Wobbly parabolic G-Higgs bundles and affine flag varieties

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- $\mathcal{M}(G)$ is Hyperkähler, holomorphic symplectic $\omega \in H^0(\Omega^2_{\mathcal{M}(G)})$.
- **Hitchin system**, duality of abelian fibrations: **SYZ mirror symmetry**.

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(Hausel–Thaddeus, Donagi–Pantev)

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- What about **parabolic *G*-Higgs bundles**?

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- **Goal:** Study **BAA-branes (upward flows)** in $\mathcal{M}_{sp}(G, \alpha)$.

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If every $\varphi|_{c_i}$ is nilpotent: **strongly parabolic**.

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- $\varphi : E \to E \otimes K_C(D)$ is a traceless morphism such that

$$\varphi(E_i^j) \subseteq E_i^{j-1} \otimes K_C(D)$$

  (strongly parabolic).

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The **upward flow** of the fixed point $\mathcal{E} := (E, \varphi, Q) \in \mathcal{M}_{sp}(G, \alpha)^{\mathbb{C}^\times}$:

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- $\mathcal{E}$ smooth $\rightsquigarrow$ $W_{\mathcal{E}}^+$ complex Lagrangian (BAA-brane).

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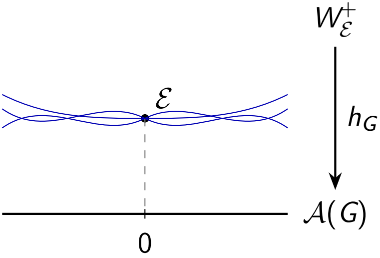
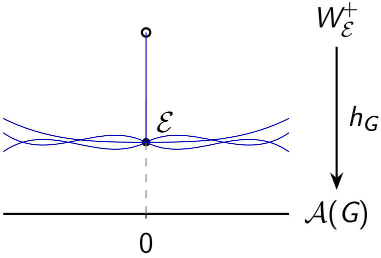
$W_{\mathcal{E}}^{+} \cap h_G^{-1}(0) = \{\mathcal{E}\}$	$W_{\mathcal{E}}^{+}$ has other nilpotents
$W_{\mathcal{E}}^{+}$ closed	$W_{\mathcal{E}}^{+}$ not closed
$h_G\|_{W_{\mathcal{E}}^{+}}$ finite cover	$\dim W_{\mathcal{E}}^{+} \cap h_G^{-1}(0) \geqslant 1$
$\mathcal{E}$ **very stable**	$\mathcal{E}$ **wobbly**





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(if $k > 0$).

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When $k = 0$, only compatible flag is Q_{tail} and $(E_0, \varphi_0, Q_{\mathrm{tail}})$ is **very stable** (Hitchin section).

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$$Q_t := 0 \subsetneq t \cdot \mathcal{O}_C|_c + K_C^{-1}((k-1)c)|_c \subsetneq E_k|_c$$

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In this family, $(E_1, \varphi_1, Q_{\mathrm{head}})$ is the other **very stable**.

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- $\mathbb{C}^\times$-action on flag variety $E|_{c_i}/B \simeq G/B$, flags Q_i are **fixed**.

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$$E = \mathcal{O} \oplus L^{-1}(D_1) \oplus L^{-2}(D_1 + D_2) \oplus \cdots \oplus L^{-n+1}(D_1 + \cdots + D_{n-1})$$

$$\varphi = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \varphi_1 & 0 & \dots & 0 & 0 \\ 0 & \varphi_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \varphi_{n-1} & 0 \end{pmatrix}.$$

- $\mathbb{C}^\times$-action on flag variety $E|_{c_i}/B \simeq G/B$, flags Q_i are **fixed**.

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First condition is (Hausel–Hitchin, 2022). Related work of (Henke, PhD thesis, 2024).

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Have **projection**

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Remark: If $D = 0$, recovers (G., 2025; arXiv:2503.01289).

Proof sketch

Strategy

Work in the **affine flag variety**

$$\mathrm{Fl}_G := G((z))/\mathcal{I}$$

where $\mathcal{I} \subseteq G[[z]]$ is the **standard Iwahori subgroup** defined by projecting to $B \subseteq G$ when $z = 0$.

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- Similar analysis over $c \notin D$ with $\mathrm{Gr}_G := G((z))/G[[z]]$.

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Definition

A **Hecke transformation** of (E, φ, Q) at $c \in C$ is another (E', φ', Q') with isomorphism over $C_0 := C \setminus c$:

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If σ lifts to $\operatorname{Aut}(C_1 \times G, B) \simeq \mathcal{I}$, result is isomorphic.

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- Higgs field φ becomes $(\varphi_0 \in H^0(E|_{C_0}(\mathfrak{g}) \otimes K_C(D)), \varphi_1 \in \mathfrak{i})$ compatible with Φ and $Q|_{C_0}$.

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Proposition

(E, φ, Q) fixed point with gen. reg. φ, there is trivialisation of $(E|_{C_1}, Q|_c)$ and $\mathbb{C}^\times$-action:

$$\lambda \cdot \sigma := \exp\left(\sum_{i=1}^{r} -\log(\lambda)\omega_i^\vee\right) \cdot \sigma$$

on $\mathrm{Fl}_G^{\varphi_1}$ such that Hecke transformation $(E'_\lambda, \varphi'_\lambda, Q'_\lambda)$ satisfies

$$(E'_\lambda, \varphi'_\lambda, Q'_\lambda) \simeq (E'_{\lambda=1}, \lambda\varphi'_{\lambda=1}, Q'_{\lambda=1}).$$

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- When $c \in D$, this $W^+_{w(E,\varphi,Q)}$ is still a point.
- When $c \notin D$ it is isomoprhic to the Springer fibre (in $G/P_{w(E,\varphi,Q)}$) over $a(c)$ as an element of the Kostant section, i.e. to the preimage of $a(c) \in \mathfrak{t} /\!\!/ W$ in $\mathfrak{t} /\!\!/ W_{L_{w(E,\varphi,Q)}}$.

Very stable covers

Proposition

Evaluating at $c \notin D$ yields the following description for the very stable covers:

$$\begin{tikzcd} W^+_{(E,\varphi,Q)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathfrak{t} /\!\!/ W_{L_{w(E,\varphi,Q)}} \arrow[r, "\sim"] \arrow[d] & H_G(G/P_{w(E,\varphi,Q)}) \arrow[d] \\ \mathcal{A}(G) \arrow[r] & \mathfrak{t} /\!\!/ W \arrow[r, "\sim"] & H_G(\mathrm{pt}) \end{tikzcd}$$

When $c \in D$ the finite cover is a one to one isomorphism.

(Left square by Hausel for $G = \mathrm{GL}_n(\mathbb{C})$, right square is a particular case of Hausel–Rychlewicz, 2022)

Thank you!