Exact Volume-Law Entangled Zero-Energy Eigenstates in a Large Class of Spin Models

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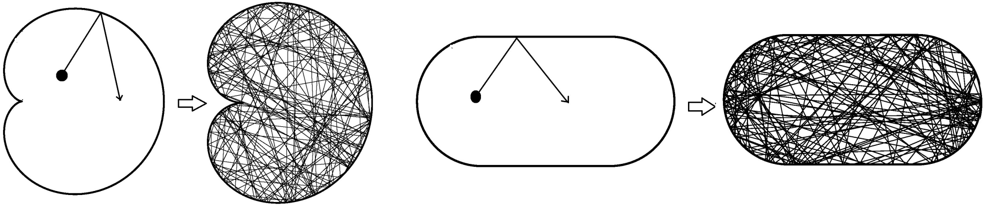
Ref.: S. Mohapatra, S. Moudgalya and Ajit C. Balram, Phys. Rev. Lett. **134**, 210403 (2025)

Outline

- Introduction: ergodicity and thermalization in isolated systems.
- Spin Hamiltonians with volume-law entangled exact eigenstates.
- Summary and future outlook.

Does an isolated classical system thermalize?

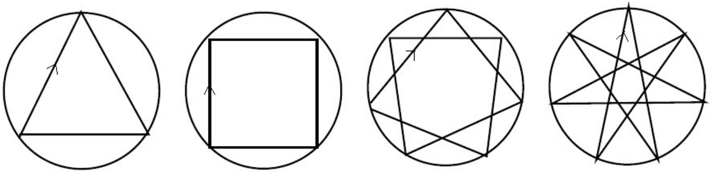
- Yes, generic classical systems are ergodic



- system forgets initial conditions $\longrightarrow$ explore full space
- chaotic motion $\longrightarrow$ ergodicity

Counterexample to thermalization: integrable systems

- No thermalization in integrable systems



- additional conservation laws/symmetries $\longrightarrow$ stable periodic orbits
- fail to explore the entire phase space
- highly fine-tuned

Thermalization in isolated quantum systems

- generic isolated quantum systems are chaotic/ergodic in nature.
- initial state evolves under unitary dynamics of the Hamiltonian

$$|\psi(t)\rangle = e^{-iHt}\,|\psi(0)\rangle$$

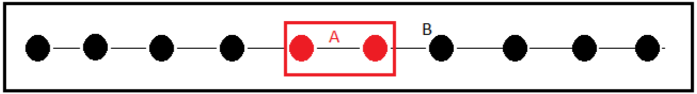
- unitary dynamics cannot make the state mixed $\rightarrow$ the entire system does not thermalize.

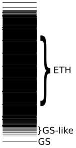
Thermalization happens locally: subsystem notion

- for any subsystem, the rest of the system can act as a bath

Thermalization is governed by ETH

- For any subsystem, the rest of the system acts as a bath





$$\lim_{t\to\infty}\rho_A(t) = Tr_B(\rho^{eq}) \approx \rho_A^{eq}, \rho^{eq} = \frac{1}{Z}e^{-\beta H}$$

- expectation value of local operators matches their thermal expectation value.

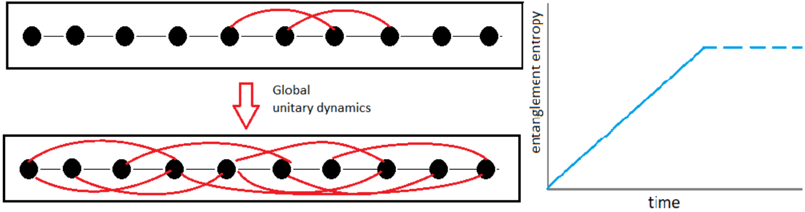
$$\langle E_\alpha|\hat{O}|E_\alpha\rangle = \frac{1}{Z}Tr(\hat{O}e^{-\beta_\alpha H})$$

Eigenstate thermalization hypothesis (ETH): every eigenstate of the thermal system is thermal

Entanglement entropy (EE)

