w/ Abert Bergeron Gelander (ABBG) '19

Q// How do invariants of spaces, e.g. Betti #'s, behave wrt BS convergence?

Recall: $\mathcal{K} = \{$ locally finite pointed simplicial complexes $(K,p)\}/$ pted isom

If $K$ is a finite complex, let $\mu_K$ be the measure on $\mathcal{K}$ that's the push forward of the counting measure under

$$V(K) \longrightarrow \mathcal{K}, \quad p \mapsto (K,p).$$

Then $(K_i)$ BS-converges to $\mu$ if $\mu_{K_i}/\mathrm{vol}(K_i) \xrightarrow{\text{weakly}} \mu$.

# verts

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Define BS-convergence of finite vol Riem manifolds similarly using $\mathcal{M}^d = \{$ pted Riem mflds $\}/\sim$ instead of $\mathcal{K}$.

Thm (Elek '08) S'pose $(K_i)$ is a sequence of bounded degree complexes that BS-converges. Then $b_k(K_i)/\mathrm{vol}(K_i)$ converges $\forall k$.

$\exists$ uniform upper bound for the # of simplices containing a vertex

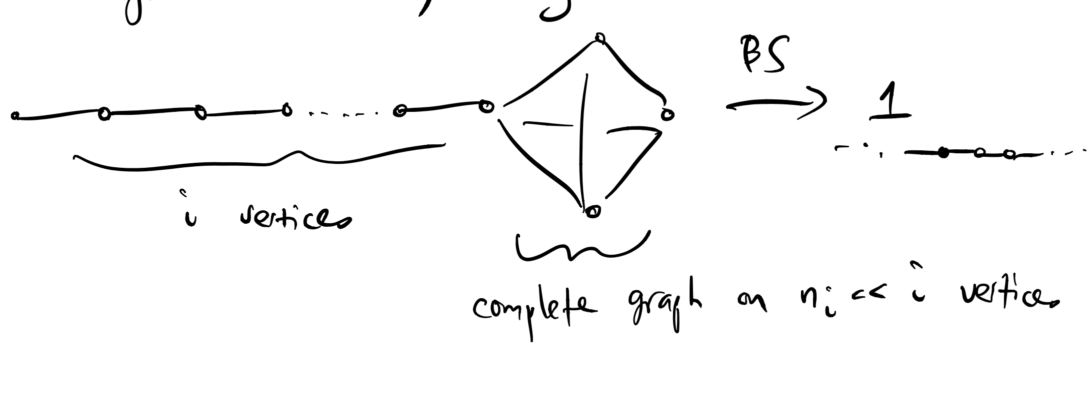
Here, vol = # vertices, $b_k = \dim H_k$, the dim of $k$th homology. $b_k/\mathrm{vol}$ is the <u>$k$th normalized Betti #</u>.

<u>Rmks</u> 1) Thm says that you can guess $b_k(K)/\mathrm{vol}(K)$ if you sample the geometry of $K$ in large balls around random vertices. The ratio is "testable".

2) If $K_i \xrightarrow{BS} \mu$ can view the thm as giving a def of "normalized Betti #'s of $\mu$." Schrödl (2018) shows how to define such an invt for an arbitrary unimodular random complex in a way that's cts under weak limits of sofic random complexes.

3) Thm is not true without the degree bound, e.g.

$K_i$:



but $\frac{b_1(K_i)}{\mathrm{vol}(K_i)} = \frac{\binom{n_i}{2} - n_i}{i + n_i - 1}$

$\leftarrow$ just has to be $\ll i^2$

$\leftarrow \sim i$

varying $n_i$ these can accumulate onto all of $\mathbb{R}$. Essentially the issue is that the Betti #'s of a complete graph aren't linearly bdd by vol.

Q// Similar thm for Riem mflds?

Need something like a degree bound, since $\exists$ Riem mflds w/ small vol, huge Bettis.

<u>Thm</u> (Elek, Bowen + ABBG) If $(M_i)$ is a BS-convergent seq of closed Riem $d$-mflds w/ sectional curvatures in $[a,b]$ and $\mathrm{inj}(M_i) \geq 2\varepsilon > 0$ then

$$\frac{b_k(M_i)}{\mathrm{vol}(M_i)} \text{ converge } \forall k.$$

Here $\mathrm{inj}(M) = \inf_{x \in M} \mathrm{inj}_M(x)$. Assumption above $\Rightarrow$ each $\varepsilon$-ball in each $M_i$ is a top ball.

