{\mathbb{R}^n} \hat{u}(\xi) e^{2\pi \mathrm{i} \xi \cdot x} \, d\xi$$

2.) Write ξ as $\xi = k + \eta$ with $k \in \mathbb{Z}^n$ and $\eta \in Z := [-\frac{1}{2}, \frac{1}{2})^n$:

$$u(x) = \int_Z \underbrace{\sum_{k \in \mathbb{Z}^n} \hat{u}(k + \eta) e^{2\pi \mathrm{i} k \cdot x}}_{=:F=F(x;\eta)} \, e^{2\pi \mathrm{i} \eta \cdot x} \, d\eta$$

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3.) Expand F in periodic eigenfunctions $\Phi_m(x;\eta)$ of $-(\nabla + 2\pi i \eta) \cdot (a_Y(\cdot)(\nabla + 2\pi i \eta))$:

$$F(x;\eta) = \sum_{m \in \mathbb{N}_0} \hat{u}_m(\eta) \Phi_m(x;\eta)$$

$w_m(x;\eta) = \Phi_m(x;\eta) e^{2\pi i \eta \cdot x}$ solves

$$-\nabla \cdot (a_Y(x) \nabla w_m(x;\eta)) = \lambda_m(\eta) \, w_m(x;\eta)$$

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$$-\nabla \cdot (a_Y(x) \nabla w_m(x;\eta)) = \lambda_m(\eta) \, w_m(x;\eta)$$

Result: The operator $\mathcal{L} = -\nabla \cdot (a_Y(\cdot)\nabla)$ acts as a multiplier:

$$\mathcal{L}u = \mathcal{L} \int_Z \sum_{m \in \mathbb{N}_0} \hat{u}_m(\eta) w_m(x;\eta) \, d\eta = \int_Z \sum_{m \in \mathbb{N}_0} \hat{u}_m(\eta) \lambda_m(\eta) w_m(x;\eta) \, d\eta$$

Bloch adaption

Rescaled Bloch expansion

$$u(x) = \int_{Z/\varepsilon} \sum_{m\in\mathbb{N}_0} \hat{u}_m^\varepsilon(\eta)\omega_m^\varepsilon(x;\eta)\,d\eta$$

$$\boxed{\begin{gathered} w_m^\varepsilon(x,\eta) := w_m(x/\varepsilon, \varepsilon\eta) \\ \lambda_m^\varepsilon(\eta) := \tfrac{1}{\varepsilon^2}\lambda_m(\varepsilon\eta) \end{gathered}}$$

The oscillatory operator $\mathcal{L}^\varepsilon := -\nabla\cdot(A^\varepsilon(\cdot)\nabla)$ acts as a multiplier:

$$\mathcal{L}^\varepsilon u = \mathcal{L}^\varepsilon \int_{Z/\varepsilon} \sum_{m\in\mathbb{N}_0} \hat{u}_m^\varepsilon(\eta)\omega_m^\varepsilon(x;\eta)\,d\eta = \int_{Z/\varepsilon} \sum_{m\in\mathbb{N}_0} \hat{u}_m^\varepsilon(\eta)\lambda_m^\varepsilon(\eta)\omega_m^\varepsilon(x;\eta)\,d\eta$$

Bloch adaption

Rescaled Bloch expansion

$$u(x) = \int_{Z/\varepsilon} \sum_{m\in\mathbb{N}_0} \hat{u}_m^\varepsilon(\eta)\omega_m^\varepsilon(x;\eta)\,d\eta$$

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Question: Analogue for adding highly oscillatory correctors?

Bloch adaption (Conca,Orive,Vanninathan '02)

Let $u \in L^2(\mathbb{R}^n;\mathbb{C})$ with Fourier transform $\hat{u}$.

$$\mathcal{A}^\varepsilon(u) := \int_{Z/\varepsilon} \hat{u}(\eta)\omega_0^\varepsilon(\cdot;\eta)\,d\eta$$

For u sufficiently smooth:

$$\mathcal{A}^\varepsilon(u) = u + \varepsilon\chi^{(1)}(\tfrac{\cdot}{\varepsilon})\cdot\nabla u + \varepsilon^2\chi^{(2)}(\tfrac{\cdot}{\varepsilon})\cdot\nabla^2 u + O(\varepsilon^3)$$

