On confluence relation

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Plan

1. Confluence relation
2. Associator relation = Confluence relation

References:

H. Furusho; *The pentagon equation and the confluence relations*, Amer. J. Math. Vol 144, No 4, (2022) 873-894.

M. Hirose and N. Sato, *Iterated integrals on* $\mathbf{P}^1 \setminus \{0, 1, \infty, z\}$ *and a class of relations among multiple zeta values*, Adv. Math. 348 (2019), 163–182.

Section 1: Confluence relation by Hirose and Sato

Setup

Let $z \neq 0, 1 \in \mathbb{C}$.

$$\mathcal{A}_z = \mathbb{Q}\langle e_0, e_z, e_1\rangle$$
$$\cup$$
$$\mathcal{A}_z^1 = \mathbb{Q} \oplus \mathcal{A}_z e_1 \oplus \mathcal{A}_z e_z$$
$$\cup$$
$$\mathcal{A}_z^0 = \mathbb{Q} \oplus \mathbb{Q}e_z \oplus \bigoplus_{a=0,z} \bigoplus_{b=1,z} e_a \mathcal{A}_z e_b$$
$$\cup$$
$$\mathcal{A}_z^{-1} = \mathbb{Q}\langle e_z\rangle \cdot \mathcal{A}_z^{-2}$$
$$\cup$$
$$\mathcal{A}_z^{-2} = \mathbb{Q} \oplus e_0 \mathcal{A}_z e_1 \oplus e_0 \mathcal{A}_z e_z$$

Put $\mathcal{A} = \mathbb{Q}\langle e_0, e_1\rangle$ and set $\mathcal{A}^i = \mathcal{A}_z^i \cap \mathcal{A}$ $(i = 1, 0)$.

Standard relations

Const $: \mathcal{A}_z \to \mathcal{A}$: alg. hom. by $e_a \mapsto \begin{cases} e_a & a = 0, 1, \\ 0 & a = z. \end{cases}$

$\partial_{z,\alpha} : \mathcal{A}_z \to \mathcal{A}_z$: lin.map ($\alpha \in \{0, z, 1\}$) defined by

$$\partial_{z,\alpha}(e_{a_n} \cdots e_{a_1}) = \sum_{i=1}^{n} (\delta_{\{a_i,a_{i+1}\},\{z,\alpha\}} - \delta_{\{a_{i-1},a_i\},\{z,\alpha\}})\, e_{a_n} \cdots \check{e}_{a_i} \cdots e_{a_1}$$

with $a_0 = 0$ and $a_{n+1} = 1$.

$$\mathfrak{I}_{\mathrm{ST}} := \{w \in \mathcal{A}_z^0 \mid \mathrm{Const}(\partial_{z,\alpha_1} \cdots \partial_{z,\alpha_r} w) = 0 \quad \text{for } r \geqslant 0, \alpha_1, \ldots, \alpha_r \in \{0, 1\}\}.$$

An element of $\mathfrak{I}_{\mathrm{ST}}$ is called a standard relation.

Iterated integral map

$$L : \mathcal{A}_z^0 \to \mathbb{C}\{\text{holomorphic functions on z}\}$$

$$e_{a_n} \cdots e_{a_1} \mapsto \int_{0<t_1<\cdots<t_n<1} \frac{dt_n}{t_n - a_n} \wedge \cdots \wedge \frac{dt_1}{t_1 - a_1}$$

Proposition([HS]):

$$L(\mathfrak{I}_{\mathrm{ST}}) = 0.$$

Question: How can we derive relations among MZV from $\mathfrak{I}_{\mathrm{ST}}$?

$$\begin{array}{ccc} \mathcal{A}_z^0 & \xrightarrow{L} & \{\text{hol.fcns.on } z\} \\ \uparrow & & \uparrow \\ \mathcal{A}^0 & \xrightarrow{L} & \mathbb{C} \end{array}$$

Confluence relations

$\lambda : (\mathcal{A}_z^0, ш) \to (\mathcal{A}^0, ш)$: alg.hom is defined by

$$\mathcal{A}_z^0 \xleftarrow[\simeq]{ш} \mathcal{A}_z^{-2} \otimes (\mathcal{A}_z^0 \cap \mathbb{Q}\langle e_1, e_z\rangle) \xrightarrow{\mathrm{id}\otimes\tau_z} \mathcal{A}_z^{-2} \otimes (\mathcal{A}_z^0 \cap \mathbb{Q}\langle e_0, e_z\rangle) \xrightarrow{ш}$$

$$\mathcal{A}_z^{-1} \xleftarrow[\simeq]{ш} \mathbb{Q}\langle e_z\rangle \otimes \mathcal{A}_z^{-2} \xrightarrow{\mathrm{Const}\otimes\mathrm{id}} \mathcal{A}_z^{-2} \xrightarrow{e_z=e_1} \mathcal{A}^0,$$

where $\tau_z : (\mathcal{A}_z, \cdot) \to (\mathcal{A}_z, \cdot)$ is anti-automor. st. $\begin{cases} e_0 \mapsto & e_z - e_1, \\ e_1 \mapsto & e_z - e_0, \\ e_z \mapsto & e_z. \end{cases}$

Definition:

$$\mathfrak{I}_{\mathrm{CF}} := \lambda(\mathfrak{I}_{\mathrm{ST}})$$

An element of $\mathfrak{I}_{\mathrm{CF}}$ is called a confluence relation.