- Thermalization goes hand in hand with entanglement

Bipartite entanglement entropy: $S = -\text{Tr}\rho_A \log \rho_A$

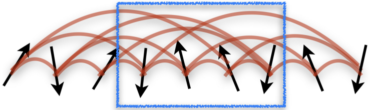


- This entropy of entanglement causes the subsystems to equilibrate

Entanglement entropy-based diagnostics of thermalization

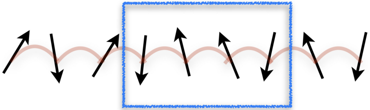
- chaotic/ergodic: volume law of EE

$$S_{ent}(A) \propto vol(A)$$

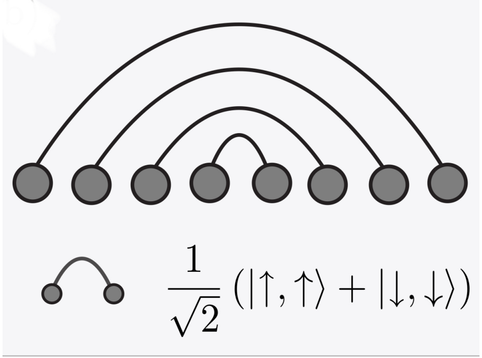


- MBL: area law of EE

$$S_{ent}(A) \propto area(A)$$



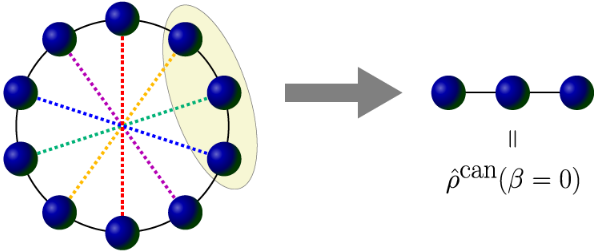
Examples of states with volume-law entanglement: 1) rainbow scars



XYZ spin chains with next-nearest neighbor $z-z$ interaction

Langlett *et al.*, Phys. Rev. B **105**, L060301 (2022)

Examples of states with volume-law entanglement: 2) EAP state



$$H_1(L) = \sum_{i=1}^{L} \left(J_1 \sigma_i^x \sigma_{i+1}^y + J_2 \sigma_i^y \sigma_{i+1}^z \right)$$

dotted line is the Bell state $|\Psi^+\rangle_{i,j} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{i,j}$

Y. Chiba and Y. Yoneta, Phys. Rev. Lett. **133**, 246605 (2024)

Generic construction: definition of spin-chain Hamiltonians

- Consider a broad class of spin-S chains of size L with a translation invariant interaction [(with periodic boundary conditions (PBC)] of the form

$$H(L)=\sum_{\alpha,i=1}^{L} J_{\alpha}\hat{H}^{\alpha}_{i+j_1,\cdots,i+j_n}=\sum_{\alpha,i=1}^{L} J_{\alpha}\prod_{k=1}^{n}(S^{\mu}_{i+j_k})^{q}, \quad (1)$$

$S^{\mu}_{j}(\mu=x,y,z)\to$ spin-S operator at site j and q is a non-negative integer.
$\hat{H}^{\alpha}_{i+j_1,\cdots,i+j_n}\to$ interaction between any $n-$spins at sites $\{i+j_1,\cdots,i+j_n\}$ with $0\leq j_k<L$.

- The Hamiltonian is defined in the Hilbert space $\mathcal{H}_L$ of dimension $\mathcal{D}_L=(2S+1)^L$ on the global computational basis $|\vec{S}\rangle=\bigotimes_{i=1}^{L}|s_i\rangle$ ($\mathcal{H}_L\equiv\text{span}\{|\vec{S}\rangle\}$), where $|s_i\rangle$ defines the local Hilbert space $(2S+1)-$ of dimension at site i.
- for example: for a spin-1/2 Hamiltonian, the local Hilbert space at site i can be defined by eigenvectors of σ^z_i : $\{|s_i\rangle\}=\{|0\rangle,|1\rangle\}$.
The global computational basis for 2 sites: $\mathcal{H}_L=\{|\vec{S}\rangle\}=\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$

Generic construction: an example

$$H(L) = J_1 \sum_i (S_i^x)^2 S_{i+n}^y + J_2 \sum_i S_i^z$$

- $\hat{H}_{i,i+n}^{\alpha_1} = (S_i^x)^2 S_{i+n}^y$
- $\hat{H}_i^{\alpha_2} = S_i^z$
- $\sum_i S_i^x S_i^y$
- $\sum_i S_i^y (S_{i+1}^z S_{i+1}^x)^3$

