1 Lecture 1: Examples of U-modules

What does a module look like? A module is a vector space with some (linear) operators. To work with a vector space, specify a basis, and to work with linear operators, specify their matrices with respect to that basis.

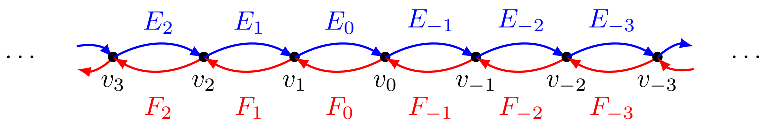
1.1 The module $L(\varepsilon_1)$ for $\mathbf{U} = U_t(\mathfrak{sl}_\infty)$

Let $E_{ij} \in M_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{C})$ denote the matrix with 1 in the (i, j) entry and 0 elsewhere.
Let

$$L(\varepsilon_1) \quad \text{be a vector space with basis} \quad \{v_i \mid i \in \mathbb{Z}\}.$$

For $i \in \mathbb{Z}$, define

$$E_i = E_{i,i+1}, \qquad F_i = E_{i+1,i}, \qquad K_i = 1 + (t-1)E_{ii} + (t^{-1}-1)E_{i+1,i+1}.$$



1.2 The module $L(\varepsilon_1)$ for $\mathbf{U} = U_t(L\mathfrak{sl}_n)$

Let $n \in \mathbb{Z}_{>2}$.
Let $E_{ij} \in M_{\mathbb{Z}\times\mathbb{Z}}(\mathbb{C})$ denote the matrix with 1 in the (i, j) entry and 0 elsewhere.
Let

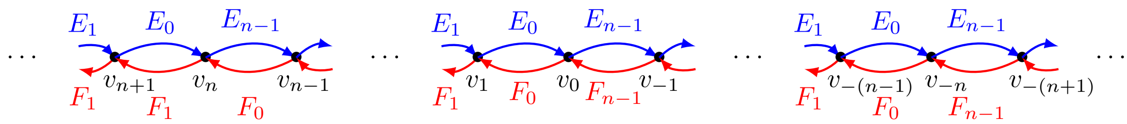
$$L(\varepsilon_1) \quad \text{be a vector space with basis} \quad \{v_i \mid i \in \mathbb{Z}\}.$$

For $i \in \mathbb{Z}/n\mathbb{Z}$, define

$$E_i = \sum_{\substack{k\in\mathbb{Z} \\ k=i \bmod n}} E_{k,k+1}, \qquad F_i = \sum_{\substack{k\in\mathbb{Z} \\ k=i \bmod n}} E_{k+1,k},$$

and

$$K_i = 1 + \sum_{\substack{k\in\mathbb{Z} \\ k=i \bmod n}} (t-1)E_{kk} + (t^{-1}-1)E_{k+1,k+1}.$$



1.3 Coiling $L(\varepsilon_1)$ for $U_t(L\mathfrak{sl}_n)$

Let $n \in \mathbb{Z}_{>2}$.
Let $E_{ij} \in M_{\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}}(\mathbb{C}[\epsilon, \epsilon^{-1}])$ denote the matrix with 1 in the (i, j) entry and 0 elsewhere.
Let $\{v_0, v_1, \ldots, v_{n-1}\}$ be a basis of $\mathbb{C}^n$. Then the vector space

$$\mathbb{C}^n[\epsilon, \epsilon^{-1}] = \mathbb{C}[\epsilon, \epsilon^{-1}] \otimes_{\mathbb{C}} \mathbb{C}^n \qquad \text{has } \mathbb{C}\text{-basis} \qquad \{\epsilon^\ell v_i \mid i \in \mathbb{Z}/n\mathbb{Z}, \ell \in \mathbb{Z}\}.$$

Define $\mathbb{C}[\epsilon, \epsilon^{-1}]$-linear endomorphisms of $\mathbb{C}^n[\epsilon, \epsilon^{-1}]$ by

$$E_i = E_{i,i+1}, \qquad F_i = E_{i+1,i}, \qquad K_i = 1 + (t-1)E_{ii} + (t^{-1}-1)E_{i+1,i+1}, \qquad \text{for } i \in \{1, \ldots, n-1\},$$

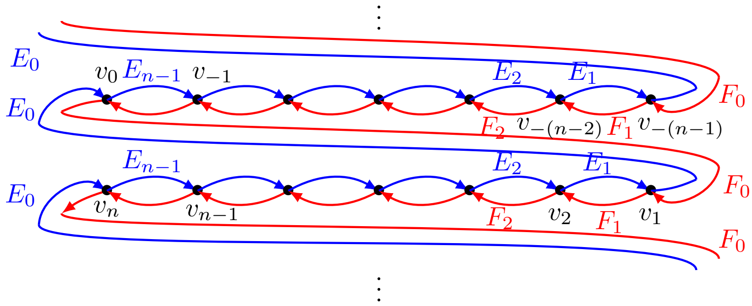
and define E_0, F_0, K_0 by

$$E_0 = \epsilon E_{0,1}, \qquad F_0 = \epsilon^{-1} E_{1,0}, \qquad K_0 = 1 + (t-1)E_{0,0} + (t^{-1}-1)E_{1,1},$$

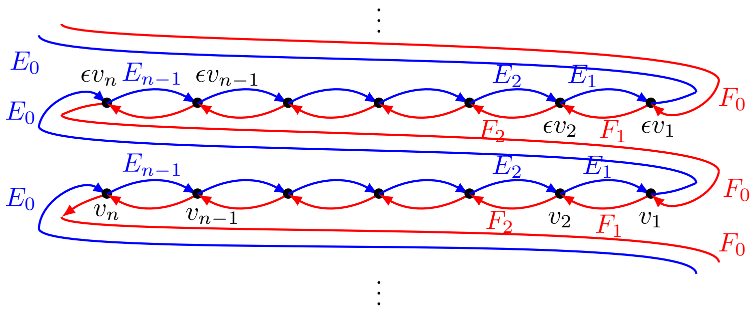
Then

$$\begin{array}{ccc} \mathbb{C}^n[\epsilon,\epsilon^{-1}] & \to & L(\varepsilon_1) \\ \epsilon^\ell v_i & \mapsto & v_{i-\ell n} \end{array} \qquad \text{is an isomorphism of } U_t(L\mathfrak{sl}_n)\text{-modules.}$$

Pictorially,



is isomorphic to



1.4 The module $L^{\mathrm{fin}}(u_1 - a)$ for $U_t(L\mathfrak{sl}_n)$

Let $a \in \mathbb{C}$.
Let $n \in \mathbb{Z}_{>1}$ and let $E_{ij} \in M_{\mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}}(\mathbb{C})$ denote the matrix with 1 in the (i,j) entry and 0 elsewhere. Let

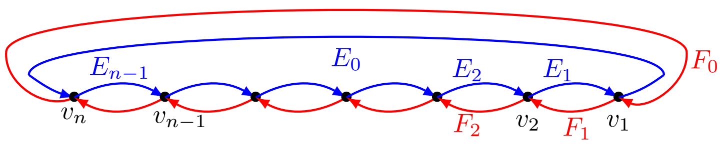
$$\mathbb{C}^n \quad \text{be a vector space with basis} \quad \{v_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}.$$

For $i \in \{1,\ldots,n\}$, define

$$E_i = E_{i,i+1}, \quad F_i = E_{i+1,i}, \quad K_i = 1 + (t-1)E_{ii} + (t^{-1}-1)E_{i+1,i+1}, \qquad \text{for } i \in \{1,\ldots,n-1\},$$

and define E_0, F_0, K_0 by

$$E_0 = aE_{0,1}, \qquad F_0 = a^{-1}E_{1,0}, \qquad K_0 = 1 + (t-1)E_{0,0} + (t^{-1}-1)E_{1,1},$$



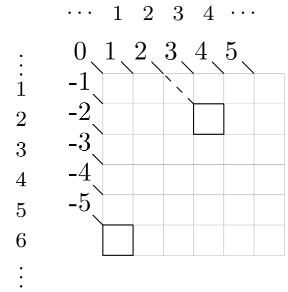
HW: Assume that t is not a root of unity. Prove that $L^{\mathrm{fin}}(u_1 - a)$ is an irreducible $U_t(L\mathfrak{sl}_n)$-module.
HW: Assume that t is not a root of unity. Prove that if $a_1, a_2 \in \mathbb{C}$ with $a_1 \neq a_2$ then

$$L^{\mathrm{fin}}(u_1 - a_1) \qquad \text{is not isomorphic to} \qquad L^{\mathrm{fin}}(u_1 - a_2).$$

1.5 Skew shapes and column strict tableaux

A *box* is an element of $\mathbb{Z}^2$ (rows and columns are indexed as for matrices). The *content* of box $= (i,j)$ is the diagonal number of the box (i,j),

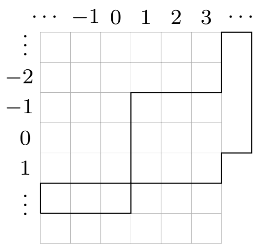
$$c(\text{box}) = j - i.$$



$b = (2,4)$ has content $c(b) = 2$,
$b = (6,1)$ has content $c(b) = -5$.

