Inner functions revisited.

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Contains joint work in progress with Mahendra Nadkarni

Happy Birthday SG!

1. Inner functions on the disc

Inner function of $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$:
analytic function $\phi : \mathbb{D} \hookleftarrow$ s.t. for m-a.e. $x \in \mathbb{T}$,
$\phi(r\chi(x)) \xrightarrow[r\to 1-]{} \phi(\chi(x)) \in \partial\mathbb{D}$ where $\chi(x) := e^{2\pi ix}$.
Restriction of f to $\mathbb{T} := \mathbb{R}/\mathbb{Z} \cong [0,1)$: measurable map
$\tau = \tau(\phi) : \mathbb{T} \hookleftarrow$ defined by $\chi(\tau x) = \phi(\chi(x))$.

- $(\mathbb{T}, m = \text{Leb}, \tau)$ a nonsingular transformation (abbr NST) by:

Boole's formula on $\mathbb{D}$ [Nordgren 1968],

$$\pi_z \circ \tau^{-1} = \pi_{\phi(z)}$$

where $d\pi_z(x) := p_z(x)dx$ with $p_z(x) := \text{Re}\,\frac{\chi(x)+z}{\chi(x)-z} = \frac{1-|z|^2}{|\chi-z|^2}$.

2. Transfer operator & acip

Transfer operator for NST (X, m, T): $\widehat{T} : L^1(m) \circlearrowleft$ s.t.:
$\int_X \widehat{T}f \cdot g dm = \int_X f \cdot g \circ T dm$.

- For $\mu \ll m$, $\mu \circ T^{-1} = \mu \iff \widehat{T}(\frac{d\mu}{dm}) = \frac{d\mu}{dm}$.
- For $\phi : \mathbb{D} \circlearrowleft$ inner with $\tau = \tau(\phi)$, by Boole's formula

$\widehat{\tau}(p_z) = p_{\phi(z)}$ and $\pi_z \circ \tau^{-1} = \pi_z$ iff $\phi(z) = z$.

Denjoy-Wolff theorem For $\phi : \mathbb{D} \circlearrowleft$ inner non Möbius,
$\exists\ \mathfrak{d} = \mathfrak{d}_\phi \in \overline{\mathbb{D}}$ so that $\phi^n(z) \xrightarrow[n\to\infty]{} \mathfrak{d}\ \forall\ z \in \mathbb{D}$.

Corollary $(\mathbb{T}, m, \tau(\phi))$ has an absolutely continuous invariant probability (abbr. acip) iff $\mathfrak{d}_\phi \in \mathbb{D}$.

Proof of $\Rightarrow$. Let μ be a an acip for $\tau(\phi)$ with $|\mathfrak{d}_\phi| = 1$, then
$\widehat{\tau}^n p_z = p_{\phi^n z} = \frac{1-|\phi^n(z)|^2}{|\chi-\phi^n(z)|^2} \xrightarrow[n\to\infty]{} 0\ \forall\ \chi(x) \neq \mathfrak{d}$ &.

$$\int_{\mathbb{T}} \mu(A|\mathcal{I})dm \xleftarrow[n\to\infty]{\text{ergodic theorem}} \int_{\mathbb{T}} (\frac{1}{n}\sum_{k=0}^{n-1} 1_A \circ T^k)dm \xrightarrow[n\to\infty]{} 0\ \forall\ A \in \bigcup_{\epsilon>0} \mathcal{B}(\mathbb{T} \setminus B(\mathfrak{d}, \epsilon))$$

and $\mu \equiv 0$. ☑

3. Conservative, ergodic, exact NSTs

Wandering set for NST (X, m, T):
$W \in \mathcal{B}_+$ such that $\{T^{-n}W :\ n \in \mathbb{N}\}$ are disjoint sets.