Results of Hirose and Sato

Theorem ([HS]):

- $L(\mathfrak{I}_{\mathrm{CF}}) = 0.$
- Confluence relations imply the double shuffle relations.

Conjecture ([HS]):

Confluence relations might exhaust all the relations among MZVs.

Section 2:
Associator relation is equivalent to Confluence relation

Review: Associator

Definition ([Drinfeld,'91] and [F,'10]):

An associator is a pair (μ, φ) with $\mu \in \mathbb{K}^\times$ ($\mathbb{K}$: a field of ch= 0) and a series $\varphi(f_0, f_1) \in \mathbb{K}\langle\langle f_0, f_1\rangle\rangle = \widehat{U\mathfrak{f}_2}$ satisfying the following associator relations

- $\varphi \in \exp[\hat{\mathfrak{f}}_2, \hat{\mathfrak{f}}_2]$,
- $(\varphi | f_0 f_1) = \frac{\mu^2}{24}$,
- $\varphi_{345}\varphi_{512}\varphi_{234}\varphi_{451}\varphi_{123} = 1$ in $\widehat{U\mathfrak{P}_5}$,

where $\mathfrak{P}_5$ is the Lie algebra generated by t_{ij} $(i, j \in \mathbb{Z}/5)$ with the relations

- $t_{ij} = t_{ji}, \quad t_{ii} = 0, \quad \sum_{j\in\mathbb{Z}/5} t_{ij} = 0 \quad (\forall i \in \mathbb{Z}/5)$,
- $[t_{ij}, t_{kl}] = 0$ for $\{i, j\} \cap \{k, l\} = \emptyset$.

Main Theorem： Associator$\Longleftrightarrow$ Confluence

Regard $U\mathfrak{f}_2 = \mathbb{Q}\langle f_0, f_1\rangle$ with the dual of $\mathcal{A} = \mathbb{Q}\langle e_0, e_1\rangle$ by the pairing $< e_i \mid f_j >= \delta_{ij}$ for $i, j = 0, 1.$

Theorem (by F)

Let $\varphi \in \exp[\hat{\mathfrak{f}}_2, \hat{\mathfrak{f}}_2]$. *Then it is an associator if and only if it satisfies the confluence relations, i.e.* $< l \mid \varphi >= 0$ *for any* $l \in \mathfrak{I}_{\mathrm{CF}}$.

Corollary (by F)

$GRT_1 = \{\varphi \in \exp[\hat{\mathfrak{f}}_2, \hat{\mathfrak{f}}_2] \mid < l|\varphi >= 0$ *for* $l \in \mathfrak{I}_{\mathrm{CF}}$ *or* $l = e_0e_1\}$.

preparation: Bar construction

Let $(A^\bullet = \oplus_{q=0}^{\infty} A^q, d)$ be DGA.

Its reduced bar complex is given by

$$\bar{B}^\bullet(A) := \oplus_{r=0}^{\infty} (\bar{A}^\bullet)^{\otimes r}$$

with $\bar{A}^\bullet = \oplus_{i=0}^{\infty} \bar{A}^i$ where $\bar{A}^0 = A^1/dA^0$ and $\bar{A}^i = A^{i+1}$.

It forms a CDGHA with the deconcatication coproduct and the differential

$$d = d_{\text{int}} + d_{\text{ext}}$$

$d_{\text{int}}[a_1|\cdots|a_k] = \sum_{i=1}^{k} (-1)^i [Ja_1|\cdots|Ja_{i-1}|da_i|a_{i+1}|\cdots|a_k]$
$d_{\text{ext}}[a_1|\cdots|a_k] = \sum_{i=1}^{k} (-1)^{i-1} [Ja_1|\cdots|Ja_{i-1}|Ja_i \cdot a_{i+1}|a_{i+2}|\cdots|a_k]$.
where $Ja = (-1)^{p-1}a$ for $a \in \bar{A}^p$.

Step 1: Moduli spaces

$$\mathcal{M}_{0,4} := \mathrm{Conf}_4(\mathbb{P}^1)/\mathrm{PGL}(2) \simeq \{z \in \mathbb{P}^1 \mid z \neq 0, 1, \infty\}$$
$$(p_1, p_2, p_3, p_4) \leftrightarrow (0, z, 1, \infty)$$
$$\mathcal{M}_{0,5} := \mathrm{Conf}_5(\mathbb{P}^1)/\mathrm{PGL}(2) \simeq \{(z, w) \in (\mathbb{P}^1)^2 | z, w \neq 0, 1, \infty, z \neq w\}$$
$$(p_1, p_2, p_3, p_4, p_5) \leftrightarrow (0, w, z, 1, \infty)$$
$$\mathcal{X}(z) := \mathbb{P}^1 \setminus \{0, 1, z, \infty\} = \{w \in \mathbb{P}^1 \mid w \neq 0, z, 1, \infty\}$$

$$\mathcal{B} := H^0 \bar{B}\left(\Omega^{\bullet}_{\mathrm{DR}}(\mathcal{M}_{0,5})\right) \simeq U\mathfrak{P}_5^*$$
$$\mathcal{A} \simeq H^0 \bar{B}\left(\Omega^{\bullet}_{\mathrm{DR}}(\mathcal{M}_{0,4})\right)$$
$$\mathcal{A}_z \simeq H^0 \bar{B}\left(\Omega^{\bullet}_{\mathrm{DR}}(\mathcal{X}(z))\right)$$