A *skew shape* is a finite subset of $\mathbb{Z}^2$ such that

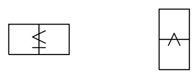
If $r \in \mathbb{Z}_{>0}$ and $(i,j), (i+r, j+r) \in \nu/\mu$ then $(i+a, j+b) \in \nu/\mu$ for $a, b \in \{0, 1, \ldots, r\}$.

$\nu/\mu =$ 

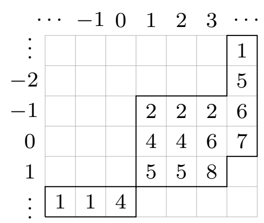
$$= \left\{ \begin{matrix} (-3,4), (-2,4), \\ (-1,1), (-1,2), (-1,3), (-1,4), \\ (0,1), (0,2), (0,3), (0,4), \\ (1,1), (1,2), (1,3), \\ (2,-2), (2,-1), (2,0), \end{matrix} \right\}$$

A *column strict tableau of shape ν/μ filled from* $\{1, \ldots, n\}$ is a function $T\colon \nu/\mu \to \{1, \ldots, n\}$ such that

(a) if $(i,j), (i+1,j) \in \nu/\mu$ then $T(i,j) > T(i+1,j)$,
(b) if $(i,j), (i,j+1) \in \nu/\mu$ then $T(i,j) \leq T(i,j+1)$.



For example,

$T =$ 

is a column strict tableau filled from $\{1, 2, \ldots, 9\}$.

Let

$$B(\nu/\mu) = \{\text{column strict tableaux of shape } \nu/\mu \text{ filled from } \{1, \ldots, n\}\}.$$

The set $B(\nu/\mu)$ is empty if ν/μ contains a column of length $> n$. If $B(\nu/\mu)$ is nonempty then it contains he *column reading tableau* $T^+ \in B(\nu/\mu)$ determined by

(a) if $(i,j) \in \nu/\mu$ and $(i-1,j)) \in \nu/\mu$ then $T^+(i,j) = 1$,
(b) if $(i,j), (i,j+1) \in \nu/\mu$ then $T^+(i,j+1) = T^+(i,j) + 1$.

1.6 The $\mathbf{U}'$-module $L^{\mathrm{fin}}(\nu/\mu)$

Let $\mathbf{U}' = U_t(L\mathfrak{sl}_n)$ and let

$$L^{\mathrm{fin}}(\nu/\mu) \qquad \text{be the vector space with basis} \qquad \{v_T \mid T \in B(\nu/\mu)\}.$$

For $T \in B(\nu/\mu)$, define

$$\gamma_T(u_0, u_1, \ldots, u_{n-1}, u_n) = \prod_{b\in\nu/\mu} \frac{1 - u_{T(b)} t^{2c(b)+T(b)-1}}{1 - u_{T(b)-1} t^{2c(b)+T(b)}}.$$

For $i \in \{1, \ldots, n-1\}$, let $\gamma_T^{(i)}(u)$ be $\gamma_T(u_0, u_1, \ldots, u_{n-1}, u_n)$ evaluated at $u_i = u$ and $u_j = 0$ at $j \neq i$,

$$\gamma_T^{(i)}(u) = \gamma_T(0, \ldots, 0, u, 0, \ldots, 0).$$

For $i \in \{1, \ldots, n\}$ and $u, w, z \in \mathbb{C}^\times$, define operators $\mathbf{q}^{(i)}(u)$, $\mathbf{x}_i^+(t^a)$ and $\mathbf{x}_i^-(t^a)$ on $L^{\mathrm{fin}}(\nu/\mu)$ by

$$\mathbf{q}_+^{(i)}(u) v_T = q^{\deg(\gamma_T^{(i)})} \frac{\gamma_T^{(i)}(t^{-1}u)}{\gamma_T^{(i)}(tu)} v_T, \qquad \text{and} \qquad \mathbf{q}_-^{(i)}(u) v_T = q^{-\deg(\gamma_T^{(i)})} \frac{\gamma_T^{(i)}(tu)}{\gamma_T^{(i)}(t^{-1}u)} v_T,$$

and

$$\mathbf{x}_i^+(t^a) v_T = \begin{cases} (const) v_{\tilde{e}_{i,a}T}, & \text{if } T \text{ has a box } b \text{ with } T(b) = i \text{ and } c(b) = a, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathbf{x}_i^-(t^a) v_T = \begin{cases} (const) v_{\tilde{f}_{i,a}T}, & \text{if } T \text{ has a box } b \text{ with } T(b) = i+1 \text{ and } c(b) = a, \\ 0, & \text{otherwise,} \end{cases}$$

where

$\tilde{e}_{i,a}T$ is T except with i changed to $i+1$ in a box of content a,

$\tilde{f}_{i,a}T$ is T except with $i+1$ changed to i in a box of content a,

2 Lecture 2: The affine Weyl group

2.1 The finite Weyl group W_{fin}

Before we start the game we wish to play we are given some symbols δ, Λ_0 and $\alpha_1^\vee, \ldots, \alpha_n^\vee$ and some integers denoted

$$\begin{matrix} a_0, a_1, \ldots, a_n \in \mathbb{Z}_{>0}, \\ a_0^\vee, a_1^\vee, \ldots, a_n^\vee \in \mathbb{Z}_{>0}, \end{matrix} \qquad \text{and} \qquad C_{ij},\ \text{for } i,j \in \{1,\ldots,n\}.$$

Fix a $\mathbb{C}$-vector space $\mathfrak{a}$ and its dual $\mathfrak{a}^*$ so that

$$\begin{matrix} \mathfrak{a} \text{ has } \mathbb{C}\text{-basis } \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}, \\ \mathfrak{a}^* \text{ has } \mathbb{C}\text{-basis } \{\omega_1, \ldots, \omega_n\} \end{matrix} \qquad \text{with} \qquad \langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}.$$

Let

$$\mathfrak{h}^* \text{ be the vector space with basis } \quad \{\delta, \omega_1, \ldots, \omega_n, \Lambda_0\}.$$

For $j \in \{1, \ldots, n\}$ define

$$\alpha_j = C_{1j}\omega_1 + \cdots + C_{nj}\omega_n$$

and

$$s_j \colon \mathfrak{h}^* \to \mathfrak{h}^* \qquad \text{by} \qquad s_i(a\delta + \lambda + \ell\Lambda_0) = a\delta + \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i + \ell\Lambda_0,$$

for $a, \ell \in \mathbb{C}$ and $\lambda \in \mathfrak{a}^*$. The *finite Weyl group*

$$W_{\mathrm{fin}} \subseteq GL(\mathfrak{h}^*) \qquad \text{is generated by } s_1, \ldots, s_n.$$

2.2 The affine Weyl group

Let

$$\mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}} = \mathbb{Z}\text{-span}\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}.$$

For $\mu^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}}$ define $t_{\mu^\vee} \colon \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$t_{\mu^\vee}(a\delta + \lambda + \ell\Lambda_0) = (a + (\lambda - \ell\tfrac{1}{2}\mu^\vee)(\mu^\vee))\delta + \lambda + \ell\mu^\vee + \ell\Lambda_0,$$

where $a, \ell \in \mathbb{C}$, $\lambda \in \mathfrak{a}^*$ and $\mu^\vee$ is viewed as an element of $\mathfrak{a}^*$ by the isomorphism

$$\begin{matrix} \mathfrak{a} & \xrightarrow{\sim} & \mathfrak{a}^* \\ a_j^\vee \alpha_j^\vee & \mapsto & a_j \alpha_j \end{matrix}$$

The *affine Weyl group* is

$$W^{\mathrm{ad}} = \mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}} \rtimes W_{\mathrm{fin}} = \{t_{\mu^\vee} w \mid \mu^\vee \in \mathfrak{a}_{\mathbb{Z}},\ w \in W_{\mathrm{fin}}\}$$

with

$$t_{\mu^\vee} t_{\nu^\vee} = t_{\mu^\vee + \nu^\vee} \qquad \text{and} \qquad w t_{\mu^\vee} = t_{w\mu^\vee} w, \qquad \text{for } \mu^\vee, \nu^\vee \in \mathfrak{a}_{\mathbb{Z}} \text{ and } w \in W_{\mathrm{fin}}.$$

