# Relativistic spin hydrodynamics

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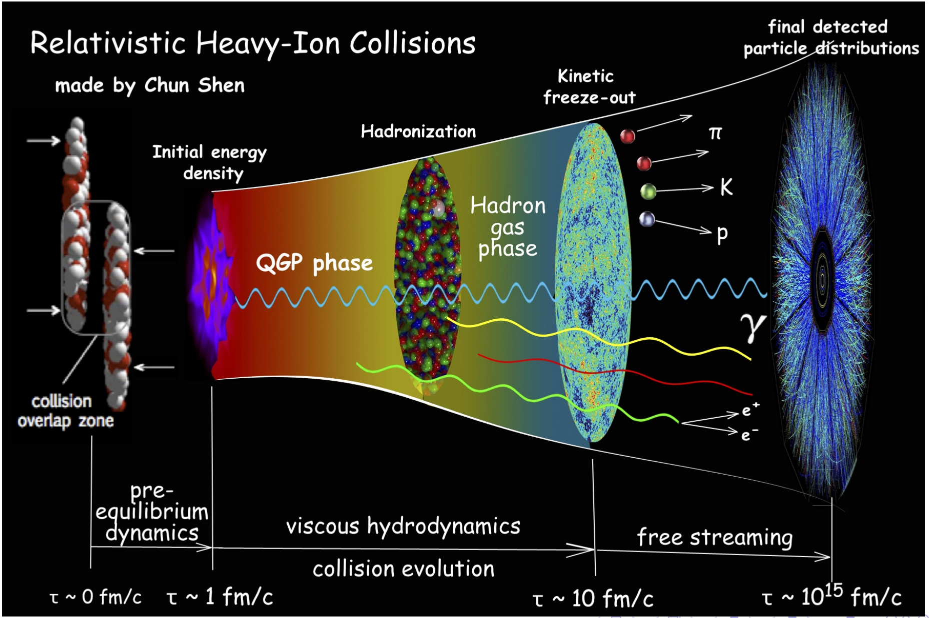
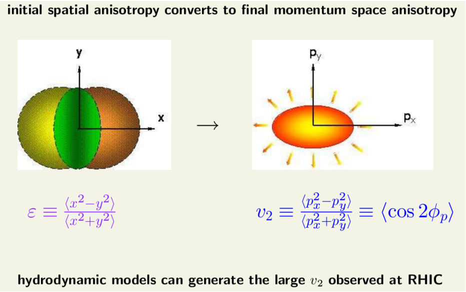

Relativistic Heavy-Ion Collisions
made by Chun Shen
final detected particle distributions
Kinetic freeze-out
Hadronization
Initial energy density
Hadron gas phase
QGP phase
π
K
p
γ
e+
e-
collision overlap zone
pre-equilibrium dynamics
viscous hydrodynamics
free streaming
collision evolution
τ ~ 0 fm/c
τ ~ 1 fm/c
τ ~ 10 fm/c
τ ~ 10^15 fm/c

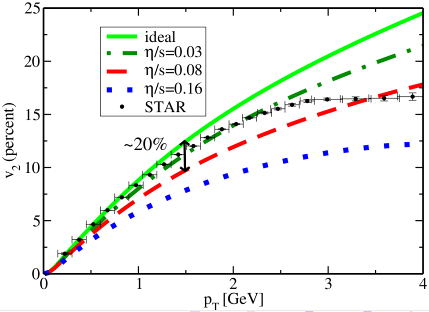
# Anisotropic flow and viscosity

- Relativistic hydrodynamics applied successfully to explain the space-time evolution.
- Role of Hydrodynamics:

Initial state spatial deformation $\xrightarrow{Hydro}$ Final state momentum anisotropy

- Viscosity degrades conversion efficiency; necessary to explain data



# Relativistic fluid dynamics

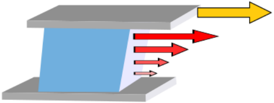
- Conservation equations for energy-momentum and charge current.