- NST (X, m, T) is:
- dissipative if $X = \bigcup$ wandering sets;
- conservative if $\nexists$ wandering sets, e.g. if $\exists$ acip;
- equivalently $\sum_{n\geq 0} \widehat{T}^n f = \infty$ a.e. for some (all) $f \in L^1(m)$, $f > 0$.
- ergodic if $A \in \mathcal{B}(X)$, $A = T^{-1}A \Rightarrow A \overset{m}{=} \varnothing, X$
- exact if $\bigcap_{n\geq 1} T^{-n}\mathcal{B}(X) \overset{m}{=} \{\varnothing, X\}$;equivalently

$\|\widehat{T}^n u\|_1 \xrightarrow[n\to\infty]{} 0\ \forall\ u \in L^1(m)_0$.

4. Conservative, ergodic, exact inner functions

☺ For $z,\ \omega \in \mathbb{D}$,

$$\|p_z - p_\omega\|_1 = \tfrac{4}{\pi}\sin^{-1}[z,\omega] \ \text{ where } [z,\omega] := \left|\tfrac{z-\omega}{1-\overline{\omega}z}\right|.$$

Thus for $\phi : \mathbb{D} \hookleftarrow$ inner, $(\mathbb{T}, m, \tau)$ is exact iff

(⚓) $$[\phi^n(z), \phi^n(\omega)] \xrightarrow[n\to\infty]{} 0 \ \forall\ z, \omega \in \mathbb{D}.$$

$\because$ (⚓) iff $\|\widehat{\tau}^n(p_z - p_\omega)\|_1 \xrightarrow[n\to\infty]{} 0$.

If ϕ is not Möb & $\mathfrak{d}_\phi \in \mathbb{U}$, then $(\mathbb{T}, pi_{\mathfrak{d}}, \tau)$ is an exact PPT.

Dichotomy For $\phi : \mathbb{D} \hookleftarrow$ inner, non Möbius, either ⚓ holds or $\exists$ $F : \mathbb{D} \to \mathbb{R}$ non constant so that $F \circ \phi \equiv F$.

Thus $(\mathbb{T}, m, \tau)$ ergodic $\implies$ $(\mathbb{T}, m, \tau)$ exact.

Now $(\mathbb{T}, m, \tau)$ is conservative iff $\sum_{n\ge 0} p_{\phi^n(z)} = \infty$ a.e., i.e. $\sum_{n\ge 0}(1 - |\phi^n(z)|) = \infty$ for some (all) $z \in \mathbb{D}$.

Thus, using Blaschke's theorem, for $\phi : \mathbb{D} \hookleftarrow$ inner, not Möbius:
$\tau(\phi)$ conservative $\implies$ $\tau(\phi)$ ergodic $\implies$ $\tau(\phi)$ exact.

5. Inner functions on $\mathbb{R}^{2+}$

Inner function of $\mathbb{R}^{2+} := \{z \in \mathbb{C} :\ \mathrm{Im} z > 0\}$:
analytic function $T : \mathbb{R}^{2+} \hookleftarrow$ s.t. for m-a.e. $x \in \mathbb{R}$,
$T(x+iy) \xrightarrow[y\to 0+]{} T(x) \in \mathbb{R}$.

$(\mathbb{R}, m, T)$ is NST by
Boole's formula on $\mathbb{R}^{2+}$ [Boole 1857] for rational inner fns.

$$P_z \circ T^{-1} = P_{Tz} \text{ where } dP_z(x) := \varphi_z(x)dx \text{ with } \varphi_z(x) := \tfrac{1}{\pi}\mathrm{Im}\tfrac{1}{x-z}.$$

For $T : \mathbb{R}^{2+} \hookleftarrow$ inner, $T(z) = \alpha_T z + \beta_T + \int_{\mathbb{R}} \frac{1+tz}{t-z} d\mu_T(t)$
$(\alpha_T \ge 0,\ \beta_T \in \mathbb{R},\ \mu_T \in \mathfrak{M}(\mathbb{R})\ \&\ \mu_T \perp m)$. $\alpha_T > 0 \ \Rightarrow\ m \circ T^{-1} = \frac{1}{\alpha_T} m$.

