Today The cts analogue of unimodular random graphs.

Unless otherwise specified, material is from Abert – B '16.

Let $\mathcal{M}^d = \left\{ \begin{matrix} \text{connected, complete} \\ \text{pointed Riem} \\ d\text{-manifolds } (M,p) \end{matrix} \right\} /$ pointed isometry, endowed

with the smooth topology, where $(M_i, p_i) \longrightarrow (M,p)$ if $\exists R_i \to \infty$ and embeddings $\varphi_i : B_M(p, R_i) \longrightarrow M_i$ s.t. $\varphi_i(p) = p_i$ and if g_i, g are the Riem metrics then $\varphi_i^* g_i \xrightarrow{C^\infty} g$.

Fact $\mathcal{M}^d$ is Polish, i.e. has the top of a complete separable metric space. See Candel Álvarez-Lopez, Barral Lijó, and AB.

Let $\mathcal{M}^d_2 = \{ (M,p,q),\ M \text{ as above} \} /$ doubly ptd isom

Def A (σ-finite) Borel measure μ on $\mathcal{M}^d$ is unimodular if $\forall$ nonnegative Borel $f : \mathcal{M}^d_2 \to \mathbb{R}$,

$$\int_{(M,p)\in\mathcal{M}^d} \int_{q\in M} f(M,p,q)\, d\mathrm{vol}_M \, d\mu = \int_{(M,p)\in\mathcal{M}^d} \int_{q\in M} f(M,q,p)\, d\mathrm{vol}_M \, d\mu$$

mass transport principle (MTP)

Indicates a compatibility of μ and the Riem measures vol_M: the basepts p of a μ-random (M,p) are distributed in M according to vol_M.

The random elt of a unimodular prob measure on $\mathcal{M}^d$ is a unimodular random manifold (URM).

<u>Ex</u> Spse M is a finite vol Riem d-mfld, and let μ_M be the measure on $\mathcal{M}^d$ obtained by pushing forward vol_M under

$$M \longrightarrow \mathcal{M}^d, \quad p \longmapsto (M,p)$$

Both sides of the MTP are $=$ to $\int_{M\times M} f(M,p,q)\, d\mathrm{vol}_M^2$, by Fubini. So, "$M$ equipped w/ a random basept is a URM."

More generally, if $\pi: N \to M$ is a regular cover of a fin vol M, the pushforward of vol_M under

$$M \longrightarrow \mathcal{M}^d, \quad p \longmapsto (N,\tilde{p}), \quad \tilde{p} \in \pi^{-1}(p).$$

is unimodular.

well defined since points in $\mathcal{M}^d$ are equiv classes wr to pted isom.

URM = N equipped w/ a random basept chosen from some fundamental domain.

$\underline{\text{Ex}}$ Space X is a Riem drmfld s.t. $\text{Isom}(X)$ acts transitively. Then any basept $p \in X$ give the same elt $X \in \mathcal{M}^d$.

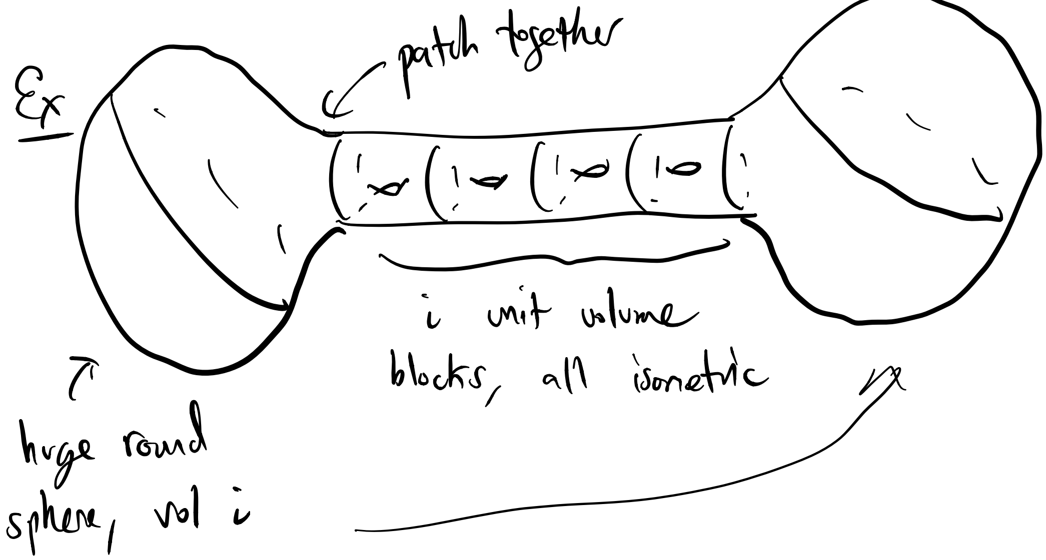
$\underline{\text{Prop}}$ The atomic measure 1_X is unimodular $\iff$ $G = \text{Isom}(X)$ is a unimodular Lie group.