Any spin-1/2 Hamiltonian can be recast to Eq. (1)

Does not exhaust the possibilities for higher spin-S Hamiltonians

Generic construction: criteria that needs to be met

If each $\hat{H}^{\alpha}_{i+j_1,\cdots,i+j_n}$ in the Hamiltonian $H(L)$ anti-commutes with an operator $\mathcal{C}\mathcal{K}$, where $\mathcal{C}=\prod_{i=1}^{L}\mathcal{C}_i$ with $\mathcal{C}_i$ onsite invertible operators, and $\mathcal{K}$ is the complex conjugation operator, i.e.,

$$\{\hat{H}^{\alpha}_{i+j_1,\cdots,i+j_n},\mathcal{C}\mathcal{K}\}=0 \text{ for } i\in\{1,\cdots,L\}, \tag{2}$$

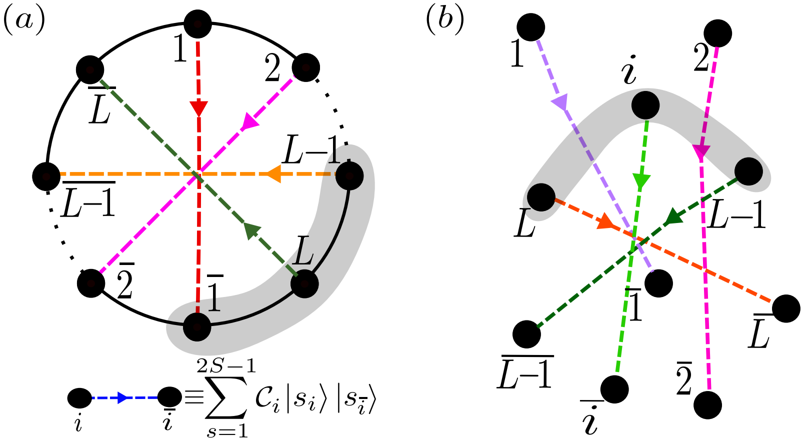
then the state

$$|\Lambda\rangle_{\mathcal{C}}=\frac{1}{\sqrt{\mathcal{D}_N}}\sum_{|\vec{S}\rangle\in\mathcal{H}_L}\mathcal{C}|\vec{S}\rangle_{1,\cdots,L}\otimes|\vec{S}\rangle_{\bar{1},\cdots,\bar{L}}=\frac{1}{\sqrt{\mathcal{D}_L}}\bigotimes_{i=1}^{L}\sum_{s=1}^{2S+1}\mathcal{C}_i|s_i\rangle|s_{\bar{i}}\rangle, \tag{3}$$

where $\bar{i}\equiv i+L$, is an exact zero-energy eigenstate of the Hamiltonian $H(2L)$ defined on a system of size $2L$ with PBC. Owes to a pair-wise cancellation $\hat{H}^{\alpha}_{i+j_1,\cdots,i+j_n}|\Lambda\rangle_{\mathcal{C}}=-\hat{H}^{\alpha}_{\overline{i+j_1},\cdots,\overline{i+j_n}}|\Lambda\rangle_{\mathcal{C}}$

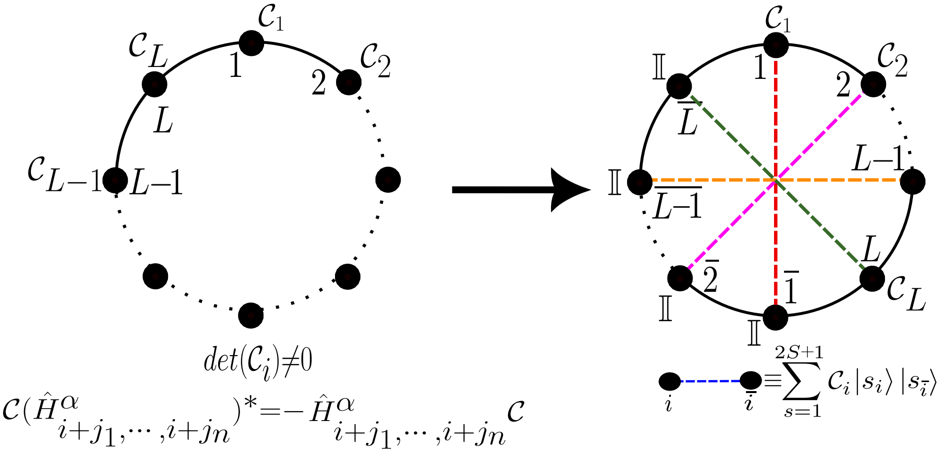
S. Mohapatra, S. Moudgalya and Ajit C. Balram, Phys. Rev. Lett. **134**, 210403 (2025)