2.3 The Heisenberg group

The matrices of s_i and $t_{\mu^\vee}$, with respect to the basis $\{\delta, \omega_1, \ldots, \omega_n, \Lambda_0\}$ of $\mathfrak{h}^*$, are

$$s_i = \left(\begin{array}{c|cccccc|c}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \hline
0 & 1 & \cdots & 0 & -\alpha_i(h_1) & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & -\alpha_i(h_2) & 0 & \cdots & 0 \\
\vdots & & & & \vdots & & & \vdots \\
0 & 0 & \cdots & 1 & -\alpha_i(h_{i-1}) & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & -\alpha_i(h_{i+1}) & 1 & \cdots & 0 \\
\vdots & & & & \vdots & & & \vdots \\
0 & 0 & \cdots & 0 & -\alpha_i(h_n) & 0 & \cdots & 0 \\ \hline
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1
\end{array}\right), \qquad \text{for } i \in \{1, \ldots, n\}, \text{ and} \tag{2.1}$$

$$t_{\mu^\vee} = \left(\begin{array}{c|ccc|c}
1 & k_1 & \cdots & k_n & -\frac{1}{2}\langle \mu^\vee, \mu^\vee \rangle \\ \hline
\vdots & & & & \mu_1^\vee \\
0 & & 1 & & \vdots \\
\vdots & & & & \mu_n^\vee \\ \hline
0 & \cdots & 0 & \cdots & 1
\end{array}\right), \quad \text{for} \quad \begin{array}{rl} \mu^\vee & = k_1 h_1 + \cdots + k_n h_n \\ & = \frac{k_1 a_1}{a_1^\vee}\alpha_1 + \cdots + \frac{k_n a_n}{a_n^\vee}\alpha_n \\ & = \mu_1^\vee \omega_1 + \cdots + \mu_n^\vee \omega_n \text{ in } \mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}}, \end{array} \tag{2.2}$$

so that $-\frac{1}{2}\langle \mu^\vee, \mu^\vee \rangle = -\frac{1}{2}(\mu_1^\vee k_1 + \cdots \mu_n^\vee k_n)$. The *Heisenberg group* is the subgroup of $GL(\mathfrak{h}^*)$ consisting of transformations for which, with respect to the basis $\{\delta, \omega_1, \ldots, \omega_n, \Lambda_0\}$ of $\mathfrak{h}^*$, the matrices are

$$\left(\begin{array}{c|ccc|c}
1 & a_1 & \cdots & a_n & z \\ \hline
\vdots & & & & \gamma_1 \\
0 & & 1 & & \vdots \\
\vdots & & & & \gamma_n \\ \hline
0 & \cdots & 0 & \cdots & 1
\end{array}\right) \quad \text{with} \quad a_1, \ldots, a_n, \quad \gamma_1, \ldots, \gamma_n, \quad z \text{ in } \mathbb{R}.$$

2.4 Orbit representatives of the W^{ad} action on $(\mathfrak{h}^*)_{\mathrm{int}}$

Let

$$\mathfrak{h}_{\mathbb{Z}}^* = \mathbb{C}\delta + \mathbb{Z}\text{span}\{\omega_1, \ldots, \omega_n, \Lambda_0\}.$$

For $\ell \in \mathbb{Z}_{>0}$, define

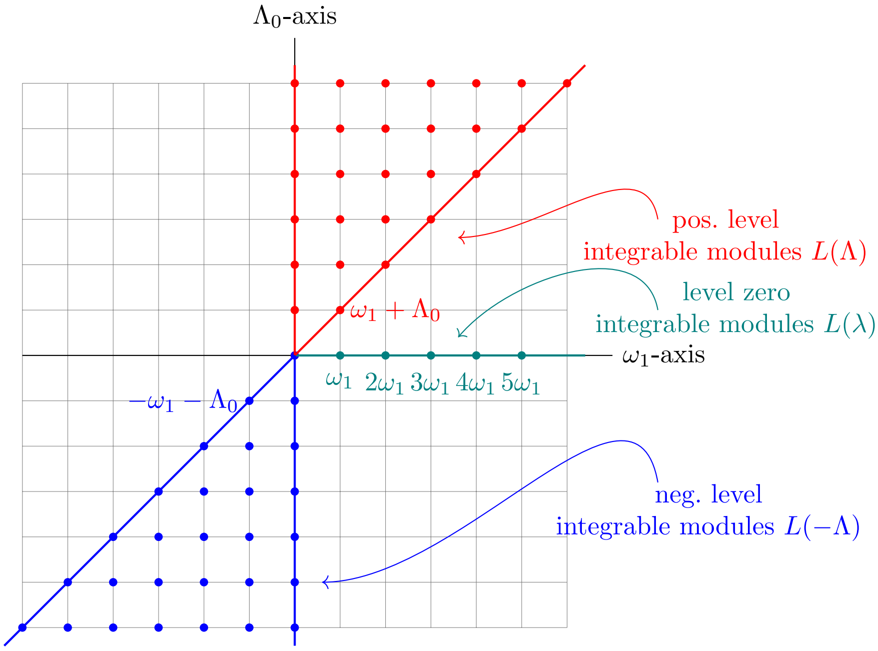
$$A_\ell = \{a\delta + \lambda + \ell\Lambda_0 \in \mathfrak{h}_{\mathbb{Z}}^* \mid a \in \mathbb{C},\ \lambda(h_\theta) \le \ell, \text{ and } \lambda(h_{\alpha_i}) \ge 0 \text{ for } i \in \{1, \ldots, n\}\},$$

$$A_0 = \{a\delta + \lambda \in \mathfrak{h}_{\mathbb{Z}}^* \mid a \in \mathbb{C}, \text{ and } \lambda(h_{\alpha_i}) \ge 0 \text{ for } i \in \{1, \ldots, n\}\} \quad \text{and}$$

$$A_{-\ell} = \{a\delta + \lambda + \ell\Lambda_0 \in \mathfrak{h}_{\mathbb{Z}}^* \mid a \in \mathbb{C},\ \lambda(h_\theta) \ge \ell, \text{ and } \lambda(h_{\alpha_i}) \le 0 \text{ for } i \in \{1, \ldots, n\}\}.$$

A set of representatives of the W^{ad} orbits on $\mathfrak{h}_{\mathbb{Z}}^*$ is

$$\mathfrak{h}_{\mathrm{int}}^* = \cdots \cup A_{-2} \cup A_{-1} \cup A_0 \cup A_1 \cup A_2 \cup \cdots.$$



2.5 Classifications of integrable modules

Let $\mathbf{U} = U_t(\mathfrak{g})$ be the quantum group corresponding to the affine Kac-Moody algebra

$$\mathfrak{g} = \mathbb{C}d \oplus \mathring{\mathfrak{g}}[\epsilon, \epsilon^{-1}] \oplus \mathbb{C}K.$$

Let $\mathbf{U}'$ be $\mathbf{U}$ but without the generator $D = t^d$.

Let $\ell \in \mathbb{Z}_{>0}$. The sets A_ℓ provide index sets for classes of $\mathbf{U}$-modules:

$$\begin{aligned}
A_0 &\quad\leftrightarrow\quad \{\text{finite dimensional } U_t(\mathring{\mathfrak{g}}) \text{ modules}\} \\
A_\ell &\quad\leftrightarrow\quad \{\text{level } \ell \text{ irreducible integrable } U_t(\mathfrak{g}) \text{ modules}\} \\
A_{-\ell} &\quad\leftrightarrow\quad \{\text{level } -\ell \text{ irreducible integrable } U_t(\mathfrak{g}) \text{ modules}\} \\
A_0 &\quad\leftrightarrow\quad \{\text{level 0 integrable extremal weight modules } U_t(\mathfrak{g}) \text{ modules}\}
\end{aligned}$$

Let

$$\mathbb{C}[u]^{\oplus n}_{\text{mon}} = \{a_1(u)\omega_1 + \cdots + a_n(u)\omega_n \mid a_i(u) \in \mathbb{C}[u] \text{ and } a_i(u) \text{ is monic}\}$$

Then

$$\mathbb{C}[u]^{\oplus n}_{\text{mon}} \leftrightarrow \{\text{irreducible finite dimensional } U_t(\mathfrak{g})\text{-modules}\}$$

2.6 Type $A_1^{(1)}$

For Type $A_1^{(1)}$ the initial data consists of the matrices $B = DC$ given by

$$B = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} = DC.$$

Since

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{then} \qquad K = \alpha_0^\vee + \alpha_1^\vee \quad \text{and} \quad \Lambda_1 = \omega_1 + \Lambda_0.$$

The matrices

$$t_{k\alpha_1^\vee} = \begin{pmatrix} 1 & -k & -k^2 \\ 0 & 1 & 2k \\ 0 & 0 & 1 \end{pmatrix}, \qquad s_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad s_0 = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{with } k \in \mathbb{Z},$$