Note: For $\mathrm{supp}(\hat{u}) \subset K \subset Z/\varepsilon$ one has

$$\|\mathcal{A}^\varepsilon(u)\|_{L^2(R^n)} = \|\hat{u}\|_{L^2(Z/\varepsilon)} = \|\hat{u}\|_{L^2(K)} = \|u\|_{L^2(\mathbb{R}^n)}$$

Homogenization problem

Oscillatory optimal control problem

Energy: min $J^\varepsilon(\theta_\varepsilon, u_\varepsilon)$

Admissible set: $\theta_\varepsilon \in U^\varepsilon_{ad} \subset L^2(\mathbb{R}^n)$

Constraint: $-\nabla \cdot (A^\varepsilon(x)\nabla u_\varepsilon(x)) + u_\varepsilon(x) = \theta_\varepsilon(x)$ in $\mathbb{R}^n$

- Admissible set U^ε_{ad} may depend on ε $\rightarrow$ next slide

1) For fixed constants $\mu_1, \mu_2 \geq 0$ and $\kappa > 0$

$$J^\varepsilon(\theta_\varepsilon, u_\varepsilon) = \frac{\mu_1}{2} E^\varepsilon(u_\varepsilon - u^\varepsilon_{d,1}) + \frac{\mu_2}{2}\|u_\varepsilon - u^\varepsilon_{d,2}\|^2_{L^2(\mathbb{R}^n)} + \frac{\kappa}{2}\|\theta^\varepsilon\|^2_{L^2(\mathbb{R}^n)}$$

2) E^ε is the energy corresponding to the oscillatory PDE-constraint,

$$E^\varepsilon(u) = \int_{\mathbb{R}^n} \nabla u(x) \cdot A^\varepsilon(x)\nabla u(x) + |u(x)|^2\, dx$$

3) For $u_{d,1}, u_{d,2}$ with **compact support** K in Fourier space

$$u^\varepsilon_{d,1} = \mathcal{A}^\varepsilon(u_{d,1}) \text{ and } u^\varepsilon_{d,2} = \mathcal{A}^\varepsilon(u_{d,2})$$

Remark: Well-prepared data are important for corrector results!

Admissible set

Assumptions

1) $U^\varepsilon_{ad} \subset L^2(\mathbb{R}^n)$ convex and closed.
2) There exists $\theta_0 \in L^2(\mathbb{R}^n)$ such that $\mathcal{A}^\varepsilon(\theta_0) \in U^\varepsilon_{ad}$ for all $\varepsilon > 0$.
3) For every $\theta \in U^\varepsilon_{ad}$ with $\theta(x) = \int_{Z/\varepsilon} \sum_{m\in\mathbb{N}_0} \hat{\theta}^\varepsilon_m(\eta)\omega^\varepsilon_m(x;\eta)\, d\eta$

$$\theta_K := \int_K \hat{\theta}^\varepsilon_0(\eta)\omega^\varepsilon_0(\cdot;\eta)\, d\eta \in U^\varepsilon_{ad}.$$

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$$\theta_K := \int_K \hat{\theta}^\varepsilon_0(\eta)\omega^\varepsilon_0(\cdot;\eta)\, d\eta \in U^\varepsilon_{ad}.$$

Example: $U^\varepsilon_{ad} = \{\theta \in L^2(\mathbb{R}^n) \,|\, \|\theta\|_{L^2(\mathbb{R}^n)} \leq L\}$

2) Choose $\theta_0 = 0$.
3) $\|\theta_K\|_{L^2(\mathbb{R}^n)} = \|\hat{\theta}^\varepsilon_0\|_{L^2(K)} \leq \sum_{m\in\mathbb{N}_0} \|\hat{\theta}^\varepsilon_m\|_{L^2(Z/\varepsilon)} = \|\theta\|_{L^2(\mathbb{R}^n)} \leq L$

Other examples:

$U^\varepsilon_{ad} = \{\theta \in L^2(\mathbb{R}^n) \,|\, \|u_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq L\}$ or $U^\varepsilon_{ad} = \{\theta \in L^2(\mathbb{R}^n) \,|\, E^\varepsilon(u_\varepsilon) \leq L\}$