Pictorial depiction of the state 1



S. Mohapatra, S. Moudgalya and Ajit C. Balram, Phys. Rev. Lett. **134**, 210403 (2025)

Pictorial depiction of the state 2



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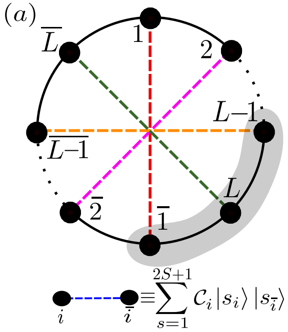
State $|\Lambda\rangle_{\mathcal{C}}$ is volume-law entangled



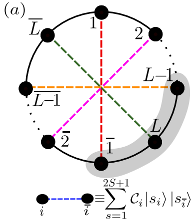
$|\Lambda\rangle_{\mathcal{C}}$ is product of "dimers" between i and $\bar{i}$

contiguous system of N sites cuts N bonds $\implies$ $\mathcal{S}_N \propto N$

S. Mohapatra, S. Moudgalya and Ajit C. Balram, Phys. Rev. Lett. **134**, 210403 (2025)

State $|\Lambda\rangle_{\mathcal{C}}$ is thermal for strictly local observables

The reduced density matrix over any contiguous subsystem is just proportional to the identity matrix, which is just the infinite-temperature Gibbs density matrix, hence the state is thermal for strictly local observables.



For tailored subsystems chosen to consist solely of sites $\{i, \bar{i}\}$, $\mathcal{S}$ is strictly zero, and non-local observables with support over these subsystems exhibit highly athermal properties.

S. Mohapatra, S. Moudgalya and Ajit C. Balram, Phys. Rev. Lett. **134**, 210403 (2025)

Corollary to the theorem

The state

$$|\Lambda\rangle_{\mathbb{I}} = \frac{1}{\sqrt{\mathcal{D}_L}} \sum_{|\vec{S}\rangle} |\vec{S}\rangle \otimes |\vec{S}\rangle = \frac{1}{\sqrt{\mathcal{D}_L}} \bigotimes_{i=1}^{L} \sum_{s_i=1}^{2S+1} |s_i\rangle |s_{\bar{i}}\rangle \tag{4}$$

is always a zero-energy eigenstate of any Hamiltonian of the form presented defined on an even number of lattice sites ($2L$) where each $\hat{H}^{\alpha}_{i+j_1,\cdots,i+j_n}$ has a purely imaginary matrix representation in the computational basis.

This is the state $|\Lambda\rangle_{\mathcal{C}}$ with $\mathcal{C}=\mathbb{I}$ for which the anti-commutation condition always holds.

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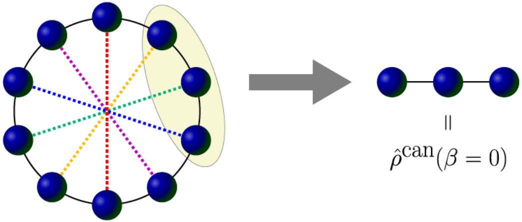
EAP state from the corollary to our theorem

$H_1(L) = \sum_{i=1}^{L} \left(J_1 \sigma_i^x \sigma_{i+1}^y + J_2 \sigma_i^y \sigma_{i+1}^z \right)$ is purely imaginary in the computational basis use corollary with $\mathcal{C} = \mathbb{I}$ to get EAP state



dotted line is the Bell state $|\Psi^+\rangle_{i,j} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{i,j}$

$|\Lambda(S=1/2)\rangle_{\mathbb{I}}$ is a zero-energy eigenstate of *any* spin-1/2 Hamiltonian with even number of sites that has an *odd* number of σ_y's in each $\hat{H}^{\alpha}_{i+j_1,\cdots,i+j_n}$