If $\lambda \in \partial\mathbb{D}$ & $A = A_\lambda : \mathbb{D} \to \mathbb{R}^{2+}$, $A(z) := i\frac{\lambda+z}{\lambda-z}$ then $\phi : \mathbb{D} \hookleftarrow$ is inner on $\mathbb{D}$ iff $A \circ \phi \circ A^{-1} : \mathbb{R}^{2+} \hookleftarrow$ is inner on $\mathbb{R}^{2+}$.

- If $\phi : \mathfrak{D} \hookleftarrow$ is inner with $\mathfrak{d} = \mathfrak{d}_\phi \in \partial\mathbb{D}$, then $T := A_{\mathfrak{d}} \circ \phi \circ A_{\mathfrak{d}}^{-1} : \mathbb{R}^{2+} \hookleftarrow$ is inner with $\alpha_T \ge 1$.

If $\alpha_T > 1$, $(\mathbb{R}, m, T)$ not ergodic. If $\alpha_T = 1$, $(\mathbb{R}, m, T)$ MPT.

$\exists$ non-ergodic, dissipative-exact & conservative examples for $\alpha_T = 1$.

6. Non-singular endomorphisms of a Lebesgue space

Let (X, m, T) be a `NST` of a polish, non-atomic probability space with transfer operator $\widehat{T}$.

The **transition kernel** for $\widehat{T}$ (aka the **pre-image measure** of (X, m, T)) is a measurable function

$x \in X_0 \mapsto \nu_x = \nu_x^{(T)} \in \mathfrak{M}(T^{-1}\{x\}) \subset C_B(X)^*$ where $X_0 \in \mathcal{B}(X)$, $m(X \setminus X_0) = 0$ so that $\widehat{T}f(x) = \int_X f d\nu_x \ \forall\ f : X \to \mathbb{R}$ bounded, Borel.

Preimage measures exist by the disintegration theorem.

`Multiplicity Proposition` For a `NST` (X, m, T) TFAE:

- (`countable to one`) $\exists$ a (finite or) countable partition mod $m : \alpha \subset \mathcal{B}(X)$ so that $T : a \to Ta$ is invertible, nonsingular $\forall\ a \in \alpha$;
- (`forward nonsingular`) $\exists\ X_0 \in \mathcal{B}(X)$, $m(X \setminus X_0) = 0$ so that $A \in \mathcal{B}(X_0), m(A) = 0 \implies m(TA) = 0$;
- ν_x is purely atomic for a.e. $x \in X$.

7. Multiplicity of inner functions

For $\phi : \mathbb{D} \hookleftarrow$ inner,

- [Heins '77] $(\mathbb{T}, m, \tau)$ is countable to one iff $\exists\ \lim_{r\to 1-} \phi'(r\chi) =: \phi'(\chi) \in \mathbb{C}$ a.e.
- $x \mapsto \nu_x^{(\tau)}$ is weak $*$ cts. $(\mathbb{T} \to C(\mathbb{T})^*)$ $\because$ $\nu_x(p_z) = p_{\phi(z)}(x)$;
- $\nu_x^{(\tau)} \perp m \because$ a.e. $\frac{d\nu_x^{(\tau)}}{dm} \xleftarrow[r\to 1-]{} \nu_x^{(\tau)}(p_{r\chi}) = p_{\phi(r\chi)}(x) \xrightarrow[r\to 1-]{} 0.$
- [Craizer '91] If $\phi : \mathbb{D} \hookleftarrow$ is inner, $\phi(0) = 0$, then $h(\tau) < \infty \iff \phi' \in N(\mathbb{D})$ & in this case $h(\tau) = \int_{\mathbb{T}} \log|\phi'(\chi)|dm$.
- [Saksman '06] $\int_{\mathbb{T}} \frac{d\nu_0^{(\tau)}(s)}{|\chi(s)-\lambda|^2} = \infty\ \forall\ \lambda \in \partial\mathbb{D} \Rightarrow \nu_x^{(\tau)}$ non-atomic $\forall\ x \neq 0$.