M-th order effective problem and main result

1) M-th order effective energy

$$J^*_M(\theta,u) = \tfrac{\mu_1}{2} E^*_M(u-u_{d,1}) + \tfrac{\mu_2}{2}\|u-u_{d,2}\|^2_{L^2(\mathbb{R}^n)} + \tfrac{\kappa}{2}\|\theta\|^2_{L^2(\mathbb{R}^n)}$$

with $E^*_M(u) = \int_{\mathbb{R}^n}\left(\sum_{k=1}^M \varepsilon^{2k-2}\langle\nabla^k u,\nabla^k u\rangle_{\mathbb{A}^*_{2k}}\right) + |u|^2$

$$\boxed{\begin{aligned} u^\varepsilon_{d,1} &= \mathcal{A}^\varepsilon(u_{d,1}) \\ u^\varepsilon_{d,2} &= \mathcal{A}^\varepsilon(u_{d,2}) \end{aligned}}$$

- Constant tensors $\mathbb{A}^*_{2k}$ of order $2k$
- $\mathbb{A}^*_2 = A^*$ classical effective matrix

2) Effective PDE-constraint

$$\left(\sum_{k=1}^M \varepsilon^{2k-2}(-1)^k \mathbb{A}^*_{2k}\nabla^{2k}u\right) + u = \theta$$

3) Effective admissible set

$$U^*_{ad} = \left\{\theta \in (\mathcal{A}^\varepsilon)^{-1}(U^\varepsilon_{ad}) \,|\, \mathrm{supp}(\hat\theta) \subset K\right\}$$

M-th order effective problem and main result

1) M-th order effective energy

$$J_M^*(\theta,u) = \tfrac{\mu_1}{2} E_M^*(u-u_{d,1}) + \tfrac{\mu_2}{2}\|u-u_{d,2}\|^2_{L^2(\mathbb{R}^n)} + \tfrac{\kappa}{2}\|\theta\|^2_{L^2(\mathbb{R}^n)}$$

with $E_M^*(u) = \int_{\mathbb{R}^n}\left(\sum_{k=1}^M \varepsilon^{2k-2}\langle\nabla^k u,\nabla^k u\rangle_{\mathbb{A}^*_{2k}}\right) + |u|^2$

$$u^\varepsilon_{d,1} = \mathcal{A}^\varepsilon(u_{d,1})$$
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$$\left(\sum_{k=1}^M \varepsilon^{2k-2}(-1)^k \mathbb{A}^*_{2k}\nabla^{2k}u\right) + u = \theta$$

3) Effective admissible set

$$U^*_{ad} = \left\{\theta \in (\mathcal{A}^\varepsilon)^{-1}(U^\varepsilon_{ad}) \,|\, \mathrm{supp}(\hat\theta) \subset K\right\}$$

Error estimate: Let (θ^*_M, u^*_M) be the solution to the above effective problem.

$$\|\theta^*_\varepsilon - \mathcal{A}^\varepsilon(\theta^*_M)\|_{L^2(\mathbb{R}^n)} + \|u^*_\varepsilon - \mathcal{A}^\varepsilon(u^*_M)\|_{L^2(\mathbb{R}^n)} \le C\varepsilon^{2M}$$

Example: $U^\varepsilon_{ad} = \{\|\theta\|_{L^2(\mathbb{R}^n)} \le L\} \Rightarrow U^*_{ad} = \{\|\theta\|_{L^2(\mathbb{R}^n)} \le L,\ \mathrm{supp}(\hat\theta) \subset K\}$

Outline of the proof

1) Expand the optimal control θ_ε^* and u_ε^* in Bloch waves. Show that θ_ε^* and u_ε^* are adaptions,

$$\theta_\varepsilon^* = \mathcal{A}^\varepsilon(\tilde{\theta}_\varepsilon^*) \text{ and } u_\varepsilon^* = \mathcal{A}^\varepsilon(\tilde{u}_\varepsilon^*).$$

2) Bloch analysis: Use analyticity of the first Bloch-eigenvalue λ_0^ε to derive the M-th order effective problem.

3) Error estimate

$$\|\tilde{\theta}_\varepsilon^* - \theta_M^*\|_{L^2(\mathbb{R}^n)} + \|\tilde{u}_\varepsilon^* - u_M^*\|_{L^2(\mathbb{R}^n)} \leq C\varepsilon^{2M}$$

by means of a variational inequality in Bloch/Fourier-space.

Important fact: Both minimization problems are constraint by the same admissible set.

