Geometric Langlands and 3d mirror symmetry II

Last time: we defined alg. 3d QFT ($\{1,2\}$-extended). We gave examples, discussed 3d $\mathcal{N}=4$ theories, esp. σ-models T_Y into T^*Y, and "vanilla" 3d mirror symmetry:

$(\mathcal{X}, \mathcal{X}^*)$ is 3d mirror dual

if $T_{\mathcal{X}} \simeq T^*_{\mathcal{X}^*}$.

(meta-conjecture).

Practical algebraic consequence

$$T_{\mathcal{X}, A} \simeq T_{\mathcal{X}^*, B}.$$

Today: add in 4d and G.

Briefly: a 4d $[1,2]$-extended $\overset{alg.}{Q}FT$ $\mathbb{T}$
is the "same" as before, but
one categorical level higher.

$$X \rightsquigarrow \mathbb{T}(X) \in DGCat.$$

$$\mathring{D}_x \rightsquigarrow \mathbb{T}(\mathring{D}_x) \in 2DGCat.$$

Same kinds of "cobordisms" are allowed

Terminology: $\mathbb{T}_1 \longrightarrow \mathbb{T}_2$ is called an
"interface"

$triv \longrightarrow \mathbb{T}$ or $\mathbb{T} \longrightarrow triv$
is a boundary condition.

In 3d bad to think of interfaces as symmetric, but in 4d usually okay.

<u>Obs</u>: $\mathrm{triv}_{4d} \longrightarrow \mathrm{triv}_{4d} \Leftrightarrow$ a 3d theory.

Main 4d analytic theory is

4d $\mathcal{N}=4$ Yang-Mills for G.

Denote this by YM_G.

Has A & B - twists $YM_{G,A}$, $YM_{G,B}$.

<u>Concretely</u>:

$$YM_{G,A}(X) = \text{D-mod}(\mathrm{Bun}_G).$$

$$YM_{G,A}(\mathring{D}_x) = \underbrace{\text{D-mod}(\mathcal{L}G)}_{\text{monoidal under conv.}}\text{-mod}$$

$$\mathrm{Vac}_x \in \mathcal{YM}_{G,A}(\mathring{D}) \simeq \mathcal{L}G\text{-mod}$$

abuse of not.

$$\|$$

$$D\text{-mod}(\mathrm{Gr}_G) \qquad \mathrm{Gr}_G := \mathcal{L}G/\mathcal{L}^+G$$

$$\mathcal{L}G = \mathrm{Maps}(\mathring{D}, G)$$

$$\mathcal{L}^+G = \mathrm{Maps}(D, G).$$

$$\mathcal{YG}_{G,A}(\mathring{D}) \xrightarrow{\mathrm{Hch}} \mathrm{DGCat}$$

$$\mathcal{C} \longmapsto \mathcal{C} \otimes D(\mathrm{Bun}_G^{\mathrm{level},x}).$$

$$D(\mathcal{L}G)$$

$$YM_{G,B}(X) = QCoh(LS_{\check{G}}(X))$$ moduli of dR local systems on X

$$YM_{G,B}(\mathring{D}) = ShvCat_{/LS_{\check{G}}(\mathring{D})}.$$

Physical S-duality: $YM_G \simeq YM^*_{\check{G}}$.

Alg/twisted S-duality: $YM_{G,A} \simeq YM_{\check{G},B}$.

Up to blurring, this is packaging

global GL

local GL

local-global compatibility.

S-duality is
a Thm as stated for $G = \mathbb{G}_m$
(Laumon, Beilinson-Drinfeld).

To pin this down, tradition is to use compatibility w/ bdry conditions.

Big one: Whit. bdry condition $\overset{\text{S-dual}}{\longleftrightarrow}$ ~~crit. level Kac-Moody~~ Neumann

AKA: princ. Nahm pole

Won't discuss.

~~A~~ construction: given $G \circlearrowright Y$

$\rightsquigarrow \mathcal{B}_Y$ for YM_G.

Classically: this $\mathcal{B}_Y$ really depends on the Hamiltonian G-space

$$T^*Y \xrightarrow{\mu} \mathfrak{g}^*.$$

For $G = G_{\mathbb{C}}^{\vee}$, $\mathcal{B}_Y \leftarrow$ 3d $N=4$ theory T_Y.

("full") 3d mirror symmetry:

$$(G \curvearrowright \mathcal{X}) \quad , \quad (\check{G}_{\mathbb{C}} \curvearrowright \mathcal{X}^*)$$

are 3d mirror dual if:

$$\mathcal{B}_{\mathcal{X}} \xleftrightarrow{\text{S-duality}} \mathcal{B}^*_{\mathcal{X}^*}.$$

Examples: 1) $G = \mathbb{G}_m \curvearrowright \mathcal{X} = \mathbb{A}^1$.

$$\check{G} = \mathbb{G}_m \curvearrowright \mathcal{X}^* = \mathbb{A}^1.$$

"Tate" example. Thm of Hilburn-R.
(for now: equivalence of cats of local operators, no global aspects).

2) $G = GL_2 \curvearrowright \mathcal{X} = GL_2$

$$\check{G} = GL_2 \curvearrowright \mathcal{X}^* = \left\{ \begin{matrix} \ell \text{ a line } + \\ \ell \longrightarrow k^2 \end{matrix} \right\}.$$

($\exists$ generalization for GL_n).

Many more examples. See Gaiotto-Witten, Hilburn-Yoo, Ben-Zvi-Sakellaridis-Venkatesh

What does 3d mirror symmetry mean here?

$$\mathcal{B}_{\mathcal{X},A}(\mathring{D}_x) \hookrightarrow \text{D-mod}(\mathcal{L}\mathcal{X}) \in \mathcal{L}G\text{-mod}.$$

$$\begin{tikzcd} \mathcal{B}_{\mathcal{X},B}(\mathring{D}_x) \arrow[r, hook'] & \text{IndCoh}(\text{Maps}(\mathring{D}_{dR}, \mathcal{X}/G)) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] \\ & \text{ShvCat}_{/LS_G(\mathring{D})} \end{tikzcd}$$

In practice:

$$\text{D-mod}(\mathcal{L}\mathcal{X}) \xleftrightarrow{LGL} \text{IndCoh}(\text{Maps}(\mathring{D}_{dR}, \check{\mathcal{X}}/\check{G})).$$

Any such assertion is conditional on LGL for $(G, \check{G})$.

Various ways to use this formalism to construct material for your research statement:

a) $G = \mathbb{G}_m$ (Helburn-R.)

b) evaluate $\mathcal{B}_{\mathcal{X}, A/B}(\mathring{D}_x)$:

$$\mathcal{YM}_{G, A/B}(\mathring{D}_x) \longrightarrow \mathrm{DGCat}$$

on $\mathrm{Vac}_{x, A/B}$.

Many such examples are work of Braverman - Finkelberg - Ginzburg - Travkin.

c) Up to you.

What we actually show:

$$D^!(\mathcal{L}\mathbb{A}^1) \simeq \mathrm{IndCoh}^*(\mathrm{Maps}(\mathring{D}_{dR}, \mathbb{A}^1/\mathbb{G}_m)).$$

$$D^*(\mathcal{L}\mathbb{G}_m) \overset{BD}{\simeq} \mathrm{QCoh}(\mathrm{LS}_{\mathbb{G}_m}(\mathring{D}))$$

Technically: $\mathrm{Maps}(\mathring{D}_{dR}, \mathbb{A}^1/\mathbb{G}_m)$ is built from ∞-ly singular non-Noetherian geometry.