$\mathrm{pr}_2 : \mathcal{M}_{0,5} \to \mathcal{M}_{0,4}$: univ family with the fiber $\mathcal{X}(z)$ at z

$$\mathrm{dec}_2 : \mathcal{B} \simeq \mathcal{A}_z \otimes \mathcal{A} \qquad j_2 : \mathcal{A}_z \hookrightarrow \mathcal{B}$$

Step 2: Upgrade Hirose-Sato's techniques

$$\begin{array}{ccc} \mathcal{B}^1 & \hookrightarrow & \mathcal{B} \\ {\scriptstyle j_2}\uparrow & & \uparrow{\scriptstyle j_2} \\ \mathcal{A}_z^1 & \hookrightarrow & \mathcal{A}_z \end{array}, \quad \begin{array}{ccc} \mathcal{B} & \xrightarrow{\widetilde{\mathrm{Const}}} & \mathcal{A} \\ {\scriptstyle j_2}\uparrow & & \| \\ \mathcal{A}_z & \xrightarrow{\mathrm{Const}} & \mathcal{A} \end{array}, \quad \begin{array}{ccc} \mathcal{B} & \xrightarrow{\tilde{\partial}_{z,\alpha}} & \mathcal{B} \\ {\scriptstyle j_2}\uparrow & & \uparrow{\scriptstyle j_2} \\ \mathcal{A}_z & \xrightarrow{\partial_{z,\alpha}} & \mathcal{A}_z \end{array} \quad (\alpha = 0, 1)$$

$$\mathfrak{I}_{\mathrm{ST}} = \{w \in \mathcal{A}_z^0 \mid \mathrm{Const}(\partial_{z,\alpha_1} \cdots \partial_{z,\alpha_r} w) = 0 \text{ for } r \geqslant 0, \alpha_1, \ldots, \alpha_r \in \{0, 1\}\}.$$

$$\tilde{\mathfrak{I}}_{\mathrm{ST}} := \{w \in \mathcal{B}^1 \mid \widetilde{\mathrm{Const}}(\tilde{\partial}_{z,\alpha_1} \cdots \tilde{\partial}_{z,\alpha_r} w) = 0 \text{ for } r \geqslant 0, \alpha_1, \ldots, \alpha_r \in \{0, 1\}\}.$$

Lemma

- $\tilde{\mathfrak{I}}_{\mathrm{ST}} \cap j_2(\mathcal{A}_z^0) = j_2(\mathfrak{I}_{\mathrm{ST}})$.
- *For any* $l \in \tilde{\mathfrak{I}}_{\mathrm{ST}}$ *and series* φ, $< l | \varphi_{351} >= 0$.

Step 3: Associator $\Longrightarrow$ Confluence

Key Formula:

For any $l \in \mathcal{A}_z^0$ and group-like series $\varphi \in \mathbb{Q}\langle\langle f_0, f_1 \rangle\rangle$,

$$< \lambda(l) \mid \varphi >=< j_2(l) \mid \varphi_{243}^{-1}\varphi_{215}\varphi_{534} >$$

If φ is an associator,

$$=< j_2(l) \mid \varphi_{351}\varphi_{124} >$$

$$=< j_2(l) \mid \varphi_{351} >$$

If $l \in \mathfrak{J}_{\mathrm{ST}}$, then $j_2(l) \in \tilde{\mathfrak{J}}_{\mathrm{ST}}$. By $< \tilde{\mathfrak{J}}_{\mathrm{ST}} \mid \varphi_{351} >= 0$,

$$= 0.$$

Step 4: Confluence $\Longrightarrow$ Associator

Assume that φ satisfies the confluence relation.

Then by key formula, for any $\boldsymbol{l} \in \mathfrak{I}_{\mathbf{ST}}$:

$$\mathbf{0} = < j_2(l) | \varphi_{243}^{-1} \varphi_{215} \varphi_{534} > = < j_2(l) | \varphi_{342} \varphi_{215} \varphi_{534} >$$
$$= < j_2(l) | \pentagon >,$$

where $\pentagon := \varphi_{153} \varphi_{342} \varphi_{215} \varphi_{534} \varphi_{421}$.

While we have $\mathbf{pr}_3(\pentagon) = \mathbf{pr}_4(\pentagon) = \mathbf{1}$.
By

$$\mathfrak{I}_{\mathbf{ST}}^{\perp} \cap \ker \mathbf{pr}_3 \cap \ker \mathbf{pr}_4 = \mathbf{1},$$

we have

$$\pentagon = \mathbf{1}.$$

$\square$