Geometric Langlands and 3d mirror symmetry

Joint with Justin Hilburn.

For physizists: compare line operators in A-twisted pure hypermultiplet 3d theory and B-twisted $U(1)$-gauged hypermultiplet

Moreover, consider these as bdry theories for twisted $U(1)$-gauge theory.

Want to discuss: algebraic geometry of 3d QFTs.

What does this possilbey mean?

Reminder: A 3d TFT
[0,3]-extended

assigns:

a # to a 3-mfld

a vector space to a 2-mfld

a category to a 1-mfld

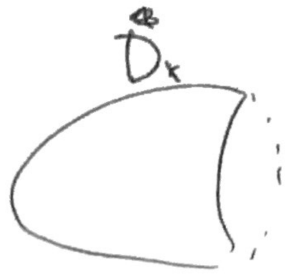
a 2-cat. to a 0-mfld.

An algebraic [1,2]-extended QFT T on a sm. proj. curve X is an assignment:

$$X \longmapsto T(X) \in \mathrm{Vect}.$$

$$x \in X \leadsto \mathring{D}_x \longmapsto T(\mathring{D}_x) \in \mathrm{DGCat}.$$

Only some cobordisms are allowed:

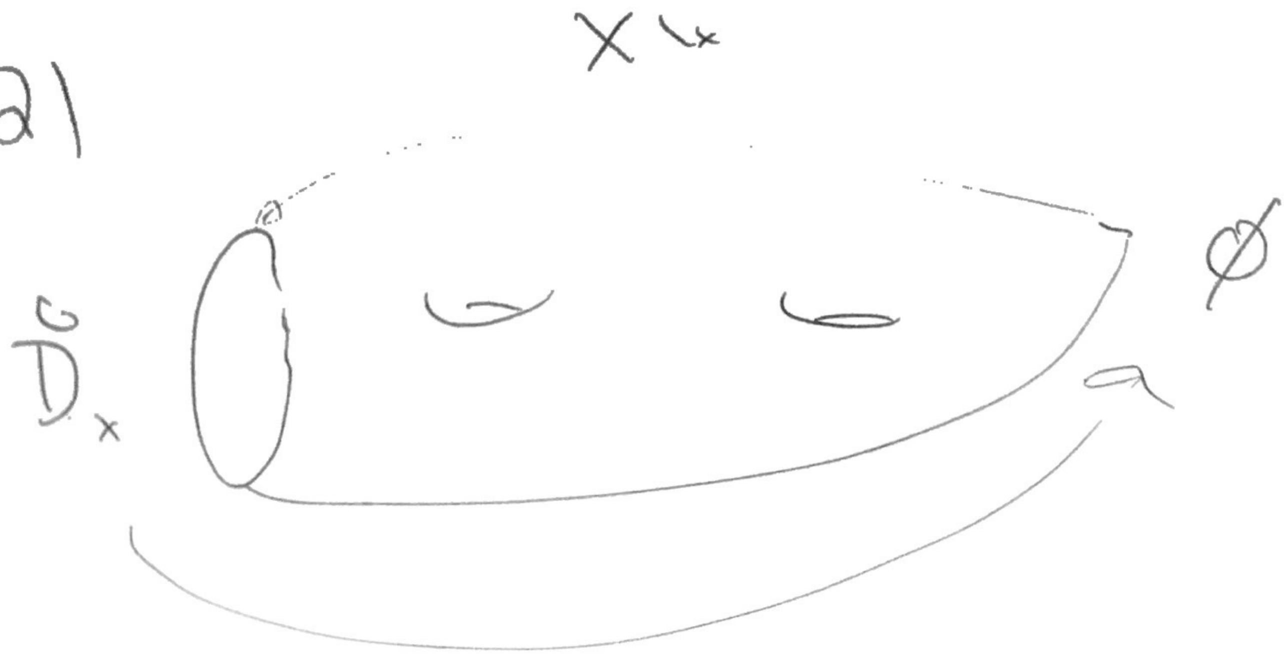


1) $\emptyset \longrightarrow \mathring{D}_x$

$\rightsquigarrow \quad T(\emptyset) = \mathrm{Vect} \longrightarrow T(\mathring{D}_x)$

amounts to an object

$\mathrm{Vac}_x \in T(\mathring{D}_x)$.