`e.g.` $\mu \in \mathcal{P}(\mathbb{T})$, $\mu := \frac{6}{\pi^2}\sum_{n\geq 1}\frac{1}{n^2}\delta_{\{\log n\}}$, then $\int_{\mathbb{T}} \frac{d\nu_0^{(\tau)}(s)}{|\chi(s)-\lambda|^2} = \infty\ \forall\ \lambda \in \partial\mathbb{D}$.

[Donohue '65] $\exists\ \phi : \mathbb{D} \hookleftarrow$ inner, $\phi(0) = 0$ & $\nu_x^{(\tau)}$ non-atomic for a.e. $x \in \mathbb{T}$.

Construction [`Saksman` '06] Define $F : \mathbb{D} \to \mathbb{R}^{2+}$ by $F(z) := \int_{\mathbb{T}} \frac{\chi+z}{\chi-z} d\mu$, then $F(0) = 1$ whence $\phi := \frac{F-1}{F+1} : \mathbb{D} \hookleftarrow$ is inner with $\phi(0) = 0$ and $\nu_0^{(\tau)} = \mu$ whence $\nu_x^{(\tau)}$ non-atomic $\forall\ x \neq 0$.

8. Representations, singularities & multiplicities

Factorization Theorem

Let $\phi:\mathbb{D}\hookleftarrow$ be inner, then

$$\phi(z) = B(z)S(z) =: \lambda \prod_{a\in F} \psi_a(z)^{\#(a)} \cdot \exp[-\int_{\mathbb{T}} \frac{\chi+z}{\chi-z} d\sigma]$$

where $F\subset\mathbb{D}$ is at most countable, $\#: F\to\mathbb{N}$ so that $\sum_{a\in F}\#(a)(1-|a|)<\infty$, $\psi_a(z) := \frac{z-a}{1-\bar{a}z}$, $\sigma\in\mathfrak{M}(\mathbb{T})$, $\sigma\perp m$ and $|\lambda|=1$.

- $B = B_F$ aka Blaschke product. A rational inner functions is a Blaschke product with $|F|<\infty$.
- $S = S_\sigma$ aka singular inner function has no zeroes in $\mathbb{D}$.
- The singular set of ϕ is

$\mathfrak{s}_\phi := \{z\in\overline{\mathbb{D}}: \nexists\, U \text{ open s.t. } z\in U \;\&\; \phi: U\to\mathbb{C} \text{ is analytic}\} = F'\cup\chi(\text{spt}\sigma)$.

- If $m(\chi^{-1}\mathfrak{s}_\phi\}) = 0$, then there is a partition α mod 0 of $\mathbb{T}$ into open intervals so that $\tau(\phi): a\to Ta$ is strictly increasing & real analytic $\forall\; a\in\alpha$, and hence countable to one.

Question Does $\tau(\phi)$ countable to one $\Rightarrow$ $m(\chi^{-1}\mathfrak{s}_\phi\}) = 0$?

9. Adapted pair and Doeblin-Fortet operators

Adapted pair: pair of Banach spaces $(\mathcal{C},\mathcal{L})$ satisfying

$\mathcal{L}\subset\mathcal{C}$, $\|\cdot\|_{\mathcal{C}}\le\mathcal{L}$, $\overline{(\mathcal{L})}_{\mathcal{C}} = \mathcal{C}$ & $\mathcal{L}$-closed, bounded sets are $\mathcal{C}$-compact.

Doeblin-Fortet (DF) operator wrt adapted pair $(\mathcal{C},\mathcal{L})$: $P\in\text{hom}(\mathcal{C},\mathcal{C})\cap\text{hom}(\mathcal{L},\mathcal{L})$ satisfying for some $\kappa\ge 1$, $R,H\in\mathbb{R}_+$ and $r\in(0,1)$:

(i) $\|P^n x\|_{\mathcal{C}}\le H\|x\|_{\mathcal{C}}\;\forall n\in\mathbb{N}, x\in\mathcal{L}$, &

(ii) $\|P^\kappa x\|_{\mathcal{L}}\le r\|x\|_{\mathcal{L}} + R\|x\|_{\mathcal{C}}\;\forall\; x\in\mathcal{L}$.