Recall G is $\underline{\text{unimodular}}$ if the unique (up to scale) left invt measure on G is the unique (up to scale) right invt measure.

We'll consider the set $\mathcal{P}(\mathcal{M}^d)$ of prob measures on $\mathcal{M}^d$ with the weak topology. Unimodularity is a closed condition, so any weak limit of unimodular prob measure is an another. In particular:

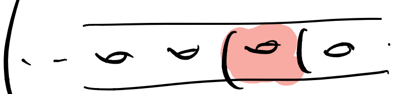
<u>Def</u> S'pose M_i is a sequence of fin vol Riem d-mflds. We say (M_i) <u>Benjamini–Schramm converges</u> if the measures $\mu_{M_i}/\mathrm{vol}(M_i)$ converge weakly in $\mathcal{P}(\mathcal{M}^d)$

<u>Ex</u>



$\Big\downarrow$ BS

$$\downarrow$$

$$\frac{2}{3}\left(1_{\mathbb{R}^2} \right) + \frac{1}{3}\Bigg($$

$$\Bigg)$$

pick basept randomly

<u>Ex</u> (From IRS)

S'pose $G = \mathrm{Isom}\,X$ acts transitively on a d-mfld X and X is simply connected. Then the map

$$\Gamma \in \mathrm{Sub}_G^{df} \longrightarrow \left(\Gamma \backslash X,\ \Gamma p_X\right)$$

discrete, torsion free sgps

projection of some fixed basept $p_X \in X$

pushes forward any G-invt measure on Sub_G^{df} to a unimodular measure on $\mathcal{M}^d$, and induces a weak homeo

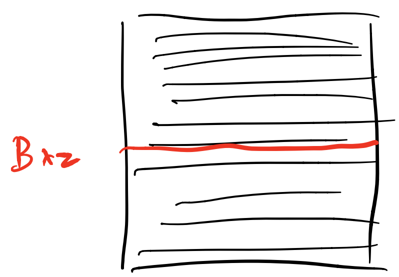
$$\left\{ \begin{matrix} \text{discrete, torsion} \\ \text{free IRSs of } G \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{unimodular} \\ \text{random} \\ X\text{-mnflds} \end{matrix} \right\}$$

$\uparrow$

Riem mnflds of the form $\Gamma \backslash X$.

$\underline{\text{Ex}}$ (Foliated spaces)

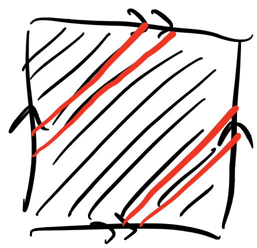
A d-dim' $\underline{\text{foliated space}}$ is a space X that's the union of d-mnflds L called $\underline{\text{leaves}}$, where every pt in X has a nbhd of the form



$\cong B \times Z$, $B \subset \mathbb{R}^d$ open

where each $B \times z$ is an open subset of a leaf. $\underline{\text{E.g.}}$

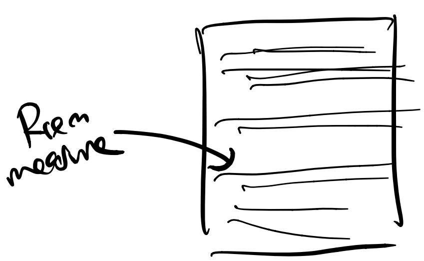
$T^2 =$



where leaves are the projection of lines of slope α in $\mathbb{R}^d$, and are circles if $\alpha \in \mathbb{Q}$, a line otherwise.

The foliated space X is <u>Riemannian</u> if each leaf comes with a Riem metric, and the metrics vary smoothly in the transverse direction.

A measure μ on X is <u>completely invariant</u> if locally it is the integral of the Riem measures on the leaves against some measure on the transverse space Z.



$B \times Z$

E.g. vol_{T^2} is locally the integral of Lebesgue measure on leaves against Lebesgue measure in the orthogonal direction.

Prop If μ is a finite completely inv't measure on a Riem foliated space, the leaf map

$$X \longrightarrow \mathcal{M}^d, \quad x \longmapsto (\underset{\substack{\uparrow \\ \text{leaf through } x}}{L_x}, x)$$

is Borel and pushes forward μ to a unimodular measure on $\mathcal{M}^d$.

=

The foliated structure of $\mathcal{M}^d$

The maps $M \xrightarrow{i} \mathcal{M}^d$, $i(p) = (M,p)$, where M is a Riem d-mnfld, have images that almost form the leave of a foliation of $\mathcal{M}^d$, except that

$$i(M) \cong M/\mathrm{Isom}(M),$$

which may not be a d-mfld. (If $\mathrm{Isom}(M)$ acts transitively, it's a pt.)