| Ideal | Dissipative |
|---|---|
| $T^{\mu\nu} = \epsilon u^\mu u^\nu - P\Delta^{\mu\nu}$ | $T^{\mu\nu} = \epsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$ |
| $N^\mu = n u^\mu$ | $N^\mu = n u^\mu + n^\mu$ |
| Unknowns: $\underbrace{\epsilon,\ P,\ n,\ u^\mu}_{1+1+1+3} = 6$ | $\underbrace{\epsilon,\ P,\ n,\ u^\mu,\ \Pi,\ \pi^{\mu\nu},\ n^\mu}_{1+1+1+3+1+5+3} = 15$ |
| Equations: $\underbrace{\partial_\mu T^{\mu\nu} = 0,\quad \partial_\mu N^\mu = 0,\quad EOS}_{4 + 1 + 1} = 6$ | |
| Closed set of equations | 9 more equations required |

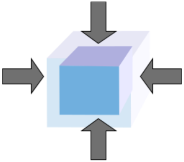
- Here $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ and Landau frame chosen: $T^{\mu\nu} u_\nu = \epsilon u^\mu$.
- Equations required for dissipative currents $\Pi$, $\pi^{\mu\nu}$ and $n^\mu$.

# Dissipation effects

- In simple terms: $\pi^{yx} = 2\,\eta\,\partial^{\langle y}u^{x\rangle}, \quad \Pi = -\zeta\,\partial\cdot u, \quad n^x = \kappa\,\partial^{\langle x\rangle}a$
  - Shear viscosity: fluid's resistance to shear forces