☺ For $b>1$, let $k_b := \{f\in L^2(\mathbb{T},m): \|f\|_{k_b}^2 := \sum_{n\in\mathbb{Z}} b^{|n|}|\hat{f}(n)|^2<\infty\}$, then k_b is a Hilbert space and $(L^p(m), k_b)$ is an adapted pair for $1\le p<\infty$.

10. Spectral gaps for inner functions

Spectral gap theorem [A-Nadkarni, '22]]

Let $\phi:\mathbb{D}\hookleftarrow$ be inner, not Möbius with $\phi(0)=0$, then
$\forall\ b>1,\ |\phi'(0)|<\rho<1,\ \exists\ M>0$ so that with $\tau=\tau(\phi)$

⚽ $$\|\widehat{\tau}^N u-\mathbb{E}(u)\|_b\le M\rho^N\|u\|_b\ \forall\ u\in k_b,\ N\ge 0$$

where $\mathbb{E}(u):=\int_{\mathbb{T}}udm=\widehat{u}(0)$.

- $\widehat{\tau}$ is a DF operator wrt $(L^2(m),k_b)\ \forall\ b>1$.

As in [Butterley-Canestrari-Jain, Arxiv2205.12607v3] consider

$$\mathfrak{r}(\phi):=\inf\{\rho>0:\ \widehat{\tau}\ \ni ⚽ \text{ with } \rho \text{ on } \mathcal{L} \text{ with } \|\cdot\|_{\mathcal{L}}\ge\|\cdot\|_1\},$$

- then $\mathfrak{r}(\phi)\ge|\phi'(0)|$, $\because\ \exists\ z\in\mathbb{D}$ s.t. $|\phi^n(z)|^{\frac{1}{n}}\xrightarrow[n\to\infty]{}|\phi'(0)|$ &

$$M\rho^n\ge\|\widehat{\tau}^n(p_z)-1\|_1\ge\epsilon|\phi^n(z)|=|\phi'(0)|^{n+o(n)}.$$

[Ivrii-Urbanski, '22] Spectral gaps for $\widehat{\tau}$ on the Dirichlet space $\mathcal{D}(\mathbb{D})$ and the Sobolev space $W^{\frac{1}{2},2}(\mathbb{T})$.

11. Perturbations of transfer operators

Let $\phi:\mathfrak{D}\hookleftarrow$ be non-Möbius inner with $\phi(0)=0$ & $\tau=\tau(\phi)$.
For $\Psi:\mathbb{T}\to\mathbb{R}$ bounded measurable, define the operators
$\Pi_z=\Pi_{z,\Psi}:L^1(m)\hookleftarrow\ (z\in\mathbb{C})$; and $P_t=P_{t,\Psi}:L^1(m)\hookleftarrow\ (t\in\mathbb{R})$ by

$$\Pi_z f:=\widehat{\tau}(e^{z\Psi}f)\ \&\ P_t f:=\Pi_{it}f=\widehat{\tau}(e^{it\Psi}f).$$

Analyticity of Perturbation [A- Nadkarni '22]

If $\Psi=\psi\circ\chi:\mathbb{T}\to\mathbb{R}$ with $\psi\in k_B$ (with $B>1$) then
$\forall\ 1<b<B, z\in\mathbb{C}\mapsto\Pi_z\in\hom(k_b,k_b)$ is holomorphic with
$\frac{d^n\Pi_z}{dz^n}(f)=\Pi_z(\Psi^n f)=:\Pi_z^{(n)}(f)$.

Corollary: a spectral theorem see [Nagaev, '57]

Let $1<b<B$, $\Psi=\psi\circ\chi:\mathbb{T}\to\mathbb{R}$, $\psi\in k_B$ and $\psi(0)=0=\mathbb{E}(\Psi)$.
$\exists\ \epsilon>0,\ K>0,\ \theta\in(0,1)$ & analytic functions
$\lambda:B(0,\epsilon)\to B_{\mathbb{C}}(0,1),\ N:B(0,\epsilon)\to\hom(k_b,k_b)$ s.t.