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An <u>$\varepsilon$-net</u> in $M$ is a maximal subset $S \subset M$ w/ $d(x,y) \geq \varepsilon$ $\forall x, y \in S$.

The $\varepsilon$-balls $B(x, \varepsilon)$, $x \in S$, cover $M$, and the <u>nerve</u> $N(S)$ is the complex with vertex set $S$ and a $k$-simplex for every $k$-fold intersection of the balls $B(x, \varepsilon)$

nerve

<u>Nerve Lemma</u> If $\mathrm{inj}(M) \geq 2\varepsilon$, then $N(S)$ is homotopy equiv to $M$.

($\Rightarrow$ Bettis are same.)

<u>Pf of Thm</u> We have $M_i \xrightarrow{BS} \mu$, WTS $\frac{b_k}{vol}$ converge.

Elek
Bowen

Randomly choose an $\varepsilon$-net $S_i \subset M_i$ by superimposing a bunch of Poisson process, and construct the nerve $N(S_i)$ of the corresponding cover by $\varepsilon$-balls. Then $N(S_i)$ is h.e. to $M_i$, and the assumptions on inj and curvature $\Rightarrow$ $N(S_i)$ has bounded degree. Show these random complexes BS-converge and use Elek's thm.

There's a gap above, since it's quite subtle to show that the $\varepsilon$-nets produced vary cnts if you vary the mfld. We circumvent this problem by constructing random almost-nets, and show they're good enough. $\square$

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Suppose $G$ is a noncompact simple Lie group, $X = G/K$ the symmetric space, e.g. $X = \mathbb{H}^n$ or $X = SL_n\mathbb{R}/SO(n)$

An <u>X-mnfld</u> is a quotient $M = \Gamma \backslash X$.

<u>Gromov</u> $\exists$ some $C = C(X)$ s.t. if $M$ is a fin vol $X$-mnfld, we have

$$b_k(M) \leq C \cdot vol(M) \quad \forall k.$$

<u>Thm</u> (ABBG) S'pose $X \neq \mathbb{H}^3$ and $M_i$ is a BS-convergent seq of fin vol $X$-mnflds. Then

$$\frac{b_k(M_i)}{vol(M_i)} \quad \text{converge } \forall k.$$

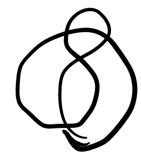
<u>Rmks</u> When $X = \mathbb{H}^2$, this follows from Gauss Bonnet, since any hyp surface $S$ satisfies

$$(\star) \quad \frac{b_1(S)}{vol\, S} \underset{GB}{=} \frac{2-\chi(S)}{2\pi\chi(S)} = \frac{1}{\pi\chi(S)} - \frac{1}{2\pi}$$

If $(S_i)$ BS converges then either $\chi(S_i) \longrightarrow \infty$ or are eventually constant, so in either case ☆ converges.

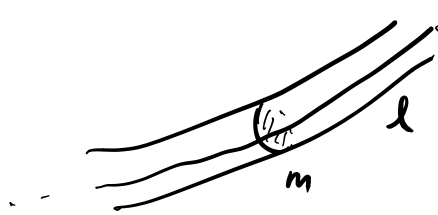
Ex, when $X = \mathbb{H}^3$

Since $K$ is the figure 8 knot, $M = S^3 \setminus K$. Then $M$ admits a complete hyp metric of fin vol, as do the Dehn fillings



$$M_i = M(1, i)$$

for large $i$. Here, we create $M_i$ by attaching a solid torus to $S^3 \setminus N(K)$ along $\partial N(K) \cong T^2$, so that the meridian of the solid torus is glued to $m + i\ell \in H_1(\partial N(K))$.


$\ell = 0 \in H_1(S^3 \setminus N(K)) \cong \mathbb{Z}m$.

Work of Thurston $\Rightarrow$

$$M_i \xrightarrow{BS} \mu^M / \mathrm{vol}(M)$$

But $b_1(M) = 1$ and $b_1(M_i) = 0$ since we're gluing $0 \in H_1(\text{solid torus})$ to a generator m + il for $H_1(S^3 \setminus N(K))$. Then

BS-converge

| X-mnfld | $M_1$ | $M$ | $M_2$ | $M$ | $M_3$ | $M$ | ... |
|---|---|---|---|---|---|---|---|
| $b_1/\mathrm{vol}$ | 0 | $\frac{1}{\mathrm{vol}\, M}$ | 0 | $\frac{1}{\mathrm{vol}\, M}$ | ... | | |

don't converge.