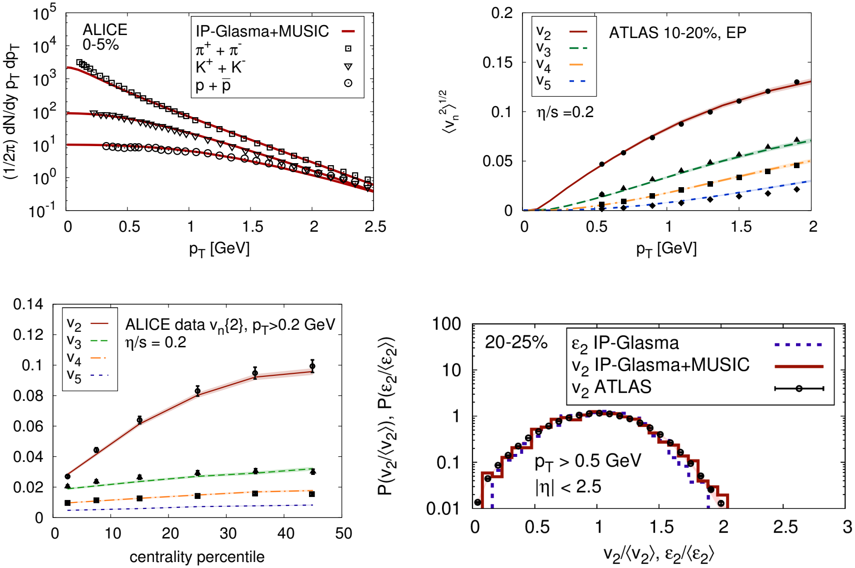


  - Bulk viscosity: fluid's resistance to compression

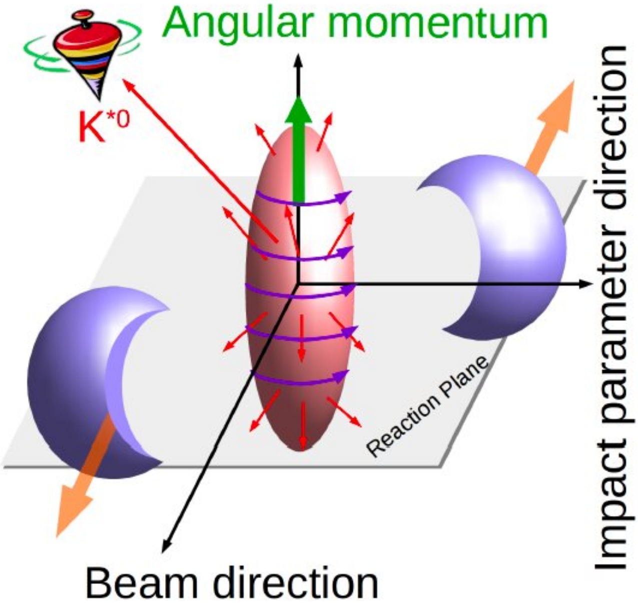


- Charge/heat conductivity: fluid's resistance to flow of charge/heat.

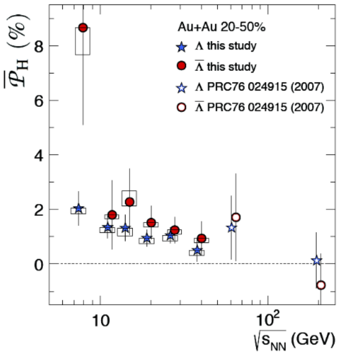
# Successes of hydro models in heavy-ion collisions




[C. Gale, S. Jeon, B. Schenke, P. Tribedy and R. Venugopalan, PRL **110**, 012302 (2013)]


Angular momentum
K*0
Impact parameter direction
Reaction Plane
Beam direction

STAR Collaboration, *Global Lambda hyperon polarization in nuclear collisions*, Nature 548 62-65, 2017





First evidence of a quantum effect in (relativistic) hydrodynamics

Adopted from F. Becattini 'Subatomic Vortices'

# Polarization by rotation

Take an ideal gas in a rigidly rotating vessel. At thermodynamical equilibrium (Landau) the gas will also have a velocity field

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$$

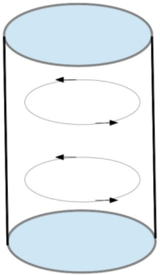
WARNING The potential term has a + sign as it stems from both centrifugal and Coriolis potentials

For the *comoving* observer the equilibrium particle distribution function will be given by:

$$f(\mathbf{x}', \mathbf{p}') \propto \exp[-\mathbf{p}'^2/2mT + m(\boldsymbol{\omega} \times \mathbf{x}')^2/2T]$$

If we calculate the distribution function seen by the external inertial observer

$$f(\mathbf{x}, \mathbf{p}) \propto \exp[-\mathbf{p}^2/2mT + \mathbf{p} \cdot (\boldsymbol{\omega} \times \mathbf{x})/T] = \exp[-\mathbf{p}^2/2mT + \boldsymbol{\omega} \cdot \mathbf{L}/T]$$



F. Becattini

It seems quite *natural* to extend this to particle with spin

$$f(\mathbf{x},\mathbf{p},\mathbf{S}) \propto \exp[-\mathbf{p}^2/2mT + \boldsymbol{\omega}\cdot(\mathbf{L}+\mathbf{S})/T]$$

which implies that particles (and antiparticles) are *POLARIZED*, in a rotating ideal gas, along the direction of the angular velocity vector by an amount

$$P \simeq \frac{S+1}{3}\frac{\hbar\omega}{KT}$$

For a gas at STP with $\omega = 1000$ Hz, $P \sim 10^{-11}$

For relativistic nuclear collisions:

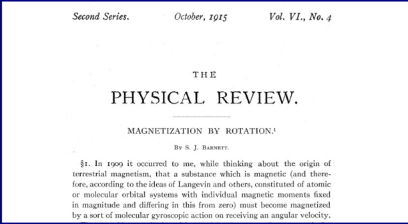
$$\frac{\hbar\omega}{KT} \approx \frac{c}{12\text{fm}200\text{MeV}} \approx 0.08$$

F. Becattini

# Barnett effect

S. J. Barnett, *Magnetization by Rotation*,

Phys. Rev.. 6, 239–270 (1915).