(⚕) $$\|P_{t,\Psi}^n h-\lambda(t)^n N(t)h\|_{k_b}\le K\theta^n\|h\|_{k_b}\quad\forall\ |t|<\epsilon,\ n\ge1,\ h\in k_b$$

where $\forall|t|<\epsilon,\ N(t):k_b\to N(t)k_b$ is a projection & $\dim N(t)k_b=1$.

12. Expansion of the eigenvalue and CLTs

Let $\phi : \mathfrak{D} \hookleftarrow$ be non-Möbius inner with $\phi(0) = 0$ & $\tau = \tau(\phi)$ and let $\Psi = \psi \circ \chi : \mathbb{T} \to \mathbb{R}$ with $\psi \in k_B$.

[Nagaev, '57] & [Rousseau-Egele, '83]

- $\mathbb{E}(\frac{\Psi_n^2}{n}) \xrightarrow[n\to\infty]{} \mathbb{E}(E(g^2\|\tau^{-1}\mathcal{B}) - E(g\|\tau^{-1}\mathcal{B})^2) =: \sigma^2 \geq 0$ where $\Psi_n := \sum_{k=0}^{n-1} \Psi \circ \tau^k$ & $g := \sum_{n\geq 0} \widehat{\tau}^n \Psi$;
- $\sigma^2 = 0 \iff \exists\ h,\ g = h \circ \tau \Rightarrow \Psi = g - \widehat{\tau} g = h \circ \tau - h$;;
- If $\sigma^2 > 0$, then $\lambda(t) = 1 - \frac{\sigma^2 t^2}{2} + o(t^2)$ as $t \to 0$ & (Ψ, τ) satisfies the CLT:

$$\mathbb{E}(\exp[\tfrac{it\Psi_n}{\sqrt{n}}]) = \mathbb{E}(P^n_{\frac{t}{\sqrt{n}}} \mathbb{1}) = \lambda(\tfrac{t}{\sqrt{n}})^n \mathbb{E}(N(\tfrac{t}{\sqrt{n}})) + O(\theta^n) \xrightarrow[n\to\infty]{} \exp[-\tfrac{\sigma^2 t^2}{2}].$$

- If $\sigma^2 > 0$ & $|\lambda(t)| < 1\ \forall\ t \neq 0$, then (below) (Ψ, τ) satisfies the LLT.

CLT's for: (χ, τ) in [Nicolau-Soler i Gilbert, arxiv:math/2006.12105] & for (Ψ, τ), Ψ a multiplier for $W^{\frac{1}{2},2}$ in [Ivrii-Urbanski, '22] .

13. Aperiodicity

Let $\phi : \mathfrak{D} \hookleftarrow$ be non-Möbius inner with $\phi(0) = 0$ & $\tau = \tau(\phi)$ and let $\Psi = \psi \circ \chi : \mathbb{T} \to \mathbb{R}$ with $\psi \in k_B$.

Call Ψ periodic if $\exists\ t \in \mathbb{R},\ \lambda \in \mathbb{C},\ |\lambda| = 1$ & $g : \mathbb{T} \to \mathbb{C}$ measurable, not constant, so that $e^{it\psi} = \lambda g \circ \tau / g$ and aperiodic otherwise. WLOG $|g| = 1$ a.e. .

Proposition Ψ is periodic iff $\exists\ t \in \mathbb{R},\ \lambda \in \mathbb{C},\ |\lambda| = 1$ and $g \in L^1(m)$ so that $P_t(g) = \lambda g$.

Proof uses that conditional expectation is an orthogonal projection.

- Thus if Ψ is aperiodic,

$|\lambda(t)| \leq lim_{n\to\infty} \|P_t\|^{\frac{1}{n}}_{\text{hom}(k_b,k_b)} < 1\ \forall\ t \neq 0$.

Eigenvalue rigidity proposition

Suppose that $t \in \mathbb{R},\ \lambda \in \partial\mathbb{D},\ \mathfrak{z} \in L^1$are so that $P_t \mathfrak{z} = \lambda \mathfrak{z}$, then $\mathfrak{z} \in k_b$.