Application S'pose X has rank $\geq 2$, e.g. $X = SL_n\mathbb{R}/SO(n)$ $n \geq 3$.

**Thm** (ABBGNRS '17) If $(M_i)$ is any sequence of fin vol $X$-mfld (pairwise nonisometric) then $M_i \xrightarrow{BS} 1_X$.

(This is the geometric version of $\mu_{\Gamma_i} \to 1_e$) as in Talk 2.

**Cor** If $(M_i)$ is as above

$$(\star) \qquad \frac{b_k(M_i)}{\mathrm{vol}(M_i)} \longrightarrow \beta_k^{(2)}(X)$$

$k^{th}$ $L^2$-Betti of $X$

**Rmk** $(\star)$ follows for cofinal covering towers $M_1 \leftarrow M_2 \leftarrow M_3 \leftarrow \cdots$ of compact $X$-mfld by DeGeorge–Wallach '78. Such towers exist $\forall X$ (Borel, Malcev)

**Pf of Cor** Any such $M_i \xrightarrow{BS} 1_X$. Interleave $(M_i)$ w/ a cofinal covering tower and apply ABBG. $\square$

# The Proof of ABBG

Let $M_i$ be a BS-convergent seq of $X$-mflds, $X \neq \mathbb{H}^2, \mathbb{H}^3$. WTS $b_k(M_i)/\mathrm{vol}(M_i)$ converge.

Let $\varepsilon > 0$ be small, and set

thin part $\to$ $(M_i)_{<\varepsilon} = \{ x \in M_i \mid \mathrm{inj}_{M_i}(x) < \varepsilon \}$

thick part $\to$ $(M_i)_{\geq \varepsilon} = M \setminus (M_i)_{<\varepsilon}$.

Step 1 One can show $(M_i)_{\geq\varepsilon}$ BS-converges. Construct random almost nets in the $(M_i)_{\geq\varepsilon}$, take nerve complexes to get a BS-convergent sequence of complexes $(K_i)$. Work of Gelander $\Rightarrow$ $K_i$ is h.e. to $(M_i)_{\geq\varepsilon}$. (This is more subtle since $(M_i)_{\geq\varepsilon}$ has $\partial$.)

Elek $\Rightarrow$ normalized Bettis of $(M_i)_{\geq\varepsilon}$ converge. Not hard to see that $b_k((M_i)_{\geq\varepsilon})/\mathrm{vol}(M_i)$ converges too.

Step 2 Show that adding back in the thin parts doesn't change $b_k$ significantly.

$\text{rank}_{\mathbb{R}} X = 1$ $\Rightarrow$ The Margulis Lemma $\Rightarrow$ each component of the thin part is either a $D^{n-1}$-bundle over $S^1$ ($n = \dim X$) or a product $\partial \times [0,\infty)$.

Gluing on such a component to the thick part changes the Bettis by at most 1, and if $\dim \geq 4$, the number of components is sublinear in volume.

So,
$$\frac{b_k((M_i)_{\geq \varepsilon})}{\text{vol}\, M_i}, \quad \frac{b_k(M_i)}{\text{vol}(M_i)}$$

converge to same value.

$\underline{\operatorname{rank}_{\mathbb{R}} X \geq 2}$ ABBGNRS $\Longrightarrow$ $M_i \xrightarrow{BS} 1_X$,

$$\Longrightarrow \quad \frac{\mathrm{vol}\left((M_i)_{<\varepsilon}\right)}{\mathrm{vol}(M_i)} \longrightarrow 0 .$$

We adapt arguments of Gromov to show that

$$b_k\left((M_i)_{<\varepsilon}\right),\ b_k\left(\partial (M_i)_{<\varepsilon}\right) \leq \overbrace{C \cdot \mathrm{vol}\left((M_i)_{<\varepsilon}\right)}^{\ll \mathrm{vol}(M_i)}$$

so gluing $(M_i)_{<\varepsilon}$ onto $(M_i)_{\geq \varepsilon}$ changes the Bettis by an amount $\ll \mathrm{vol}(M_i)$. $\square$