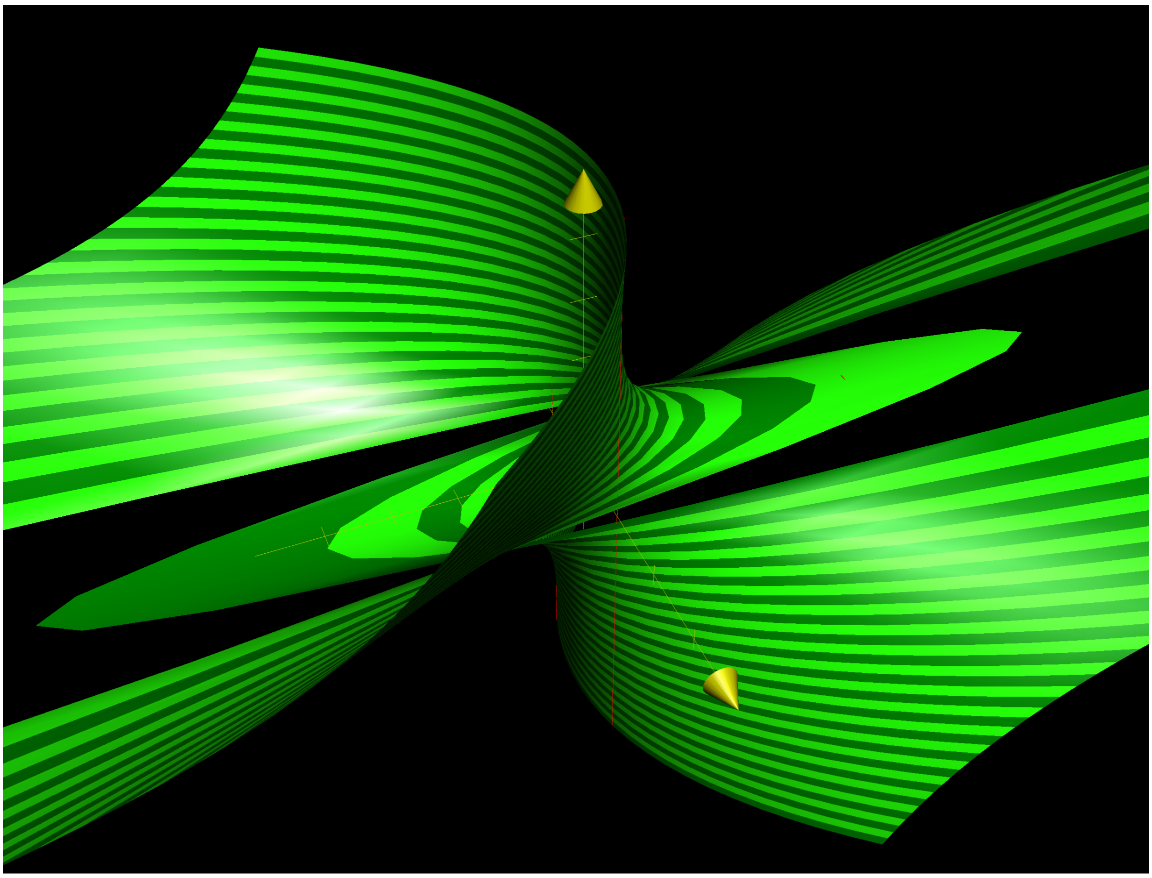
generate the action of W^{ad} on $\mathfrak{h}^*$, with respect to the basis $\{\delta, \omega_1, \Lambda_0\}$ of $\mathfrak{h}^*$, For example, if $\ell \in \mathbb{R}_{\neq 0}$ then

$$\text{the } W^{\mathrm{ad}}\text{-orbit of } (2\omega_1 + \ell\Lambda_0) \text{ is contained in the parabola } \quad \{y\delta + x\omega_1 + \ell\Lambda_0 \mid y = \tfrac{1}{-4\ell}(x^2 - 4)\}.$$

and if $\ell = 0$ then

$$\text{the } W^{\mathrm{ad}}\text{-orbit of } (2\omega_1 + 0\Lambda_0) \text{ is contained in the two lines } \quad \{y\delta + x\omega_1 \mid x = 2 \text{ or } x = -2\}.$$

These parabolas and lines are displayed in the following picture.



3 Lecture 3: Extremal weight modules

Let

$$\mathfrak{h}^*_{\mathbb{Z}} = \mathbb{C}\delta + \mathbb{Z}\text{-span}\{\Lambda_0, \dots, \Lambda_n\}.$$

A set of representatives for the W^{ad}-orbits on $\mathfrak{h}^*_{\mathbb{Z}}$ is $(\mathfrak{h}^*)_{\mathrm{int}} = (\mathfrak{h}^*)^+_{\mathrm{int}} \cup (\mathfrak{h}^*)^0_{\mathrm{int}} \cup (\mathfrak{h}^*)^-_{\mathrm{int}}$, where

$$\begin{aligned}
(\mathfrak{h}^*)^+_{\mathrm{int}} &= \mathbb{C}\delta + \mathbb{Z}_{\geq 0}\text{-span}\{\Lambda_0, \dots, \Lambda_n\}, \\
(\mathfrak{h}^*)^0_{\mathrm{int}} &= \mathbb{C}\delta + 0\Lambda_0 + \mathbb{Z}_{\geq 0}\text{-span}\{\omega_1, \dots, \omega_n\}, \\
(\mathfrak{h}^*)^-_{\mathrm{int}} &= \mathbb{C}\delta + \mathbb{Z}_{\leq 0}\text{-span}\{\Lambda_0, \dots, \Lambda_n\}.
\end{aligned} \tag{3.1}$$

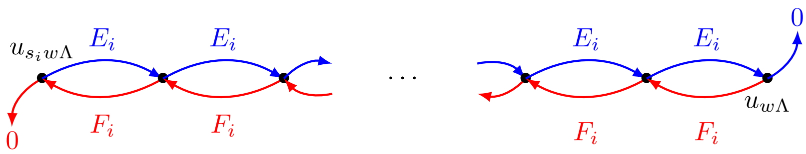
For $\widehat{\mathfrak{sl}}_2$ these sets are pictured (mod δ) in (6).

3.1 Extremal weight modules $L(\Lambda)$

Let $\Lambda \in \mathfrak{h}^*_{\mathrm{int}}$. The *extremal weight module* $L(\Lambda)$ is the **U**-module

$$\begin{aligned}
&\text{generated by } \{u_{w\Lambda} \mid w \in W\} \qquad \text{with relations} \quad K_i(u_{w\Lambda}) = q^{\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda}, \\
&E_i u_{w\Lambda} = 0, \quad \text{and} \quad F_i^{\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda} = u_{s_i w\Lambda}, \qquad \text{if } \langle w\Lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\geq 0}, \\
&F_i u_{w\Lambda} = 0, \quad \text{and} \quad E_i^{-\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda} = u_{s_i w\Lambda}, \qquad \text{if } \langle w\Lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\leq 0},
\end{aligned} \tag{3.2}$$

for $i \in \{0, \cdots, n\}$. Pictorially, if $\langle w\Lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\geq 0}$ then there is a chain of length $\langle w\Lambda, \alpha_i^\vee\rangle$ from $u_{w\Lambda}$ to $u_{s_i w\Lambda}$,

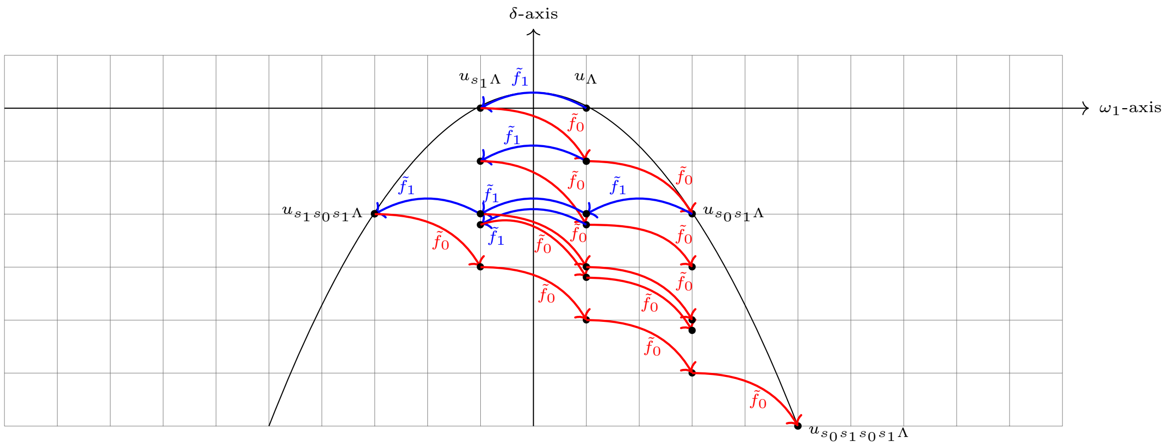


The module $L(\Lambda)$ has a crystal, denoted $B(\Lambda)$. The crystal is a labeling set for a (weight) basis of $L(\Lambda)$.