Proof uses Yosida's ergodic theorem for DF operators.

Aperiodicity Proposition [A-Nadkarni, '22]

If ϕ is not a finite Blaschke product and $\psi \in k_B,\ \psi(0) = 0,\ \psi \not\equiv 0$, then $\Psi = \psi \circ \chi$ is aperiodic.

Proof uses Seidel's theorem on inner function singularities.

14. Local Limits

Local Limit Theorem

Let $\phi : \mathfrak{D} \hookleftarrow$ be non-Möbius inner with $\phi(0) = 0$ & $\tau = \tau(\phi)$ and let $\Psi = \psi \circ \chi : \mathbb{T} \to \mathbb{R}$ with $\psi \in k_B$ be aperiodic,
then for $I \subset \mathbb{R}$ an interval, and $k_n \in \mathbb{Z}$, $\frac{k_n}{\sigma\sqrt{n}} \to \kappa \in \mathbb{R}$ as $n \to \infty$,

$$\sigma\sqrt{n}\,\widehat{\tau}^n(1_{[\Psi_n \in k_n + I]}) \xrightarrow[n\to\infty]{} \frac{|I|}{\sqrt{2\pi}} e^{-\frac{\kappa^2}{2}}.$$

where $\sigma^2 = \lim_{n\to\infty} \mathbb{E}(\frac{\Psi_n^2}{n})$ and $|I|$ is the length of I.

15. Sketch of proof of local limits

Fourier method of proof as in [L. Breiman, '68].
Let $h \in L^1(\mathbb{R})$, $h \geq 0$ $\hat{h} \in L^1(\mathbb{R})$, $\hat{h} \equiv 0$ off $[-M, M]$.
Let $\epsilon > 0$ $\theta \in (0,1)$ be as in ⚕. By aperiodicity $\exists\ \eta \in (\theta, 1)$ so that $\sup_{\epsilon \leq |t| \leq M} |\lambda(t)| \leq \eta < 1$.

$$\sigma\sqrt{n}\cdot\widehat{\tau}^n(h(\Psi_n - k_n)) = \frac{\sigma\sqrt{n}}{\sqrt{2\pi}} \int_{-M}^{M} \hat{h}(x) e^{-ik_n x} P_x^n \mathbb{1}\, dx$$

$$= \frac{\sigma\sqrt{n}}{\sqrt{2\pi}} \int_{|x|\leq\epsilon} \hat{h}(x) e^{-ik_n x} \lambda(x)^n N(\tfrac{x}{\sigma\sqrt{n}}) \mathbb{1}\, dx + O(\sqrt{n}\eta^n)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{|x|\leq\epsilon\sigma\sqrt{n}} \hat{h}(\tfrac{x}{\sigma\sqrt{n}}) e^{-i\frac{k_n}{\sigma\sqrt{n}}x} \lambda(\tfrac{x}{\sigma\sqrt{n}})^n N(\tfrac{x}{\sigma\sqrt{n}}) \mathbb{1}\, dx + o(1)$$

$$\xrightarrow[n\to\infty]{} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{h}(0) e^{-\frac{x^2}{2}} e^{-i\kappa x} dx = \int_{\mathbb{R}} h(x) dx \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\kappa^2}{2}}.$$

For $\omega \in \mathbb{T}$, $k_n \in \mathbb{R}$, $\frac{k_n}{\sigma\sqrt{n}} \xrightarrow[n\to\infty]{} \kappa$, define Radon measures Λ_n on $\mathbb{R}$ by $\Lambda_n(h) := \sigma\sqrt{n}\widehat{\tau}^n(h(\Psi_n - k_n))(\omega)$. Above $\overset{[L.Breiman,'68]}{\Longrightarrow}$
$\Lambda_n \xrightarrow[n\to\infty]{\text{weak * in} C_c(\mathbb{R})^*} \frac{1}{\sqrt{2\pi}} e^{-\frac{\kappa^2}{2}} \cdot \text{Leb}$. The theorem follows from this. ☑

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127. The End

Thank you for listening.