Second Series. October, 1915 Vol. VI., No. 4

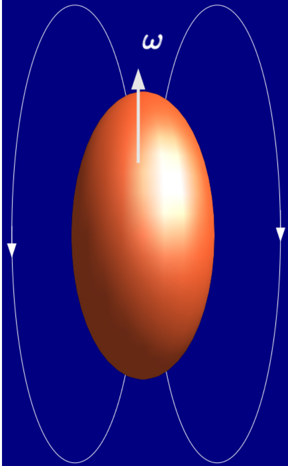
THE

PHYSICAL REVIEW.

MAGNETIZATION BY ROTATION.[1]

By S. J. Barnett.

§1. In 1909 it occurred to me, while thinking about the origin of terrestrial magnetism, that a substance which is magnetic (and therefore, according to the ideas of Langevin and others, constituted of atomic or molecular orbital systems with individual magnetic moments fixed in magnitude and differing in this from zero) must become magnetized by a sort of molecular gyroscopic action on receiving an angular velocity.




Spontaneous magnetization of an uncharged body when spun around its axis, in quantitative agreement with the previous polarization formula

$$M = \frac{\chi}{g}\omega$$

It can be seen as a dissipative transformation of the orbital angular momentum into spin of the constituents. The angular velocity decreases and a small magnetic field appears; this phenomenon is accompanied by a heating of the sample. Requires a spin-orbit coupling.

F. Becattini

# Converse: Einstein-De Haas effect

*the only Einstein's non-gedanken experiment*

A. Einstein, W. J. de Haas, Koninklijke Akademie van Wetenschappen te Amsterdam, Proceedings, 18 I, 696-711 (1915)


Rotation of a ferromagnet originally at rest when put into an external H field

An effect of angular momentum conservation:
spins get aligned with H (irreversibly) and this must be compensated by a on overall orbital angular momentum

F. Becattini

# Hydrodynamics with angular momentum conservation

- Apply relativistic hydrodynamics to understand particle polarization.
- Hydrodynamic evolution should ensure total angular momentum conservation.
- Formulation of relativistic hydrodynamics with angular momentum conservation necessary.
- Polarization processes are dissipative in nature.
- Dissipation should also be incorporated in the formulation.
- Topic of present discussion: formulation of ideal and dissipative hydrodynamics with angular momentum conservation.
- Relativistic kinetic theory framework.

# Angular momentum conservation: particles

- Orbital angular momentum of a particle with momentum $\vec{p}$:

$$\vec{L} = \vec{x} \times \vec{p} \quad \Rightarrow \quad L_i = \varepsilon_{ijk}\, x_i\, p_j$$

- One can obtain the dual tensor:

$$L_{ij} \equiv \varepsilon_{ijk}\, L_k \quad \Rightarrow \quad L_{ij} = x_i\, p_j - x_j\, p_i$$

- We know that both definitions are equivalent.
- In absence of external torque, i.e., $\dfrac{d\vec{L}}{dt} = 0$, we have: $\partial_i L_{ij} = 0$.
- This treatment valid for non-relativistic point particles.
- For fluids, particle momenta $\rightarrow$ "generalized fluid momenta"

The energy-momentum tensor

# Angular momentum conservation: fluid

- The orbital angular momentum for relativistic fluids is defined as

$$L^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}$$

- Keeping in mind the energy-momentum conservation, $\partial_\mu T^{\mu\nu} = 0$:

$$\partial_\lambda L^{\lambda,\mu\nu} = T^{\mu\nu} - T^{\nu\mu}$$

- Obviously, for symmetric $T^{\mu\nu}$, orbital angular momentum is automatically conserved. Classically $T^{\mu\nu}$ always symmetric.

- For medium constituent with intrinsic spin, different story

$$J^{\lambda,\mu\nu} = L^{\lambda,\mu\nu} + S^{\lambda,\mu\nu} + LS \text{ couplings}$$

- Ensure total angular momentum conservation: $\partial_\lambda J^{\lambda,\mu\nu} = 0$.