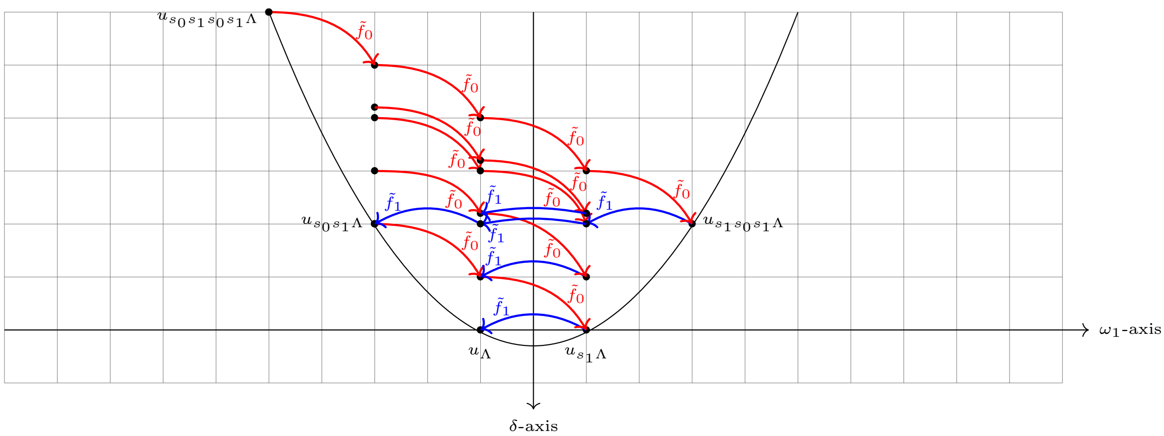
Some properties of the $L(\Lambda)$ are:

- If $\Lambda \in (\mathfrak{h}^*)^+_{\mathrm{int}}$ then $L(\Lambda)$ is the simple **U**-module of highest weight Λ.
- If $\Lambda \not\in (\mathfrak{h}^*)^+_{\mathrm{int}}$ then $L(\Lambda)$ is not a highest weight module.
- If $\Lambda \in (\mathfrak{h}^*)^-_{\mathrm{int}}$ then $L(\Lambda)$ is the simple **U**-module of lowest weight Λ.
- If $\Lambda \not\in (\mathfrak{h}^*)^-_{\mathrm{int}}$ then $L(\Lambda)$ is not a lowest weight module.

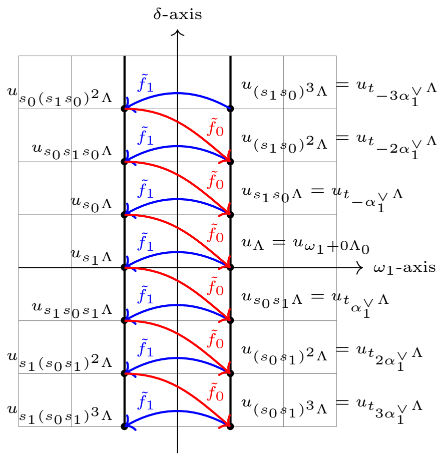
PLATE B: Pictures of $B(\omega_1+\Lambda_0)$, $B(\omega_1+0\Lambda_0)$ and $B(-\omega_1-\Lambda_0)$ for $\widehat{\mathfrak{sl}}_2$



Initial portion of the crystal graph of $B(\omega_1+\Lambda_0)$ for $\widehat{\mathfrak{sl}}_2$



Final portion of the crystal graph of $B(-\omega_1-\Lambda_0)$ for $\widehat{\mathfrak{sl}}_2$



Middle portion of the crystal graph of $B(\omega_1+0\Lambda_0)$ for $\mathfrak{g}=\widehat{\mathfrak{sl}}_2$

3.2 Bruhat orders

Let $\mathfrak{a}_{\mathbb{R}}^* = \mathbb{R}\text{-span}\{\alpha_1, \ldots, \alpha_n\}$. An *alcove* is a fundamental region for the action of W^{ad} on $(\mathbb{R}\delta + \mathfrak{a}_{\mathbb{R}}^* + \Lambda_0)/\mathbb{R}\delta$. There is a bijection

$$\begin{array}{ccc} W^{\mathrm{ad}} & \longleftrightarrow & \{\text{alcoves}\} \\ 1 & \longmapsto & \{x + \Lambda_0 \in \mathfrak{a}_{\mathbb{R}}^* + \Lambda_0 \mid x(h_i) > 0 \text{ for } i \in \{0, \ldots, n\}\} \end{array} \qquad (3.3)$$

An element $w \in W^{\mathrm{ad}}$ is *dominant* if

$$w(\rho + \Lambda_0) \in \mathbb{R}_{\geq 0}\text{-span}\{\omega_1, \ldots, \omega_n\} + \Lambda_0, \qquad \text{where} \quad \rho = \omega_1 + \cdots + \omega_n.$$

In the identification (3.3) of elements of W^{ad} with alcoves, the dominant elements of W^{ad} are the alcoves in the dominant Weyl chamber.

Let $x, w \in W^{\mathrm{ad}}$ and let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced word for w in the generators $s_0, \ldots, s_n$. The *positive level Bruhat order on* W^{ad} is defined by

$$x \overset{+}{\trianglelefteq} w \quad \text{if } x \text{ has a reduced word which is a subword of } w = s_{i_1} \cdots s_{i_\ell}$$

The *negative level Bruhat order* on W^{ad} is defined by $x \overset{-}{\le} w$ if $x \overset{+}{\trianglerighteq} w$.

The *level 0 Bruhat order* on W^{ad} is determined by

(a) $\overset{0}{\trianglelefteq}$ for dominant elements: If x, w are dominant then $x \overset{0}{\trianglelefteq} w$ if and only if $x \overset{+}{\trianglelefteq} w$,

(b) $\overset{0}{\trianglelefteq}$ translation invariance: If $\mu^\vee \in \mathfrak{a}_{\mathbb{Z}}^{\mathrm{ad}}$ and $x, w \in W$ then $x \overset{0}{\trianglelefteq} w$ if and only if $xt_{\mu^\vee} \overset{0}{\trianglelefteq} wt_{\mu^\vee}$.

3.3 Demazure submodules

Let $\mathbf{U}^+$ be the subalgebra of $\mathbf{U}$ generated by $E_0, \ldots, E_n, K_0, \ldots, K_n, C, D$.

Let $w \in W^{\mathrm{ad}}$. The *Demazure module* $L(\Lambda)_w^+$ is the $\mathbf{U}^+$-submodule of $L(\Lambda)$ given by

$$L(\Lambda)_w^+ = \mathbf{U}^+ u_{w\Lambda} \qquad \text{and} \qquad \mathrm{char}(L(\Lambda)_w^+) = \sum_{p \in B(\Lambda)_w^+} e^{\mathrm{wt}(p)},$$

since $L(\Lambda)_w^+$ has a crystal $B(\Lambda)_w^+$.

3.4 Demazure operators

The *BGG-Demazure operator* on $\mathbb{C}[\mathfrak{h}_{\mathbb{Z}}^*] = \mathbb{C}\text{-span}\{X^\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*\}$ is given by

$$D_i = (1 + s_i)\frac{1}{1 - X^{-\alpha_i}}, \qquad \text{for } i \in \{0, 1, \ldots, n\}.$$

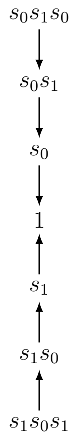
Let $\Lambda \in (\mathfrak{h}^*)_{\mathrm{int}}$, $w \in W^{\mathrm{ad}}$ and $i \in \{0, 1, \ldots, n\}$.

$$\text{If } \Lambda \in (\mathfrak{h}^*)_{\mathrm{int}}^+ \quad \text{then} \qquad D_i \mathrm{char}(L(\Lambda)_w^+)) = \begin{cases} \mathrm{char}(L(\Lambda)_{s_i w}^+), & \text{if } s_i w \overset{+}{\trianglerighteq} w, \\ \mathrm{char}(L(\Lambda)_w^+), & \text{if } s_i w \overset{+}{\trianglelefteq} w; \end{cases}$$