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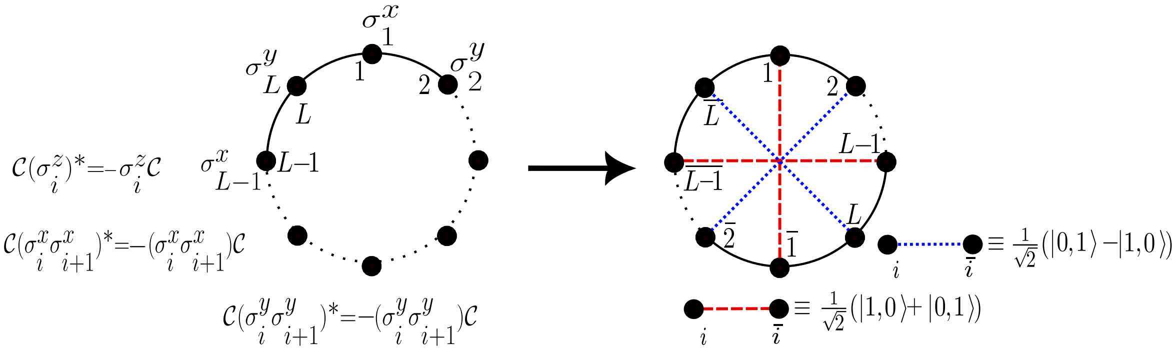
A simple illustrative example of spin-1/2 XY chain

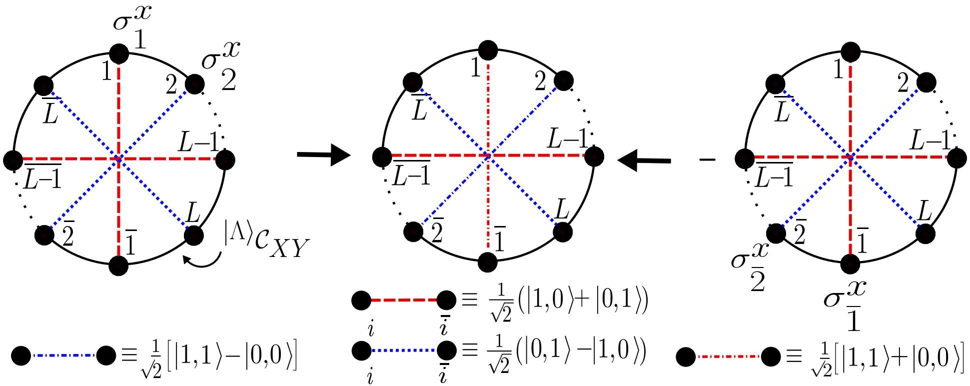
- $H_{XY}(L) = \sum_{i=1}^{L} \left(J_1 \sigma_i^x \sigma_{i+1}^x + J_2 \sigma_i^y \sigma_{i+1}^y + h\sigma_i^z\right)$, $\{\hat{H}^\alpha\} = \{\{\sigma_i^x \sigma_{i+1}^x\}, \{\sigma_i^y \sigma_{i+1}^y\}, \{\sigma_i^z\}\}$
- When $L = even$, $\mathcal{C}_{XY} = \prod_{i=1}^{L/2} \sigma_{2i-1}^x \sigma_{2i}^y$ satisfies Eq. (2)

$\implies |\Lambda\rangle_{\mathcal{C}_{XY}} = \bigotimes_{i=1}^{L-1} |\phi^+\rangle_{i,\bar{i}} |\phi^-\rangle_{i+1,\overline{i+1}}$ where $|\Phi^\pm\rangle_{i,j} = \frac{(|01\rangle \pm |10\rangle)_{i,j}}{\sqrt{2}}$



