An introduction to KZ associator

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Plan

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- KZ-associator (a.k.a Drinfeld associator) Φ_{KZ}
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MPL

Definition 1. For $m, k_1, \ldots, k_m > 1$, $z \in \mathbb{C}$, multiple polylogarithm (MPL) is the complex function defined by

$$\mathrm{Li}_{k_1,\ldots,k_m}(z) = \sum_{0<n_1<\cdots<n_m} \frac{z^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}}$$

- It converges on $\{z \in \mathbb{C} \mid |z| < 1\}$.
- $\lim_{z\to 1} \mathrm{Li}_{k_1,\ldots,k_m}(z) = \zeta(k_1, \ldots, k_m)$: MZV $(k_m > 1)$.
- $\frac{d}{dz}\mathrm{Li}_{k_1,\ldots,k_m}(z) = \begin{cases} \frac{1}{z}\mathrm{Li}_{k_1,\ldots,k_{m-1},k_m-1}(z) & (k_m > 1), \\ \frac{1}{1-z}\mathrm{Li}_{k_1,\ldots,k_{m-1}}(z) & (k_m = 1). \end{cases}$

KZ (Knizhnik-Zamolodchikov) equation

Definition 2. KZ equation is the diff.eqn. over $\mathbf{P}^1_{\mathbb{C}} \backslash \{0, 1, \infty\}$:

$$dG = \omega_{\mathrm{KZ}}(z) \cdot G(z)$$

with

$$\begin{cases} G(z) \in \mathbb{C}\langle\langle A, B\rangle\rangle, \\ \omega_{\mathrm{KZ}}(z) = \frac{dz}{z} A + \frac{dz}{z-1} B \end{cases}$$

Lemma 3. Let $G(z), H(z) \in \mathbf{SolKZ}$ and $G(z) \in \mathbb{C}\langle\langle A, B\rangle\rangle^\times$. Then $G(z)^{-1} \cdot H(z)$ is constant.

Proof.

$$\frac{d}{dz}\{G(z)^{-1}H(z)\} = -G(z)^{-1}\{\frac{d}{dz}G(z)\}G(z)^{-1}H(z) + G(z)^{-1}\frac{d}{dz}H(z) =$$
$$-G(z)^{-1}\omega_{\mathrm{KZ}}(z)G(z)G(z)^{-1}H(z) + G(z)^{-1}\omega_{\mathrm{KZ}}(z)H(z) = 0$$

□

Lemma 4. Let $a \neq 0, 1, \infty$. Then there exists uniquely $G_a(z) \in \mathrm{SolKZ}$ st.$G_a(a) = 1$.

Proof. Actually

$$\begin{aligned}
G_a(z) &:= \mathcal{P}\exp \int_a^z \omega_{\mathrm{KZ}} \\
&:= 1 + \int_a^z \omega_{\mathrm{KZ}}(t) + \int_a^z \int_a^{t_2} \omega_{\mathrm{KZ}}(t_2) \wedge \omega_{\mathrm{KZ}}(t_1) + \cdots \\
&= 1 + \int_a^z \frac{dt}{t} A + \int_a^z \frac{dt}{t-1} B + \int_a^z \int_a^{t_2} \frac{dt_2}{t_2} \wedge \frac{dt_1}{t_1} AA \\
&\quad + \int_a^z \int_a^{t_2} \frac{dt_2}{t_2} \wedge \frac{dt_1}{t_1 - 1} AB + \cdots .
\end{aligned}$$

□

Lemma 5. Let $a, b \neq 0, 1, \infty$. We have

$$G_b(z) \cdot G_a(b) = G_a(z).$$

Lemma 6. There exists uniquely $G_0(z) \in \mathbf{SolKZ}$ st.$G_0(z) \approx z^A$ $(z \to 0)$. Here it means that $P(z) := G_0(z) \cdot \{1 - \frac{\log z}{1!}A + \frac{(\log z)^2}{2!}A^2 - \cdots\}$ is analytic in a nbd of $z = 0$ and $P(0) = 1$.

Proof. Put $P(z) = 1 + \sum_{W:\text{words}} P_W(z)W$. Then by KZ eqn $P_W(z) \in z\mathbb{Q}[[Z]]$ can be constructed inductively

$$\begin{cases} \frac{d}{dz}P_{AWA}(z) = \frac{1}{z}P_{WA}(z) - \frac{1}{z}P_{AW}(z), \\ \frac{d}{dz}P_{AWB}(z) = \frac{1}{z}P_{WB}(z), \\ \frac{d}{dz}P_{BWA}(z) = \frac{1}{z-1}P_{WA}(z) - \frac{1}{z}P_{BW}(z), \\ \frac{d}{dz}P_{BWB}(z) = \frac{1}{z-1}P_{WB}(z), \\ \frac{d}{dz}P_{A}(z) = 0, \\ \frac{d}{dz}P_{B}(z) = \frac{1}{z-1}. \end{cases}$$

□

Lower degree: $G_0(A,B)(z) =$
$1 + (\log z)A + \log(1-z)B + \frac{(\log z)^2}{2}A^2 - Li_2(z)AB +$
$\{Li_2(z) + (\log z)\log(1-z)\}\,BA + \frac{\{\log(1-z)\}^2}{2}B^2 + \cdots$

Lemma 7. There exists uniquely $G_1(z) \in \mathbf{SolKZ}$ st. $G_1(z) \approx (1-z)^B \ (z \to 1)$.