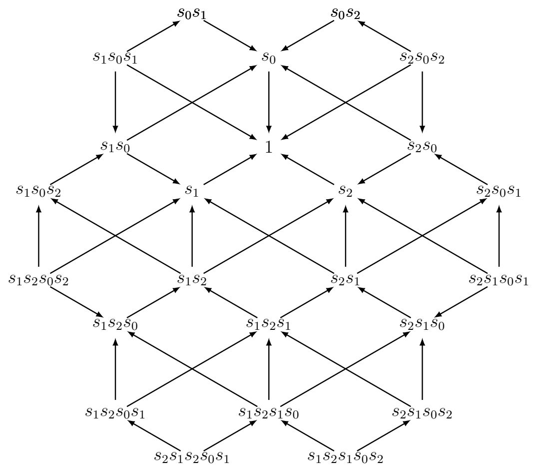
$$\text{if } \lambda \in (\mathfrak{h}^*)_{\mathrm{int}}^0 \quad \text{then} \qquad D_i \mathrm{char}(L(\lambda)_w^+) = \begin{cases} \mathrm{char}(L(\lambda)_{s_i w}^+), & \text{if } s_i w \overset{0}{\trianglerighteq} w, \\ \mathrm{char}(L(\lambda)_w^+), & \text{if } s_i w \overset{0}{\trianglelefteq} w; \end{cases}$$

$$\text{if } \Lambda \in (\mathfrak{h}^*)_{\mathrm{int}}^- \quad \text{then} \qquad D_i \mathrm{char}(L(\Lambda)_w^+) = \begin{cases} \mathrm{char}(L(\Lambda)_{s_i w}^+), & \text{if } s_i w \overset{-}{\ge} w, \\ \mathrm{char}(L(\Lambda)_w^+), & \text{if } s_i w \overset{-}{\le} w; \end{cases}$$

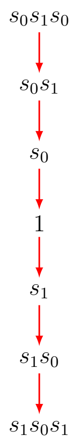
PLATE A: Bruhat orders on the affine Weyl group (partial relations)



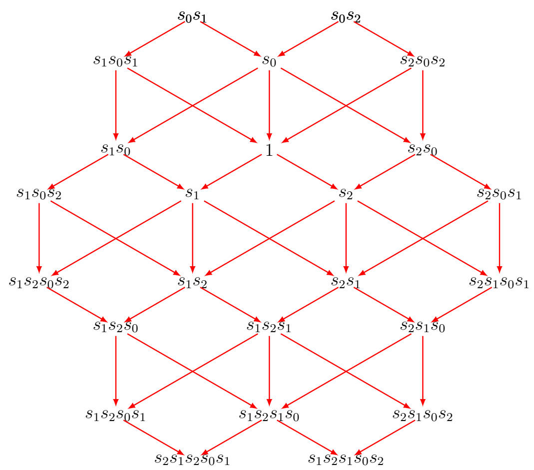
postive level Bruhat order for $\widehat{\mathfrak{sl}}_2$
1 is minimal



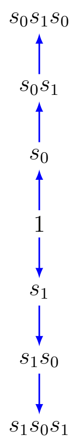
postive level Bruhat order for $\widehat{\mathfrak{sl}}_3$
1 is minimal



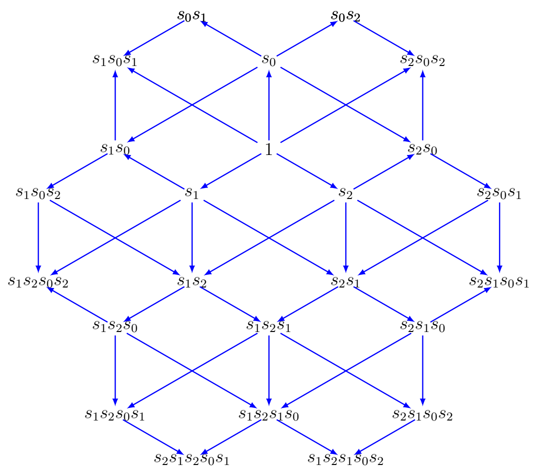
level zero Bruhat order for $\widehat{\mathfrak{sl}}_2$
translation invariant



level zero Bruhat order for $\widehat{\mathfrak{sl}}_3$
translation invariant



negative level Bruhat order for $\widehat{\mathfrak{sl}}_2$
1 is maximal



negative level Bruhat order for $\widehat{\mathfrak{sl}}_3$
1 is maximal

3.5 Crystals

Let $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}\text{-span}\{\delta, \omega_1, \ldots, \omega_n, \Lambda_0\}$. The universal crystal in mind is the set

$$B(\mathrm{univ}) = \left\{ \text{piecewise linear paths } p\colon \mathbb{R}_{[0,1]} \to \mathfrak{h}^*_{\mathbb{R}} \;\middle|\; \begin{array}{l} p(0) = 0, \\ p(1) \in \mathfrak{h}^*_{\mathbb{Z}} \end{array} \right\}$$

with root operators $\tilde{e}_0, \ldots, \tilde{e}_n$ and $\tilde{f}_0, \ldots, \tilde{f}_n$ defined by Littelmann (see [Ra06, §5] for an exposition). For $\Lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, the *straight line path from* 0 *to* Λ is

$$p_\Lambda\colon \mathbb{R}_{[0,1]} \to \mathfrak{h}^*_{\mathbb{R}} \qquad \text{given by} \qquad p_\Lambda(t) = t\Lambda, \qquad \text{for } t \in \mathbb{R}_{[0,1]}.$$

Let $w \in W^{\mathrm{ad}}$.

$$\text{If } \Lambda \in (\mathfrak{h}^*)^+_{\mathrm{int}} \quad \text{then} \quad \begin{array}{l} B(\Lambda) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} p_\Lambda \mid k \in \mathbb{Z}_{\geq 0} \text{ and } i_1, \ldots, i_k \in \{0, 1, \ldots, n\}\}, \\ \\ B(\Lambda)^+_w = \{p \in B(\Lambda) \mid \text{the initial direction of } p \text{ is } \trianglelefteq w\}. \end{array}$$

$$\text{If } \Lambda \in (\mathfrak{h}^*)^-_{\mathrm{int}} \quad \text{then} \quad \begin{array}{l} B(\Lambda) = \{\tilde{e}_{i_1} \cdots \tilde{e}_{i_k} p_\Lambda \mid k \in \mathbb{Z}_{\geq 0} \text{ and } i_1, \ldots, i_k \in \{0, 1, \ldots, n\}\}, \\ \\ B(\Lambda)^+_w = \{p \in B(\Lambda) \mid \text{the initial direction of } p \text{ is } \leq w\}. \end{array}$$

3.6 Weyl character formula

The Weyl character formulas are formulas for the characters of the extremal weight modules $L(\Lambda)$ for the cases when $\Lambda \in (\mathfrak{h}^*)^+_{\mathrm{int}}$ or $\Lambda \in (\mathfrak{h}^*)^-_{\mathrm{int}}$.

Let $\mathring{\rho} = \omega_1 + \cdots + \omega_n$ and $h^\vee = a_0^\vee + a_1^\vee + \cdots + a_n^\vee$ and

$$\rho = \Lambda_0 + \Lambda_1 + \cdots + \Lambda_n = \omega_1 + \cdots + \omega_n + (a_0^\vee + a_1^\vee + \cdots + a_n^\vee)\Lambda_0 \;= \mathring{\rho} + h^\vee \Lambda_0.$$

Letting

$$q = e^{-\delta},$$

the *Weyl denominators* are

$$a^+_\rho = e^{\mathring{\rho} + h^\vee \Lambda_0} \prod_{r \in \mathbb{Z}_{>0}} \Big((1 - q^r)^n \cdot \prod_{\alpha \in R^+} (1 - q^{r-1} e^{-\alpha})(1 - q^r e^{\alpha}) \Big)$$

and

$$a^-_\rho = e^{-\mathring{\rho} - h^\vee \Lambda_0} \prod_{r \in \mathbb{Z}_{>0}} \Big((1 - q^{-r})^n \cdot \prod_{\alpha \in R^+} (1 - q^{-(r-1)} e^{\alpha})(1 - q^{-r} e^{-\alpha}) \Big)$$

and the *Weyl character formulas* are

$$\begin{array}{lll} \text{if } \Lambda \in (\mathfrak{h}^*)^+_{\mathrm{int}} & \text{then} & \mathrm{char}(L(\Lambda)) = \dfrac{1}{a^+_\rho} \displaystyle\sum_{w \in W} \det(w) e^{w(\Lambda + \rho)}, \\ \text{if } \Lambda \in (\mathfrak{h}^*)^-_{\mathrm{int}} & \text{then} & \mathrm{char}(L(\Lambda)) = \dfrac{1}{a^-_\rho} \displaystyle\sum_{w \in W} \det(w) e^{w(\Lambda - \rho)}. \end{array}$$

The *Weyl denominator formula* is equivalent to $\mathrm{char}(L(0)) = 1$.