Expansion of the solution

$$\theta^*_\varepsilon(x) = \int_{Z/\varepsilon} \sum_{m\in\mathbb{N}_0} \hat{\theta}^*_{\varepsilon,m}(\eta)\omega^\varepsilon_m(x;\eta)\,d\eta$$

$$u^*_\varepsilon(x) = \int_{Z/\varepsilon} \sum_{m\in\mathbb{N}_0} \hat{u}^*_{\varepsilon,m}(\eta)\omega^\varepsilon_m(x;\eta)\,d\eta$$

PDE-constraint

$$-\nabla\cdot(A^\varepsilon(x)\nabla u^*_\varepsilon) + u^*_\varepsilon = \theta^*_\varepsilon$$

$$\Rightarrow \hat{u}^*_{\varepsilon,m}(\eta) = \frac{\hat{\theta}^*_{\varepsilon,m}(\eta)}{1+\lambda^\varepsilon_m(\eta)}$$

Expansion of the solution

$$\theta^*_\varepsilon(x) = \int_{Z/\varepsilon} \sum_{m\in\mathbb{N}_0} \hat{\theta}^*_{\varepsilon,m}(\eta)\omega^\varepsilon_m(x;\eta)\,d\eta$$

$$u^*_\varepsilon(x) = \int_{Z/\varepsilon} \sum_{m\in\mathbb{N}_0} \hat{u}^*_{\varepsilon,m}(\eta)\omega^\varepsilon_m(x;\eta)\,d\eta$$

PDE-constraint

$$-\nabla\cdot(A^\varepsilon(x)\nabla u^*_\varepsilon) + u^*_\varepsilon = \theta^*_\varepsilon$$

$$\Rightarrow \hat{u}^*_{\varepsilon,m}(\eta) = \frac{\hat{\theta}^*_{\varepsilon,m}(\eta)}{1+\lambda^\varepsilon_m(\eta)}$$

Step 1: Reduction of the expansion

$\hat{\theta}^*_{\varepsilon,m}(\eta) = 0$ for $m \geq 1$ or $\eta \notin K$

Proof:

- $J^\varepsilon(\theta_\varepsilon, u_\varepsilon) = \frac{\mu_1}{2}E^\varepsilon(u_\varepsilon - u^\varepsilon_{d,1}) + \frac{\mu_2}{2}\|u_\varepsilon - u^\varepsilon_{d,2}\|^2_{L^2(\mathbb{R}^n)} + \frac{\kappa}{2}\|\theta^\varepsilon\|^2_{L^2(\mathbb{R}^n)}$
- Bloch coefficients of $u^\varepsilon_{d,1}$, $u^\varepsilon_{d,2}$ vanish for $m \geq 1$ or $\eta \notin K$

$$\Rightarrow J^\varepsilon(\underbrace{(\theta_\varepsilon)_K}_{\in U^\varepsilon_{ad}}, (u_\varepsilon)_K) \leq J^\varepsilon(\theta_\varepsilon, u_\varepsilon)$$

Bloch analysis

$$\theta_\varepsilon^*(x) = \int_K \hat{\theta}_{\varepsilon,0}^*(\eta)\omega_0^\varepsilon(x;\eta)\,d\eta = \mathcal{A}^\varepsilon(\tilde{\theta}_\varepsilon^*)$$
$$u_\varepsilon^*(x) = \int_K \frac{\hat{\theta}_{\varepsilon,0}^*(\eta)}{1+\lambda_0^\varepsilon(\eta)}\omega_0^\varepsilon(x;\eta)\,d\eta = \mathcal{A}^\varepsilon(\tilde{u}_\varepsilon^*)$$

with

$$\boxed{\begin{aligned}\tilde{\theta}_\varepsilon^*(x) &= \int_K \hat{\theta}_{\varepsilon,0}^*(\eta)e^{2\pi i\eta\cdot x}\,d\eta \\ \tilde{u}_\varepsilon^*(x) &= \int_K \frac{\hat{\theta}_{\varepsilon,0}^*(\eta)}{1+\lambda_0^\varepsilon(\eta)}e^{2\pi i\eta\cdot x}\,d\eta\end{aligned}}$$

Taylor expansion of λ_0^ε

$$\lambda_0^\varepsilon(\eta) = \underbrace{(2\pi)^2\mathbb{A}_2^*\cdot\eta^{\otimes 2} + \varepsilon^2(2\pi)^4\mathbb{A}_4^*\cdot\eta^{\otimes 4} + \cdots + \varepsilon^{2M-2}(2\pi)^{2M}\mathbb{A}_{2M}^*\cdot\eta^{\otimes 2M}}_{=:P_M(\eta)} + O(\varepsilon^{2M})$$