Proof. Actually $G_1(A,B)(1-z) = G_0(B,A)(z)$. □

KZ-associator (a.k.a. Drinfeld associator)

Definition 8. The KZ-associator is defined to be
$\Phi_{KZ} := \Phi_{KZ}(A,B) := G_1(z)^{-1} \cdot G_0(z) \in \mathbb{C}\langle\langle A,B\rangle\rangle$.

It is constant (independent of z).

Lemma 9. $\Phi_{KZ} = \lim_{\epsilon\to 0} \epsilon^{-B} \cdot \mathcal{P}\exp \int_{\epsilon}^{1-\epsilon} \omega_{KZ} \cdot \epsilon^{A}$

Proof.

$$
\begin{aligned}
RHS &= \lim_{\epsilon\to 0} \epsilon^{-B} \cdot G_{\epsilon}(1-\epsilon) \cdot \epsilon^{A} \\
&= \lim_{\epsilon\to 0} \epsilon^{-B} \cdot G_0(1-\epsilon) \cdot G_0(\epsilon)^{-1} \cdot \epsilon^{A} \\
&= \lim_{\epsilon\to 0} \epsilon^{-B} \cdot G_1(1-\epsilon) \cdot \Phi_{KZ} \cdot G_0(\epsilon)^{-1} \cdot \epsilon^{A} = LHS \quad \square
\end{aligned}
$$

Corollary 10. Its coefficient is given by MZV;

$$\Phi_{\mathrm{KZ}} = 1 + \sum (-1)^m \zeta(k_1, \ldots, k_m) A^{k_m-1} B \cdots A^{k_1-1} B + \cdots$$

Proof. By Lemma 9,

$$\begin{aligned}
&\langle \Phi_{\mathrm{KZ}} \mid A^{k_m-1} B \cdots A^{k_1-1} B \rangle \\
&= \lim_{\epsilon \to 0} \int_{\epsilon}^{1-\epsilon} \frac{dt}{t}^{\wedge(k_m-1)} \wedge \frac{dt}{t-1} \wedge \cdots \wedge \frac{dt}{t}^{\wedge(k_1-1)} \wedge \frac{dt}{t-1} \\
&= (-1)^m \zeta(k_1, \ldots, k_m).
\end{aligned}$$

□

Lower degree:

$\Phi_{\mathrm{KZ}}(A, B) = 1 - \zeta(2)AB + \zeta(2)BA - \zeta(3)A^2B + 2\zeta(3)ABA + \zeta(1,2)AB^2 - \zeta(3)BA^2 - 2\zeta(1,2)BAB + \zeta(1,2)B^2A + \cdots$.

See Appendix B, for general formula.

Associator relations by Drinfeld ('91)

0. group-like condition; $\Delta(\Phi_{KZ}) = \Phi_{KZ} \otimes \Phi_{KZ}$ where Δ is the coproduct of $\mathbb{C}\langle\langle A, B\rangle\rangle$.
1. 2-cycle relation; $\Phi_{KZ}(A, B)\Phi_{KZ}(B, A) = 1$.
2. 3-cycle relation;
   $$e^{\pi i A}\Phi_{KZ}(C, A)e^{\pi i C}\Phi_{KZ}(B, C)e^{\pi i B}\Phi_{KZ}(A, B) = 1$$
   with $C := -A - B$.
3. 5-cycle relation;
   $$\Phi_{KZ}(t_{12}, t_{23} + t_{24})\Phi_{KZ}(t_{13} + t_{23}, t_{34}) =$$
   $$\Phi_{KZ}(t_{23}, t_{34})\Phi_{KZ}(t_{12} + t_{13}, t_{24} + t_{34})\Phi_{KZ}(t_{12}, t_{23}).$$

Here $\{t_{ij}\}$ are generators of Drinfeld-Kohno Lie algebra: for different integers i, j, k, l,

$$t_{ii} = 0,\ t_{ij} = t_{ji},\ [t_{ij}, t_{kl}] = 0,\ [t_{ij}, t_{kl}] = 0,\ [t_{ij}, t_{ik} + t_{jk}] = 0.$$