4 Lecture 4: Level 0 representations

4.1 Extremal weight modules $L(\Lambda)$

Let $\Lambda \in \mathfrak{h}^*_{\mathrm{int}}$. The *extremal weight module* $L(\Lambda)$ is the $\mathbf{U}$-module

$$\begin{aligned} &\text{generated by } \{u_{w\Lambda} \mid w \in W\} \qquad \text{with relations} \quad K_i(u_{w\Lambda}) = q^{\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda}, \\ &E_i u_{w\Lambda} = 0, \quad \text{and} \quad F_i^{\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda} = u_{s_i w\Lambda}, \qquad \text{if } \langle w\Lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\geq 0}, \\ &F_i u_{w\Lambda} = 0, \quad \text{and} \quad E_i^{-\langle w\Lambda, \alpha_i^\vee\rangle} u_{w\Lambda} = u_{s_i w\Lambda}, \qquad \text{if } \langle w\Lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\leq 0}, \end{aligned} \tag{4.1}$$

for $i \in \{0, \cdots, n\}$. This module has a crystal, denoted $B(\Lambda)$.

4.2 Level 0 extremal weight modules $L(\lambda)$

Let

$$\lambda = m_1\omega_1 + \cdots + m_n\omega_n, \qquad \text{with} \quad m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}.$$

Let

$$x_{1,1}, \ldots, x_{m_1,1}, \qquad x_{1,2}, \ldots, x_{m_2,2}, \qquad \ldots, \qquad x_{1,n}, \ldots, x_{m_n,n},$$

be n sets of formal variables and define

$$RG_\lambda = \mathbb{C}[x_{1,1}^{\pm 1}, \ldots, x_{m_1,1}^{\pm 1}]^{S_{m_1}} \otimes \cdots \otimes \mathbb{C}[x_{1,n}^{\pm 1}, \ldots, x_{m_n,n}^{\pm 1}]^{S_{m_n}}$$

For $i \in \{1, \ldots, n\}$, define

$$\begin{aligned} e_+^{(i)}(u) &= (1 - x_{1,i}u)(1 - x_{2,i}u) \cdots (1 - x_{m_i,i}u) \quad \text{and} \\ e_-^{(i)}(u^{-1}) &= (1 - x_{1,i}^{-1}u^{-1})(1 - x_{2,i}^{-1}u^{-1}) \cdots (1 - x_{m_i,i}^{-1}u^{-1}). \end{aligned}$$

Let $\mathbf{U}'$ be the subalgebra of $\mathbf{U}$ without the generator D.

Theorem 4.1. *The extremal weight module $L(\lambda)$ is the $(\mathbf{U}' \otimes_{\mathbb{Z}} RG_\lambda)$-module generated by a single vector m_λ with relations*

$$\mathbf{x}_{i,r}^+ m_\lambda = 0, \qquad K_i m_\lambda = q^{m_i} m_\lambda, \quad C m_\lambda = m_\lambda,$$

$$\mathbf{q}_+^{(i)}(u) m_\lambda = K_i \frac{e_+^{(i)}(q^{-1}u)}{e_+^{(i)}(qu)} m_\lambda \qquad \text{and} \qquad \mathbf{q}_-^{(i)}(u^{-1}) m_\lambda = K_i^{-1} \frac{e_-^{(i)}(qu^{-1})}{e_-^{(i)}(q^{-1}u^{-1})} m_\lambda,$$

where $\mathbf{q}_+^{(i)}(u)$ and $\mathbf{q}_-^{(i)}(u^{-1})$ are generating series for loop generators of $\mathbf{U}$.

An alternative presentation of $L(\lambda)$ is as the $(\mathbf{U}' \otimes_{\mathbb{Z}} RG_\lambda)$-module generated by a single vector m_λ with relations

$$\mathbf{x}_{i,r}^+ m_\lambda = 0, \qquad K_i m_\lambda = q^{m_i} m_\lambda, \quad C m_\lambda = m_\lambda,$$

and

$$\mathbf{e}_s^{(i)} m_\lambda = 0 \quad \text{and} \quad \mathbf{e}_{-s}^{(i)} m_\lambda = 0, \qquad \text{for } i \in \{1, \ldots, n\} \text{ and } s \in \mathbb{Z}_{>m_i},$$

4.3 Finite dimensional standard modules $M^{\text{fin}}(a(u))$

A *Drinfeld polynomial* is an n-tuple of polynomials $a(u) = (a^{(1)}(u), \ldots, a^{(n)}(u))$ with $a^{(i)}(u) \in \mathbb{C}[u]$, represented as

$$a(u) = a^{(1)}(u)\omega_1 + \cdots + a^{(n)}(u)\omega_n, \qquad \text{with} \quad a^{(i)}(u) = (u - a_{1,i}) \cdots (u - a_{m_i,i})$$

so that

$$\text{the coefficient of } u^j \text{ in } a^{(i)}(u) \text{ is } \quad e^{(i)}_{m_i-j}(a_{1,i}, \ldots, a_{m_i,i}),$$

the $(m_i - j)$th elementary symmetric function evaluated at the values $a_{1,i}, \ldots, a_{m_i,i}$. Define

$$M^{\text{fin}}(a(u)) = L(\lambda) \otimes_{RG_\lambda} m_{a(u)},$$

where

$$e^{(i)}_k(x_{1,i}, x_{2,i}, \ldots)m_{a(u)} = e^{(i)}_k(a_{1,i}, \ldots, a_{m_i,i})m_{a(u)}$$

specifies the RG_λ-action on $m_{a(u)}$. In other words, the module $M^{\text{fin}}(a(u))$ is $L(\lambda)$ except that variables $x_{j,i}$ have been specialised to the values $a_{j,i}$.

4.4 Finite dimensional simple modules

Let $\mathbf{U}'$ be the subalgebra of $\mathbf{U}$ without the generator D.

Theorem 4.2. *The standard module*

$$M^{\text{fin}}(a(u)) \quad \textit{has a unique simple quotient} \quad L^{\text{fin}}(a(u))$$

and

$$\begin{array}{rcc} \{\textit{Drinfeld polynomials}\} & \longrightarrow & \{\textit{finite dimensional simple } \mathbf{U}'\textit{-modules}\} \\ a(u) = a^{(1)}(u)\omega_1 + \cdots + a^{(n)}(u)\omega_n & \longmapsto & L^{\text{fin}}(a(u)) \end{array}$$

is a bijection.

4.5 Crystals for level 0 $L(\lambda)$ and $M^{\text{fin}}(a(u))$

Let

$$\lambda = m_1\omega_1 + \cdots + m_n\omega_n, \qquad \text{with } m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}.$$

Let $k = \#\{i \in \{1, \ldots, n\} \mid m_i \neq 0\}$ and

$$S^\lambda = \{\vec{\kappa} = (\kappa^{(1)}, \ldots, \kappa^{(n)}) \mid \kappa^{(i)} \text{ is a partition with } \ell(\kappa^{(i)}) < m_i \text{ for } i \in \{1, \ldots, n\}\}.$$

Given λ there are uniquely determined

$$w \in W^{\text{ad}} \text{ and } j \in \mathbb{Z}_{\geq 0} \text{ and } \nu \in A_1 \qquad \text{such that} \qquad w(\nu + \Lambda_0) = -j\delta + \lambda + \Lambda_0.$$

Then the crystal of $L(\lambda)$ is the set

$$B(\lambda) = B(\nu + \Lambda_0)^+_w \times \mathbb{Z}^k \times S^\lambda.$$

and the crystal of $M^{\text{fin}}(a(u))$ is the set

$$B^{\text{fin}}(\lambda) = B(\nu + \Lambda_0)^+_w.$$

4.6 Character formulas

Let
$$0_q = \frac{1}{1-q} + \frac{q^{-1}}{1-q^{-1}} = \cdots + q^{-3} + q^{-2} + q^{-1} + 1 + q + q^2 + \cdots ,$$
(although $\frac{q^{-1}}{1-q^{-1}} = \frac{1}{q-1} = \frac{-1}{1-q}$, it is important to note that 0_q is *not* equal to 0, it is a doubly infinite formal series in q and q^{-1}).