- In absence of coupling terms, $\partial_\lambda S^{\lambda,\mu\nu} = 0$ (Spin Hydrodynamics).

# Spin dependent distribution function

- Introduce spin four-vector $s^\alpha$ such that in the particle rest frame, $s^\alpha = (0, \boldsymbol{s}_*)$ with the length of the spin vector defined by

$$-s^2 \equiv -s_\alpha s^\alpha = |\boldsymbol{s}_*|^2 = \mathfrak{s}^2 = \frac{1}{2}\left(1 + \frac{1}{2}\right)$$

- Start with spin-1/2 massive particles and introduce their internal angular momentum $s^{\alpha\beta}$

$$s^{\alpha\beta} = \frac{1}{m}\epsilon^{\alpha\beta\gamma\delta} p_\gamma s_\delta \qquad \Rightarrow \qquad s^\alpha = \frac{1}{2m}\epsilon^{\alpha\beta\gamma\delta} p_\beta s_{\gamma\delta}$$

- The equilibrium distribution function is

$$f^\pm_{s,\mathrm{eq}}(x,p,s) = f^\pm_{\mathrm{eq}}(x,p)\exp\left[\frac{1}{2}\omega_{\mu\nu}(x)s^{\mu\nu}\right],$$

$$f^\pm_{\mathrm{eq}}(x,p) = \exp\left[-p^\mu \beta_\mu(x) \pm \xi(x)\right].$$

[W. Florkowski, A. Kumar, R. Ryblewski, Prog. Part. Nucl. Phys. 108 (2019) 103709]

# Integration over particle degrees of freedom

- Introduce the spin integral measure

$$\mathrm{dS} = \frac{m}{\pi \mathfrak{s}}\, \mathrm{d}^4 s \; \delta(s \cdot s + \mathfrak{s}^2) \; \delta(p \cdot s).$$

- Suitably normalized

$$\int \mathrm{dS} = 2, \qquad \int \mathrm{dS}\, f^{\pm}_{s,\mathrm{eq}}(x,p,s) = f^{\pm}_{\mathrm{eq}}(x,p).$$

- Introduce the momentum integral measure

$$\mathrm{dP} = \int \frac{d^4 p}{(2\pi)^4}\, 2\,\delta(p^2 - m^2)\, \Theta(p^0) = \frac{d^3 p}{(2\pi)^3 E_p}.$$

- Next step: construct hydrodynamic quantities.

# Conserved charge current

- The equilibrium charge current is defined as

$$N^\mu_{\rm eq} = \int {\rm dP ~dS}~ p^\mu ~\left[ f^+_{s,{\rm eq}}(x,p,s) - f^-_{s,{\rm eq}}(x,p,s) \right].$$

- Using the equilibrium distribution functions

$$N^\mu_{\rm eq} = 2\sinh(\xi) \int {\rm dP}\, p^\mu e^{-p\cdot\beta} \int {\rm dS} \exp\left(\frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right).$$

- In the limit of small polarization, i.e., linear in $\omega_{\alpha\beta}$,

$$N^\mu_{\rm eq} = 2\sinh(\xi) \int {\rm dP}\, p^\mu\, e^{-p\cdot\beta} \int {\rm dS} \left(1 + \frac{1}{2}\omega_{\alpha\beta}s^{\alpha\beta}\right).$$

- Integrate over spin and momenta

$$N^\mu_{\rm eq} = n u^\mu, \quad n = 4\,\sinh(\xi)\, n_0(T), \quad n_0 = \frac{1}{2\pi^2} T^3 z^2 K_2(z),$$

where $z \equiv m/T$.