Appendix A: Rough proof of associator relations for $\Phi_{\mathbf{KZ}}$

Proof of group-like condition: Consider ΔKZ-equaton

$$dH = \left(\frac{\Delta(A)}{z} + \frac{\Delta(B)}{z-1}\right) \cdot H(z)$$

with $H(z) \in \mathbb{C}\langle\langle A, B\rangle\rangle^{\hat{\otimes}2}$.
Then for $G(z) \in \mathbf{SolKZ}$, we have both $G(\Delta(A), \Delta(B))(z)$ and $G(z)\hat{\otimes}G(z) \in \mathbf{SolKZ}$. Since

- $G_0(\Delta(A), \Delta(B))(z) \approx z^{\Delta(A)} = z^{A\otimes 1+1\otimes A} = z^A \otimes z^A$,
- $G_0(z)\hat{\otimes}G_0(z) \approx z^A \otimes z^A$,

when $z \to 0$, we have

$$G_0(\Delta(A), \Delta(B))(z) = G_0(z)\hat{\otimes}G_0(z).$$

Similarly we have $G_1(\Delta(A), \Delta(B))(z) = G_1(z)\hat{\otimes}G_1(z)$.
So we have $\Delta(\Phi_{KZ}) = \Phi_{KZ}(\Delta(A), \Delta(B)) =$
$G_1(\Delta(A), \Delta(B))(z)^{-1} \cdot G_0(\Delta(A), \Delta(B))(z) =$
$(G_1(z)\hat{\otimes}G_1(z))^{-1} \cdot (G_0(z)\hat{\otimes}G_0(z)) = \Phi_{KZ} \otimes \Phi_{KZ}$. □

Proof of 2-cycle relation: By using

- $G_0(A, B)(z) = G_1(A, B)(z)\Phi_{KZ}(A, B)$,
- $G_1(A, B)(z) = G_0(B, A)(1 - z)$,

we have

$$\begin{aligned} G_0(A, B)(z) &= G_0(B, A)(1 - z)\Phi_{KZ}(A, B) \\ &= G_1(B, A)(1 - z)\Phi_{KZ}(B, A)\Phi_{KZ}(A, B) \\ &= G_0(A, B)(z)\Phi_{KZ}(B, A)\Phi_{KZ}(A, B) \end{aligned}$$

So we have $\Phi_{KZ}(B, A)\Phi_{KZ}(A, B) = 1$. □

Proof of 3-cycle relation: Make use of 6 fundamental solutions of KZ-eqn:
$G_0(A,B)(z)$, $G_0(B,A)(1-z)$, $G_0(B,C)(1-\frac{1}{z})$,
$G_0(C,B)(\frac{1}{z})$, $G_0(C,A)(\frac{1}{1-z})$, $G_0(A,C)(\frac{z}{z-1})$
and their relation including
$G_0(A,C)(\frac{z}{z-1}) = G_0(A,B)(z)e^{\pi i A}$,
$G_0(A,B)(z) = G_0(B,A)(1-z)\Phi_{KZ}(A,B)$, etc.

Rough proof (consult Drinfeld's paper) of 5-cycle relation: Make use of 5 fundamental solutions of two variables KZ equation over
$\mathcal{M}_{0,5} \simeq \{(x,y) \in \mathbb{C}^2 \mid x, y, xy \neq 0, 1\}$.

Appendix B:
Explicit formula of coefficients of $\Phi_{\mathbf{KZ}}$

Formula of Le-Murakami ('96) and F ('03)

Put $U\mathfrak{F}_2 := \mathbb{C}\langle\langle A, B\rangle\rangle$. For a word $W \in U\mathfrak{F}_2$, let $I(W)$ be its coefficient in Φ_{KZ}. Then we have, for $k_m > 1$,

$$I(A^{k_m-1}B\cdots A^{k_1-1}B) = (-1)^m\zeta(k_1,\ldots,k_m).$$

Suppose that W is written as B^rVA^s ($r, s \geqslant 0$, $V \in A\cdot U\mathfrak{F}_2\cdot B$ or $V = 1$). Then

$$I(W) = \sum_{\substack{0\leqslant a\leqslant r\\0\leqslant b\leqslant s}} (-1)^{a+b} I\Big(\pi(B^a \sqcup\!\sqcup B^{r-a}VA^{s-b} \sqcup\!\sqcup A^b)\Big).$$

Here $\pi : U\mathfrak{F}_2 \to U\mathfrak{F}_2$ is the natural projection $U\mathfrak{F}_2 \to \mathbb{C} + A\cdot U\mathfrak{F}_2\cdot B(\subset U\mathfrak{F}_2)$ annihilating $B\cdot U\mathfrak{F}_2$ and $U\mathfrak{F}_2\cdot A$.