Approximation: $\tilde{u}_{M,\text{app}}^*(x) := \int_K \frac{\hat{\theta}_{\varepsilon,0}^*(\eta)}{1+P_M(\eta)}e^{2\pi i\eta\cdot x}\,d\eta$

Error estimate

$$\|\tilde{u}_\varepsilon^* - \tilde{u}_{M,\text{app}}^*\|_{L^2(\mathbb{R}^n)} \le C\varepsilon^{2M}$$

Observation:

$$\left(\sum_{k=1}^M \varepsilon^{2k-2}(-1)^k\mathbb{A}_{2k}^*\nabla^{2k}\tilde{u}_{M,\text{app}}^*\right) + \tilde{u}_{M,\text{app}}^* = \tilde{\theta}_\varepsilon^*$$

Problem: $\tilde{\theta}_\varepsilon^*$ is related to the original optimal control, not the effective one!

Reformulation in Bloch/Fourier space

1. **Original energy:**

$$J^\varepsilon(\theta_\varepsilon, u_\varepsilon) = \tfrac{\mu_1}{2} E^\varepsilon(u_\varepsilon - u^\varepsilon_{d,1}) + \tfrac{\mu_2}{2}\|u_\varepsilon - u^\varepsilon_{d,2}\|^2_{L^2(\mathbb{R}^n)} + \tfrac{\kappa}{2}\|\theta^\varepsilon\|^2_{L^2(\mathbb{R}^n)}$$

For $\theta_\varepsilon(x) = (\theta_\varepsilon)_K(x) = \int_K \hat\theta_{\varepsilon,0}(\eta)\omega_0^\varepsilon(x;\eta)\,d\eta$ one finds

$$\begin{aligned} J^\varepsilon(\theta_\varepsilon, u_\varepsilon) =& \hat J^\varepsilon(\hat\theta_{\varepsilon,0}) \\ =& \frac{\mu_1}{2}\int_K \left|\frac{\hat\theta_{\varepsilon,0}(\eta)}{1+\lambda_0^\varepsilon(\eta)} - \hat u_{d,1}(\eta)\right|^2 (1+\lambda_0^\varepsilon(\eta))\,d\eta \\ &+ \frac{\mu_2}{2}\int_K \left|\frac{\hat\theta_{\varepsilon,0}(\eta)}{1+\lambda_0^\varepsilon(\eta)} - \hat u_{d,2}(\eta)\right|^2 d\eta + \frac{\kappa}{2}\int_K |\hat\theta_{\varepsilon,0}(\eta)|^2\,d\eta \end{aligned}$$

2. **Effective energy:** For $\theta(x) = \int_K \hat\theta(\eta) e^{2\pi\mathrm{i}\cdot x}\,d\eta$ one finds

$$\begin{aligned} J^*_M(\theta, u) =& \hat J^*_M(\hat\theta) \\ =& \frac{\mu_1}{2}\int_K \left|\frac{\hat\theta(\eta)}{1+P_M(\eta)} - \hat u_{d,1}(\eta)\right|^2 (1+P_M(\eta))\,d\eta \\ &+ \frac{\mu_2}{2}\int_K \left|\frac{\hat\theta(\eta)}{1+P_M(\eta)} - \hat u_{d,2}(\eta)\right|^2 d\eta + \frac{\kappa}{2}\int_K |\hat\theta(\eta)|^2\,d\eta \end{aligned}$$

Admissible sets in Bloch/Fourier space

Lemma

Let $\hat{\theta} \in L^2(K;\mathbb{C})$. Consider

1) $\theta_B(x) := \int_K \hat{\theta}(\eta)\omega_0^\varepsilon(x;\eta)\,d\eta$
2) $\theta_F(x) := \int_K \hat{\theta}(\eta)e^{2\pi i\eta\cdot x}\,d\eta$

Then $\theta_B \in (U_{ad}^\varepsilon)_K \;\Leftrightarrow\; \theta_F \in U_{ad}^*$.