Desingularization Thm

$\exists$ a Riem foliated space $\mathcal{P}^d$ s.t. every unimodular measure μ on $\mathcal{M}^d$ is the pushforward of some completely inv't measure ν on $\mathcal{P}^d$ under the leaf map $\mathcal{P}^d \longrightarrow \mathcal{M}^d$.

$$x \longmapsto (L_x, x)$$

Idea: $\mathcal{P}^d = \{ (M, p, D) \mid \begin{array}{l} D \subset M \text{ is locally finite and } \nexists \text{ a} \\ \text{nontrivial isom} \\ f: M \to M,\ f(D) = D \end{array} \}$

$$\mu \text{ on } \mathcal{M}^d \rightsquigarrow \nu = \int_{(M,p)} \text{Poisson process on } M \, d\mu .$$

Rmk This thm shows how unimodularity means basepts are distributed according to the vol_M's of different M.

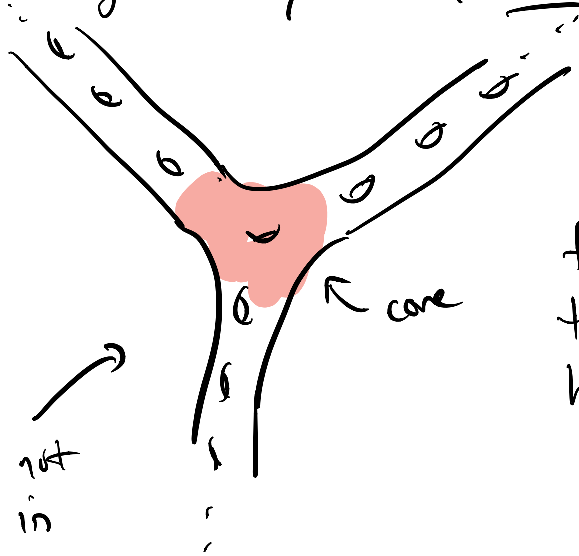
The No Core Principle

S'pose that μ is a unimodular prob measure on $\mathcal{M}^d$ and $f : \mathcal{M}^d \to \{0,1\}$ is a Borel fn. Then for μ-a.e. $(M,p) \in \mathcal{M}^d$,

$$(\star) \quad \underline{0 < \mathrm{vol}_M \{ q \in M \mid f(M,q) = 1 \} < \infty} \implies \mathrm{vol}(M) < \infty.$$

<u>Pf</u> (exercise). $\square$

Imagine $f(M,q) = 1$ when q is in some sort of geometrically defined <u>core</u> of M.



only have a fin vol core if the whole mnfld has fin vol.

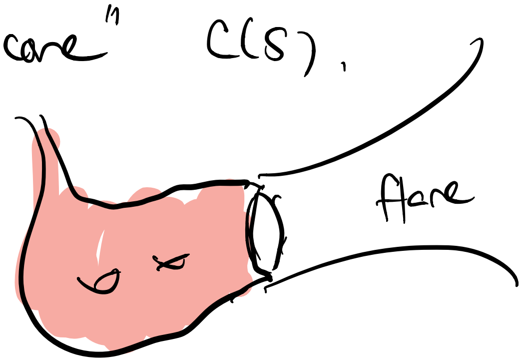
does not occur in a URM.

Cor (B-Rambault '17) A URM a.s. has either 0, 1, 2 or a Cantor set of ends.

(can also use this to classify exactly which ~~tg~~ types of surfaces appear in a URM w/ bounded curvature, following Ghys)

Cor If S is a UR hyperbolic surface w/ finitely gen'ed π_1, then a.s. S has finite volume or is $\cong \mathbb{H}^2$.

Pf Hyp surfaces ($\neq \mathbb{H}^2$) w/ fin gen π_1 and ∞-vol have a fin vol "convex core" $C(S)$.



$C(S)$ $\square$

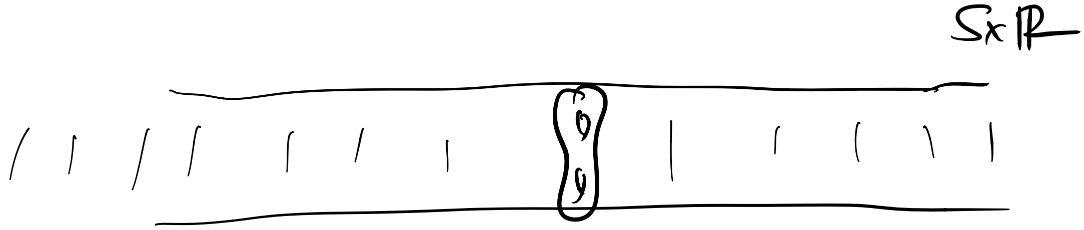
What about in 3-dims?

Thm (Abert - B) Any UR hyp 3-mfld w/ finitely gen π_1 is a.s. either

1) finite vol

2) $\mathbb{H}^3$

3) a doubly degenerate hyp 3-mfld homeo to $S \times \mathbb{R}$.



exhausted by copies of S w/ bounded area.