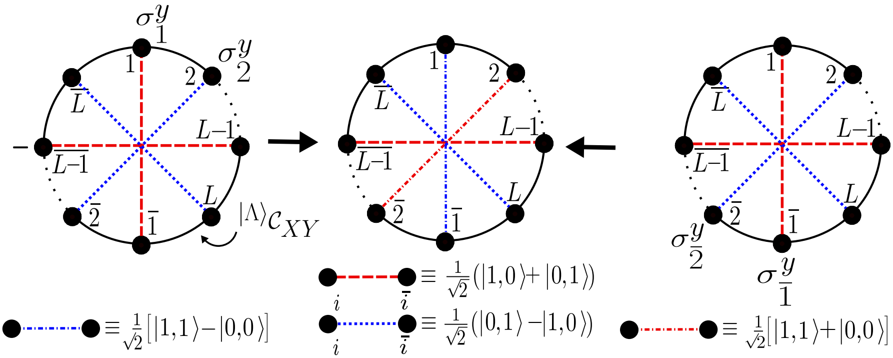
Proof of zero-energy for the XX term

- Hamiltonian for $2L$-sites with PBC: $H^{XX}(2L) = \sum_{i=1}^{2L} J_1 \sigma_i^x \sigma_{i+1}^x$
- For all i: $\sigma_i^x \sigma_{i+1}^x |\Lambda\rangle_{\mathcal{C}_{XY}} = -\sigma_{\bar{i}}^x \sigma_{\overline{i+1}}^x |\Lambda\rangle_{\mathcal{C}_{XY}} = -\sigma_{i+L}^x \sigma_{i+L+1}^x |\Lambda\rangle_{\mathcal{C}_{XY}}$



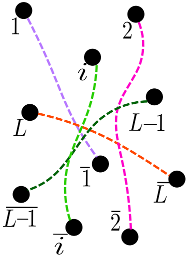
Proof of zero-energy for the YY term

- Hamiltonian for $2L$-sites with PBC: $H^{YY}(2L) = \sum_{i=1}^{2L} J_2 \sigma_i^y \sigma_{i+1}^y$
- For all i: $\sigma_i^y \sigma_{i+1}^y |\Lambda\rangle_{\mathcal{C}_{XY}} = -\sigma_{\bar{i}}^y \sigma_{\overline{i+1}}^y |\Lambda\rangle_{\mathcal{C}_{XY}} = -\sigma_{i+L}^y \sigma_{i+L+1}^y |\Lambda\rangle_{\mathcal{C}_{XY}}$



State has equal number of up and down spins so it has zero-energy for $H^Z = \sum_{i=1}^{2L} h\sigma_i^z$.

Generalization to arbitrary dimensions



- $\{\hat{H}^{\alpha}_{j_1,\cdots,j_n}, \mathcal{C}\mathcal{K}\}=0$ (with $\mathcal{C}_i=\mathcal{C}_{\bar{i}}$) $\implies (\hat{H}^{\alpha}_{j_1,\cdots,j_n} + \hat{H}^{\alpha}_{\bar{j}_1,\cdots,\bar{j}_n})|\Lambda\rangle_{\mathcal{C}}=0$ (where $j_k \in \{1,\bar{1},\cdots,L,\bar{L}\}$ and $j_k \neq \bar{j}_{k'}$ for all k, k')
- All Hamiltonians of the form $H=\sum_{\alpha,\{j\}} J_\alpha(\hat{H}^{\alpha}_{j_1,\cdots,j_n}+\hat{H}^{\alpha}_{\bar{j}_1,\cdots,\bar{j}_n})$ with arbitrary coefficients J_α, host $|\Lambda\rangle_{\mathcal{C}}$ as a zero energy eigenstate.

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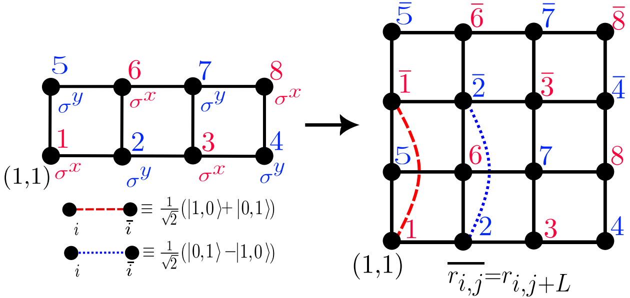
Example of two-dimensional spin-1/2 XY model