Effective admissible set:

$$U_{ad}^* = \left\{\theta \in (\mathcal{A}^\varepsilon)^{-1}(U_{ad}^\varepsilon)\,\middle|\,\mathrm{supp}(\hat{\theta}) \subset K\right\}$$

Proof: Since by definition $\theta_B = \mathcal{A}^\varepsilon(\theta_F)$,

$$\theta_F \in U_{ad}^* \Leftrightarrow \theta_F = \int_K \hat{\theta}(\eta)e^{2\pi i\eta\cdot x}\,d\eta \in (\mathcal{A}^\varepsilon)^{-1}(U_{ad}^\varepsilon) \;\Leftrightarrow\; \theta_B \in (U_{ad}^\varepsilon)_K$$

Conclusion: The minimization problems for $\hat{J}^\varepsilon$ and $\hat{J}_M^*$ are constrained by the same set

$$\hat{U}_{ad} := \{\hat{\theta} \in L^2(K;\mathbb{C})\,|\,\theta_F \in U_{ad}^*\} = \{\hat{\theta} \in L^2(K;\mathbb{C})\,|\,\theta_B \in (U_{ad}^\varepsilon)_K\}$$

Estimate for the optimal controls

Theorem

The minimizers $\hat{\theta}^*_{\varepsilon,0}$ and $\hat{\theta}^*_M$ satisfy

$$\|\hat{\theta}^*_{\varepsilon,0} - \hat{\theta}^*_M\|_{L^2(K;\mathbb{C})} \le C\varepsilon^{2M}$$

$$\hat{\theta}^*_{\varepsilon,0} = \operatorname*{argmin}_{\hat{\theta}\in\hat{U}_{ad}} \hat{J}^\varepsilon(\hat{\theta})$$

$$\hat{\theta}^*_M = \operatorname*{argmin}_{\hat{\theta}\in\hat{U}_{ad}} \hat{J}^*_M(\hat{\theta})$$

Proof: Apply the variational inequality for $\hat{J}^\varepsilon$ to $\hat{\theta}^*_M$ and for $\hat{J}^*_M$ to $\hat{\theta}^*_{\varepsilon,0}$:

$$\kappa\|\hat{\theta}^*_{\varepsilon,0} - \hat{\theta}^*_M\|_{L^2(K;\mathbb{C})} \le \tilde{C}\varepsilon^{2M}\left(\|\hat{\theta}^*_{\varepsilon,0}\|_{L^2(K;\mathbb{C})} + \|\hat{u}_{d,2}\|_{L^2(K;\mathbb{C})}\right) \le C\varepsilon^{2M},$$

where the factor $\tilde{C}\varepsilon^{2M}$ is due to $\sup_{\eta\in K}|\lambda_0^\varepsilon(\eta) - P_M(\eta)| \le \tilde{C}\varepsilon^{2M}$.

Corollary

1) $\|\tilde{\theta}^*_\varepsilon - \theta^*_M\|_{L^2(\mathbb{R}^n)} \le C\varepsilon^{2M}$
2) $\|\theta^*_\varepsilon - \mathcal{A}^\varepsilon(\theta^*_M)\|_{L^2(\mathbb{R}^n)} \le C\varepsilon^{2M}$

- $\theta^*_\varepsilon = \mathcal{A}^\varepsilon(\tilde{\theta}^*_\varepsilon)$ with $\tilde{\theta}^*_\varepsilon(x) = \int_K \hat{\theta}^*_{\varepsilon,0}(\eta)e^{2\pi i\eta\cdot x}\,d\eta$
- $\theta^*_M = \int_K \hat{\theta}^*_M(\eta)e^{2\pi i\eta\cdot x}\,d\eta$ effective optimal control

Estimate for the solutions

Recall:

- $\theta^*_\varepsilon = \mathcal{A}^\varepsilon(\tilde\theta^*_\varepsilon)$ and $u^*_\varepsilon = \mathcal{A}^\varepsilon(\tilde u^*_\varepsilon)$
- First approximation: $\|\tilde u^*_\varepsilon - \tilde u^*_{M,\mathsf{app}}\|_{L^2(\mathbb{R}^n)} \le C\varepsilon^{2M}$, where

$$\Big(\sum_{k=1}^{M} \varepsilon^{2k-2}(-1)^k \mathbb{A}^*_{2k} \nabla^{2k} \tilde u^*_{M,\mathsf{app}}\Big) + \tilde u^*_{M,\mathsf{app}} = \tilde\theta^*_\varepsilon$$

Theorem

Let u^*_M be the solution to $\left(\sum_{k=1}^{M} \varepsilon^{2k-2}(-1)^k \mathbb{A}^*_{2k} \nabla^{2k} \tilde u^*_M\right) + \tilde u^*_M = \theta^*_M$.