# Energy momentum tensor

- The energy-momentum tensor is defined as

$$T^{\mu\nu}_{\rm eq} = \int {\rm dP}\ {\rm dS}\ p^\mu p^\nu\ \left[f^+_{s,{\rm eq}}(x,p,s) + f^-_{s,{\rm eq}}(x,p,s)\right].$$

- Using equilibrium distribution functions in small polarization limit

$$T^{\mu\nu}_{\rm eq}(x) = \varepsilon u^\mu u^\nu - P\Delta^{\mu\nu}.$$

- The energy density and pressure is given by

$$\varepsilon = 4\,\cosh(\xi)\,\varepsilon_0(T), \quad P = 4\,\cosh(\xi)\,P_0(T).$$

- Integration over momenta leads to

$$\varepsilon_0(T) = \frac{1}{2\pi^2} T^4 z^2 \left[3K_2(z) + zK_1(z)\right], \quad P_0(T) = \frac{1}{2\pi^2} T^4 z^2 K_2(z) = n_0(T)T.$$

# Spin tensor

- We adopt the following intuitive definition for spin tensor

$$S_{\rm eq}^{\lambda,\mu\nu} = \int {\rm dP}\ {\rm dS}\ p^\lambda\, s^{\mu\nu} \left[f^+_{s,{\rm eq}}(x,p,s) + f^-_{s,{\rm eq}}(x,p,s)\right].$$

- Integrating over spin in small polarization limit

$$S_{\rm eq}^{\lambda,\mu\nu} = \frac{4\mathfrak{s}^2}{3m^2}\cosh(\xi)\int {\rm dP}\, p^\lambda\, e^{-p\cdot\beta}\left(m^2\omega^{\mu\nu} + 2p^\alpha p^{[\mu}\omega^{\nu]}_{\ \ \alpha}\right).$$

- Integration over momenta leads to

$$S_{\rm eq}^{\lambda,\mu\nu} = \mathcal{C}\Big[n_0\, u^\lambda\omega^{\mu\nu} + \mathcal{A}_0\, u^\lambda u^\delta u^{[\mu}\omega^{\nu]}_{\ \ \delta} + \mathcal{B}_0\Big(u^{[\mu}\Delta^{\lambda\delta}\omega^{\nu]}_{\ \ \delta} + u^\lambda\Delta^{\delta[\mu}\omega^{\nu]}_{\ \ \delta} + u^\delta\Delta^{\lambda[\mu}\omega^{\nu]}_{\ \ \delta}\Big)\Big].$$

$$\mathcal{C} = (4/3)\mathfrak{s}^2\cosh(\xi), \quad \mathcal{B}_0 = -\frac{2}{z^2}\frac{\varepsilon_0(T)+P_0(T)}{T} = -\frac{2}{z^2}s_0(T),$$

$$\mathcal{A}_0 = \frac{6}{z^2}s_0(T) + 2n_0(T) = -3\mathcal{B}_0 + 2n_0(T).$$

# Entropy four-current

- For entropy current we adopt the Boltzmann definition

$$H^{\mu}_{\rm eq} = -\int {\rm dP}\ {\rm dS}\, p^{\mu} \left[ f^{+}_{s,{\rm eq}} \left( \ln f^{+}_{s,{\rm eq}} - 1 \right) + f^{-}_{s,{\rm eq}} \left( \ln f^{-}_{s,{\rm eq}} - 1 \right) \right].$$

- Using previous definitions

$$H^{\mu}_{\rm eq} = \beta_{\alpha} T^{\mu\alpha}_{\rm eq} - \frac{1}{2} \omega_{\alpha\beta} S^{\mu,\alpha\beta}_{\rm eq} - \xi N^{\mu}_{\rm eq} + P \beta^{\mu}.$$

- Using conservation laws, $\partial_{\mu} N^{\mu}_{\rm eq} = \partial_{\mu} T^{\mu\nu}_{\rm eq} = \partial_{\lambda} S^{\lambda,\mu\nu}_{\rm eq} = 0$, we get

$$\partial_{\mu} H^{\mu}_{\rm eq} = \left(\partial_{\mu}\beta_{\alpha}\right) T^{\mu\alpha}_{\rm eq} - \frac{1}{2} \left(\partial_{\mu}\omega_{\alpha\beta}\right) S^{\mu,\alpha\beta}_{\rm eq} - \left(\partial_{\mu}\xi\right) N^{\mu}_{\rm eq} + \partial_{\mu}(P\beta^{\mu}) = 0.$$

- The above result is exact, i.e., up to all orders in $\omega^{\mu\nu}$.
- Provides support for the assumed form of $S^{\lambda,\mu\nu}_{\rm eq}$.