- $H^{2D}_{XY}(M\times L)=\sum_{\eta,i,j} J_\eta(\sigma^\eta_{r_{i,j}}\sigma^\eta_{r_{i+1,j}}+\sigma^\eta_{r_{i,j}}\sigma^\eta_{r_{i,j+1}})+h\sum_{i,j}\sigma^z_{r_{i,j}}$
  $\eta\in\{x,y\}$ and $r_{i,j}=i+M\times(j-1)$ with $i\in\{1,\cdots,M\}$ (horizontal) and $j\in\{1,\cdots,L\}$ (vertical)



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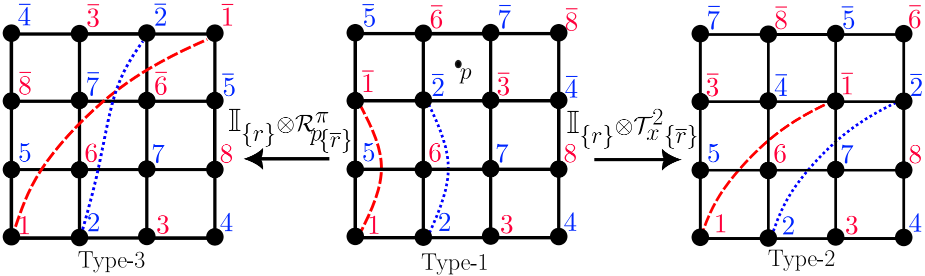
Additional classes of zero energy using lattice symmetry



The operator $\mathcal{T}^{x,M/2}_{\{\bar{r}\}}$ translates all the $\{\bar{r}\}$ sites by $M/2$ units in the horizontal direction and the operator $\mathcal{R}^{p,\pi}_{\{\bar{r}\}}$ rotates all the $\{\bar{r}\}$ sites around the point p=$[(M+1)/2,(3L+1)/2]$ by an angle π.

More states using full translations and rotations of each type.

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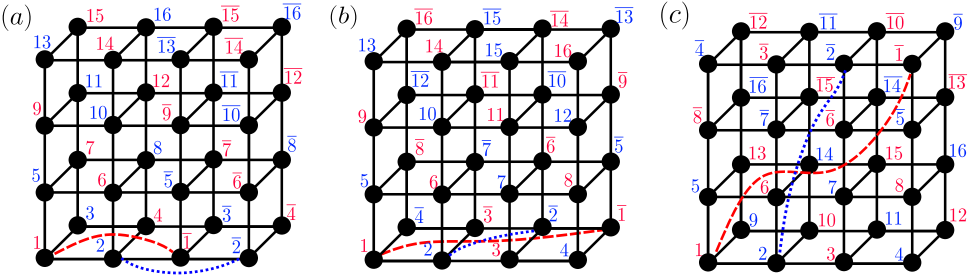
- The number of such zero energy eigenstates scales with the volume of the system $\mathcal{O}(ML)$.
- Similar construction for other geometries



Unlike the one-dimensional case, in higher dimensions, there can be special contiguous bipartitions for which the entanglement is not a volume law, although it is for most of them.

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Many other examples (some hold even for $S>1/2$)

- transverse-field Ising model,
- spin-1/2 model with three-spin interactions Udupa *et. al*, Phys. Rev. B **108**, 214430 (2023),
- PXP model Andrew N. Ivanov and Olexei I. Motrunich, Phys. Rev. Lett. **134**, 050403 (2025),
- spin-S XY model,
- spin-S Kitaev chain
- $\vdots$

S. Mohapatra, S. Moudgalya and Ajit C. Balram, Phys. Rev. Lett. **134**, 210403 (2025)

Summary and future outlook

- Constructed a large class of spin chain Hamiltonians with exact volume-law entangled excited eigenstates
- Entanglement entropy and/or expectation values of local observables alone may not be good indicators or definitive diagnostics to support or violate ETH (these criteria do not distinguish our atypical states from typical thermal eigenstates)
- Extensions to higher dimensions.
- Generalization to fermionic and bosonic systems and clustering beyond pairing.
- Algorithms to find such states given the Hamiltonian (analogous to ScarFinder)

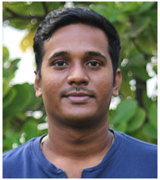
Petrova *et al.*, PRX Quantum **6**, 040333 (2025)

Ren *et al.*, PRX Quantum **6**, 040332 (2025)

Acknowledgments

- Work done with:

Sashikanta Mohapatra and Sanjay Moudgalya

- Supported by:

Thanks!