1) $\|\tilde u^*_\varepsilon - u^*_M\|_{L^2(\mathbb{R}^n)} \le C\varepsilon^{2M}$
2) $\|u^*_\varepsilon - \mathcal{A}^\varepsilon(u^*_M)\|_{L^2(\mathbb{R}^n)} \le C\varepsilon^{2M}$

Proof: Use that $\|\tilde\theta^*_\varepsilon - \theta^*_M\|_{L^2(\mathbb{R}^n)} \le C\varepsilon^{2M}$ and exploit a priori estimate for the effective equation.

Well-posedness of the effective problem

M-th order effective equation:

$$\left(-\mathbb{A}_2^*\nabla^2 u + \varepsilon^2\mathbb{A}_4^*\nabla^4 u + \cdots + (-1)^M\varepsilon^{2M-2}\mathbb{A}_{2M}^*\nabla^{2M} u\right) + u = \theta$$

Solution in Fourier space:

$$u(x) = \int_{\mathbb{R}^n} \frac{\hat{\theta}(\eta)}{1 + P_M(\eta)} e^{2\pi i \eta\cdot x}\, d\eta$$

Problem: $\mathbb{A}_4^*$ is negative semidefinite (Conca,Orive,Vanninathan '06)!

$\Rightarrow$ The effective equation is not well-posed in general!

Lemma

Let $K \subset \mathbb{R}^n$ be compact and $M \in \mathbb{N}$ fixed. Then for every $\varepsilon < \varepsilon_M$

$$P_M(\eta) \geq \frac{\lambda}{2}|\eta^2| \text{ for all } \eta \in K,$$

where λ is the ellipticity constant of $(2\pi)^2\mathbb{A}_2^*$.

- For ε below ε_M and $\operatorname{supp}(\theta) \subset K$ the equation is well posed
- A priori estimate for $\|u\|_{H^1(\mathbb{R}^n)}$ uniformly in ε.

Substitution trick

Second order effective equation

$-\mathbb{A}_2^*\nabla^2 u(x) + \varepsilon^2\mathbb{A}_4^*\nabla^4 u + u = \theta$

- $\mathbb{A}_2^*$ positive definite
- $\mathbb{A}_4^*$ negative semidefinite

Substitution trick

1-D: Replace $\varepsilon^2\mathbb{A}_4^*\partial_x^4 u = \varepsilon^2\frac{\mathbb{A}_4^*}{\mathbb{A}_2^*}\partial_x^2\left(\mathbb{A}_2^*\partial_x^2 u\right)$ to lowest order in ε by

Formally: $\varepsilon^2\mathbb{A}_4^*\partial_x^4 u = \varepsilon^2\frac{\mathbb{A}_4^*}{\mathbb{A}_2^*}\partial_x^2\left(\mathbb{A}_2^*\partial_x^2 u\right) = \varepsilon^2\frac{\mathbb{A}_4^*}{\mathbb{A}_2^*}\partial_x^2\left(u-\theta\right) + O(\varepsilon^4)$

Well-posed effective equation

$\left(-\mathbb{A}_2^* + \varepsilon^2\frac{\mathbb{A}_4^*}{\mathbb{A}_2^*}\right)\partial_x^2 u + u = \theta + \varepsilon^2\frac{\mathbb{A}_4^*}{\mathbb{A}_2^*}\partial_x^2\theta$

- The effective energy has to be transformed, too.

n-D: Decomposition $\mathbb{A}_4^*\nabla^4 = -\mathbb{B}_2^*\nabla^2(\mathbb{A}_2^*\nabla^2) + \mathbb{B}_4^*\nabla^4$ with $\mathbb{B}_2^*, \mathbb{B}_4^* \geq 0$.

Effective equation: $(-\mathbb{A}_2^* + \varepsilon^2\mathbb{B}_2^*)\nabla^2 u + \varepsilon^2\mathbb{B}_4^*\nabla^4 u + u = (\mathsf{Id} - \varepsilon^2\mathbb{B}_2^*\nabla^2)\theta$

Conclusion

- Derivation of a high order effective optimal control problem via Bloch analysis
- Bloch adaption of the effective solution required to capture the high order oscillations
- Effective admissible set given as inverse image under the Bloch adaption

[1] A.Lamacz-Keymling, I. Yousept: High-order homogenization in optimal control by the Bloch wave method. Submitted (2020)

Thank you!