# Ideal spin hydrodynamics

- Ideal 'spin'-hydrodynamic equations are given by

$$\partial_\mu N^\mu_{\rm eq} = 0, \quad \partial_\mu T^{\mu\nu}_{\rm eq} = 0, \quad \partial_\lambda S^{\lambda,\mu\nu}_{\rm eq} = 0.$$

- Contributions to the entropy production coming from the spin polarization tensor are quadratic.
- No effect on the entropy production from the polarization in the linear order.
- Standard hydrodynamic equation are not affected by polarization at linear order.
- Therefore up to linear in $\omega$, spin polarization evolves on top of such a hydrodynamic background. [W. Florkowski, B. Friman, **AJ**, R. Ryblewski and E. Speranza, PRC 97, 041901 (2018); PRD 97, 116017 (2018)]
- Next: dissipation needs to be included.

# Relativistic Boltzmann equation

- Starting from Wigner function and its Clifford-algebra decomposition, the evolution equation for $f_s^{\pm}(x,p,s)$ is obtained
  [S. Bhadury, W. Florkowski, **AJ**, A. Kumar and R. Ryblewski, arXiv:2002.03937]

- In the absence of mean fields, the distribution function satisfies

$$p^{\mu}\partial_{\mu} f_s^{\pm}(x,p,s) = C[f_s^{\pm}(x,p,s)].$$

- Assume relaxation-time approximation for the collision term

$$C[f_s^{\pm}(x,p,s)] = -(u\cdot p)\,\frac{f_s^{\pm}(x,p,s) - f_{s,\mathrm{eq}}^{\pm}(x,p,s)}{\tau_{\mathrm{eq}}}.$$

- Solving up to first-order in gradients,

$$\delta f_s^{\pm} = \frac{-\tau_{\mathrm{eq}}}{(u\cdot p)}\, e^{\pm\xi - p\cdot\beta}\Big[\big(\pm p^{\mu}\partial_{\mu}\xi - p^{\lambda}p^{\mu}\partial_{\mu}\beta_{\lambda}\big)\Big(1+\frac{1}{2}s^{\alpha\beta}\omega_{\alpha\beta}\Big) + \frac{1}{2}p^{\mu}s^{\alpha\beta}\big(\partial_{\mu}\omega_{\alpha\beta}\big)\Big].$$

# Boltzmann equation and dissipation

- Consider dissipative corrections to conserved quantities

$$\delta N^\mu = N^\mu - N^\mu_{\rm eq}, \quad \delta T^{\mu\nu} = T^{\mu\nu} - T^{\mu\nu}_{\rm eq}, \quad \delta S^{\lambda,\mu\nu} = S^{\lambda,\mu\nu} - S^{\lambda,\mu\nu}_{\rm eq}$$

- Suitable moments of the Boltzmann equation in RTA leads to

$$\partial_\mu N^\mu = -\frac{u_\mu \delta N^\mu}{\tau_{\rm eq}}, \quad \partial_\mu T^{\mu\nu} = -\frac{u_\mu \delta T^{\mu\nu}}{\tau_{\rm eq}}, \quad \partial_\lambda S^{\lambda,\mu\nu} = -\frac{u_\lambda \delta S^{\lambda,\mu\nu}}{\tau_{\rm eq}}.$$

- Conservation equations provides the matching/frame conditions:

$$u_\mu \delta N^\mu = 0, \quad u_\mu \delta T^{\mu\nu} = 0, \quad u_\lambda \delta S^{\lambda,\mu\nu} = 0.$$

- All dissipative currents are orthogonal to fluid four-velocity.
- With RTA, one has to work in Landau frame.