$\rightsquigarrow \quad T(\mathring{D}_x) \xrightarrow{H_{ch}} T(\emptyset) = \mathrm{Vect}.$

"chiral homology" for the theory T.

Example: $H_{ch}(\mathrm{vac}_x) = T(X)$.

Examples:

1) If $\mathcal{A}$ is a chiral ~~co~~/factorization alg. on X (due to Beilinson–Drinfeld; local on X analogue of a VOA)

$\rightsquigarrow$ a 3d theory $T_{\mathcal{A}}$.

$T_{\mathcal{A}}(B_x) = \mathcal{A}\text{-mod}_x$.

$vac_x \in \mathcal{A}\text{-mod}_x$ is the "vacuum" representation $\mathcal{A}_x$.

$H_{ch} : \mathcal{A}\text{-mod}_x \longrightarrow Vect$ is what BD call chiral homology.

2) Say given some "target" stack Y. Then there is a 3d theory assigns:

$$D\text{-mod}(\mathcal{L}_x Y)$$

to a $\mathring{D}_x$.

there: e.g., if $Y = \mathbb{A}^1$

$\mathcal{L}Y$ = ind-pro-co-dim'l affine space of Laurent series.

Secretly: $T_{\mathcal{A}}$ for $\mathcal{A}$ a CDO on Y

(if Y is smooth & affine)

3) With Y as before, there is a 3d theory attaching

$$\mathrm{IndCoh}(\mathrm{Maps}(\mathring{D}_{dR}, Y))$$

to $\mathring{D}$.

Example: $Y = \mathbb{B}G$, $\mathrm{Maps}(\mathring{D}_{dR}, Y) = LS_G(\mathring{D}) =$ dR local systems on $\mathring{D}$.

$Y = \mathbb{A}^1/\mathbb{G}_m$, $\mathrm{Maps}(\mathring{D}_{dR}, Y) =$ moduli of a line bundle w/ connection $(\mathcal{L}, \nabla)$ on $\mathring{D}$ + $s \in \Gamma^\nabla(\mathcal{L})$ a flat section.

:xactly the 3d theory for $\mathcal{A}$ constant comm chiral alg. w/ fiber = Fun(Y) (Y affine).

Another way these 3d algebraic theories often appear is via (hol, top)-twists of "analytic" (i.e. "normal") QFTs.

Background: Supersymmetric QFTs have twists (other "deformed" QFTs), often "nicer": holomorphic or topological.

More specifically, 3d $\underbrace{\mathcal{N}=4}_{\text{how much SUSY you have}}$ theories

$\Rightarrow$ analytic setting

T have two topological twists T_A and T_B. Often these are algebraic.

Moreover, a 3d $\mathcal{N}=4$ theory T has an "abstract mirror dual" T^* that is the "same" theory, but SUSY algebra embedding is modified.

$$(T^*)_A = T_B$$

$$(T^*)_B = T_A.$$

Where do these theories come from?

(algebraic)
σ-models with symplectic targets

AKA hyperKähler targets.

To simplify quantization issues, assume targets are T^*Y for some smooth stack Y.

Will let T_Y = σ-model with target $T^* Y$.

Briefly: $T_{Y,A}(\mathring{D}_x) = \text{D-mod}(Y)$

$T_{Y,B}(\mathring{D}_x) = \text{IndCoh}(\text{Maps}(\mathring{D}_{dR}, Y))$

3d mirror symmetry makes assertions:

$(\mathcal{X}, \mathcal{X}^*)$ two stacks "mirror dual".

s.t. $T^*_{\mathcal{X}} = T^*_{\mathcal{X}^*}$.

Predicts (pass to A-twists):

$$\text{D-mod}(\mathcal{L}\mathcal{X}) \simeq \text{IndCoh}(\text{Maps}(\mathring{D}_{dR}, \mathcal{X}^*)).$$