# Dissipative quantities

- The dissipative quantities $\delta N^\mu$, $\delta T^{\mu\nu}$, and $\delta S^{\lambda,\mu\nu}$ are defined in terms of the non-equilibrium parts of the distribution functions:

$$\delta N^\mu = \int \mathrm{dP}\ \mathrm{dS}\ p^\mu(\delta f_s^+ - \delta f_s^-),$$

$$\delta T^{\mu\nu} = \int \mathrm{dP}\ \mathrm{dS}\ p^\mu p^\nu(\delta f_s^+ + \delta f_s^-),$$

$$\delta S^{\lambda,\mu\nu} = \int \mathrm{dP}\ \mathrm{dS}\ p^\lambda s^{\mu\nu}(\delta f_s^+ + \delta f_s^-).$$

- Remember the first-order solution from Boltzmann equation

$$\delta f_s^\pm = \frac{-\tau_{\rm eq}}{(u\cdot p)} e^{\pm\xi - p\cdot\beta}\left[\left(\pm p^\mu\partial_\mu\xi - p^\lambda p^\mu\partial_\mu\beta_\lambda\right)\left(1 + \frac{1}{2}s^{\alpha\beta}\omega_{\alpha\beta}\right) + \frac{1}{2}p^\mu s^{\alpha\beta}(\partial_\mu\omega_{\alpha\beta})\right].$$

- Substituting in above definitions leads to relativistic Navier-Stokes equations for 'spin'-hydrodynamics.

# First-order dissipative spin hydrodynamics

- Dissipative correction to conserved charge current:

$$\delta N^\mu = n^\mu = \tau_{\rm eq}\ \beta_n \nabla^\mu \xi$$

- Dissipative correction to energy-momentum tensor:

$$\delta T^{\mu\nu} = \pi^{\mu\nu} - \Pi\Delta^{\mu\nu}, \quad \pi^{\mu\nu} = 2\tau_{\rm eq}\,\beta_\pi \sigma^{\mu\nu}, \quad \Pi = -\tau_{\rm eq}\,\beta_\Pi \theta$$

- Dissipative correction to spin tensor:

$$\delta S^{\lambda,\mu\nu} = \tau_{\rm eq}\Big[B_\Pi^{\lambda,\mu\nu}\,\theta + B_n^{\kappa\lambda,\mu\nu}\,(\nabla_\kappa \xi) + B_\pi^{(\kappa\delta)\lambda,\mu\nu}\sigma_{\kappa\delta} + B_\Sigma^{\eta\beta\gamma\lambda,\mu\nu}\nabla_\eta \omega_{\beta\gamma}\Big].$$

- The transport coefficients $\beta_n,\ \beta_\pi,\ \beta_\Pi$ remain unchanged up to linear order in $\omega^{\mu\nu}$.

- Form of dissipation to spin current is new. Transport coefficients corresponding to spin dissipation can be found in: [S. Bhadury, W. Florkowski, **AJ**, A. Kumar and R. Ryblewski, arXiv:2008.10976]

# Dissipative spin hydrodynamics: other relevant works

- Other parallel approaches from Wigner function formalism [N. Weickgenannt, X.-l. Sheng, E. Speranza, Q. Wang and D. Rischke, PRD 100 (2019) 056018].
- Appraoch based on Lagrangian method [D. Montenegro and G. Torrieri, PRD 100 (2019) 056011].
- A very useful review on spin hydro: [W. Florkowski, R. Ryblewski and A. Kumar, Prog.Part.Nucl.Phys. 108 (2019) 103709, arXiv:1811.04409].
- Relatively unexplored area of research.
- Much work needed in this direction.

# Thank you!