Conceptually, the set $\mathbb{Z}^k \times S^\lambda$ is the crystal of RG_λ. Letting $q = e^{-\delta}$, its character is

$$\mathrm{char}(RG_\lambda) = \Big(0_{q^{m_1}} \prod_{k=1}^{m_1-1} \frac{1}{1-q^k}\Big)\Big(0_{q^{m_2}} \prod_{k=1}^{m_2-1} \frac{1}{1-q^k}\Big) \cdots \Big(0_{q^{m_n}} \prod_{k=1}^{m_n-1} \frac{1}{1-q^k}\Big).$$

The character of the crystal $B(\nu+\Lambda_0)^+_w$ is determined by the Demazure character formulas. A pleasant way to express this character is as the evaluation of an electronic Macdonald polynomial,

$$\mathrm{char}(B(\nu+\Lambda_0)^+_w) = E_{w_0\lambda}(q, 0).$$

Putting $\mathrm{char}(RG_\lambda)$ and $\mathrm{char}(B(\nu+\Lambda_0)^+_w)$ together gives

$$\mathrm{char}(B(\lambda)) = \mathrm{char}(B(\nu+\Lambda_0)^+_w)\,\mathrm{char}(RG_\lambda).$$

5 Lecture 5: R-matrices

5.1 Braiding

A *quasi-triangular Hopf algebra* is a Hopf algebra U with an invertible element

$$\mathcal{R} = \sum_{\mathcal{R}} R_1 \otimes R_2 \qquad \text{in a completion of } U \otimes U$$

which satisfies

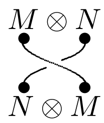
$$\text{if } a \in U \quad \text{then} \qquad \Delta^{\mathrm{op}}(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1} \tag{Rbraid}$$

and the cabling relations

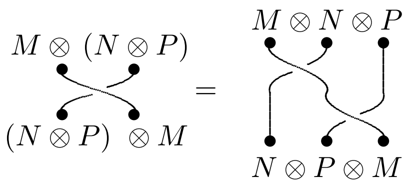
$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad \text{and} \qquad (\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}. \tag{Rcabling}$$

This is a structure that makes the category of U-modules into a braided monoidal category as follows. The relation (Rbraid) implies that for U-modules M and N, the map

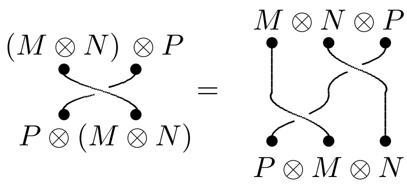
$$\begin{array}{rccc} \check{R}_{MN}\colon & M \otimes N & \longrightarrow & N \otimes M \\ & m \otimes n & \longmapsto & \sum_{\mathcal{R}} R_2 n \otimes R_1 m \end{array}$$



is a U-module isomorphism. The cabling relations (Rcabling) give that if M, N and P are U-modules then For U modules M, N, P

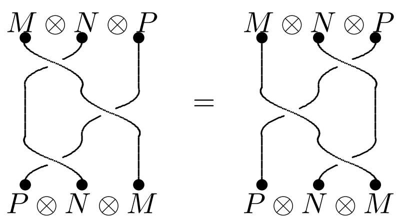


$$\check{R}_{M,N\otimes P} = (\check{R}_{MN} \otimes \mathrm{id}_P)(\mathrm{id}_N \otimes \check{R}_{MP})$$



$$\check{R}_{M\otimes N,P} = (\mathrm{id}_M \otimes \check{R}_{NP})(\check{R}_{MP} \otimes \mathrm{id}_N),$$

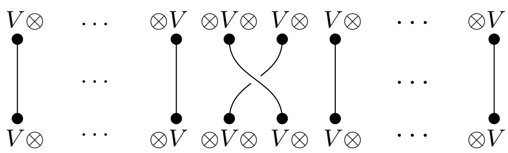
and, together, these imply the braid relation



$$(\check{R}_{MN} \otimes \mathrm{id}_P)(\mathrm{id}_N \otimes \check{R}_{MP})(\check{R}_{NP} \otimes \mathrm{id}_M) = (\mathrm{id}_M \otimes \check{R}_{NP})(\check{R}_{MP} \otimes \mathrm{id}_N)(\mathrm{id}_P \otimes \check{R}_{MN}).$$

5.2 Centralizer algebras

Let V be a U-module. Let $k \in \mathbb{Z}_{>0}$. Define

$T_i =$  , for $i \in \{1, \ldots, k-1\}$. Then T_i is an element of

$$\mathcal{Z}_k = \mathrm{End}_U(V^{\otimes k}) \qquad \text{and} \qquad \begin{array}{rl} T_iT_j = T_jT_i, & \text{if } j \notin \{i+1, i-1\}, \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, & \text{for } i \in \{1, \ldots, k-1\}. \end{array}$$

(1) If $U = U_t(L\mathfrak{sl}_n)$ and $V = L(\varepsilon_1)$ with $\dim(V) = n$ then

$$(T_i - t)(T_i + t^{-1}) = 0, \qquad \text{for } i \in \{1, \ldots, k-1\}$$

and $\mathcal{Z}_k$ is a (quotient of a) Iwahori-Hecke algebra (a t-deformation of the group algebra of the symmetric group $\mathbb{C}S_k$).
(2) If $U = U_t(L\mathfrak{so}_{2r+1})$ and $V = L(\varepsilon_1)$ with $\dim(V) = 2r+1$ then

$$(T_i - q^{-(2r+1)})(T_i - t)(T_i + t^{-1}) = 0, \qquad \text{for } i \in \{1, \ldots, k-1\}$$

and $\mathcal{Z}_k$ is a (quotient of a) BMW algebra (a t-deformation of the group algebra of the Brauer algebra).

5.3 Spectral parameters

Assume that there are automrophisms

$$\tau_\lambda \colon U \to U \qquad \text{with} \qquad \tau_\lambda \tau_\mu = \tau_{\lambda \oplus \mu}.$$

Remark 5.1. The notation $\lambda \oplus \mu$ is formal group law notation.

- For Yangians, $\lambda \oplus \mu = \lambda + \mu$;
- For quantum affine algebras, $\lambda \oplus \mu = \lambda\mu$;
- For elliptic quantum groups, $\oplus$ comes from the group law on an elliptic curve;
- For cobordism quantum groups, $\oplus$ is the universal formal group law.

The *R-matrix with spectral parameter* is

$$\mathcal{R}(\lambda \ominus \mu) = (\tau_\lambda \otimes \tau_\mu)(\mathcal{R}).$$

The spectral quantum Yang-Baxter equation (QYBE).

$$\mathcal{R}_{12}(\lambda_1 \ominus \lambda_2)\mathcal{R}_{13}(\lambda_1 \ominus \lambda_3)\mathcal{R}_{23}(\lambda_2 \ominus \lambda_3) = \mathcal{R}_{23}(\lambda_2 \ominus \lambda_3)\mathcal{R}_{13}(\lambda_1 \ominus \lambda_3)\mathcal{R}_{12}(\lambda_1 \ominus \lambda_2).$$

The spectral unitarity condition.

$$\mathcal{R}_{12}(\lambda_1 \ominus \lambda_2)\mathcal{R}_{21}(\lambda_2 \ominus \lambda_1) = 1 \otimes 1.$$

If M is a U-module, define a new U-module

$$M(\lambda) = M \qquad \text{with} \qquad a \cdot m = \tau_\lambda(a)m,$$

for $a \in U$. Then

$$\begin{array}{rccc} \check{R}_{MN}(\lambda \ominus \mu) \colon & M(\lambda) \otimes N(\mu) & \to & N(\mu) \otimes M(\mu) \\ & m \otimes n & \mapsto & \displaystyle\sum_{\mathcal{R}} \tau_\mu(R_2)n \otimes \tau_\lambda(R_1)m \end{array}$$

is a U-module morphism.

5.4 Quantum affine algebras

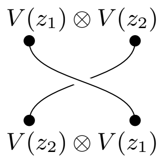
Let $\mathbf{U}'$ be the quantum affine algebra (without D).
The coproduct $\Delta\colon \mathbf{U}' \to \mathbf{U}' \otimes \mathbf{U}'$ is given by

$$\Delta(E_i) = E_i \otimes 1 + K_i^{-1} \otimes E_i, \qquad \Delta(F_i) = F_i \otimes K_i + 1 \otimes F_i, \qquad \Delta(K_i) = K_i \otimes K_i,$$

for $i \in \{0, 1, \ldots, n\}$.
There are automorphisms $\tau_z\colon \mathbf{U}' \to \mathbf{U}'$ for $z \in \mathbb{C}^\times$ given by

$$\begin{matrix} \tau_z(E_0) = zE_0, \\ \tau_z(F_0) = z^{-1}F_0, \end{matrix} \qquad \text{and} \qquad \begin{matrix} \tau_z(E_i) = E_i, \\ \tau_z(F_i) = F_i, \end{matrix} \quad \text{for } i \in \{1, \ldots, n\}.$$

Let V be a finite dimensional simple $\mathbf{U}'$-module.

Compute $\qquad \check{R}_{VV}(z_1, z_2) =$ 

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For $\mathbf{U}'$ of classical type and $V = L(\varepsilon_1)$, Jimbo 1986 has formulas in the basis

$$\{v_i \otimes v_j \mid i, j \in \{1, \ldots, N\}\ \}, \qquad \text{where } \{v_1, \ldots, v_N\} \text{ is a (weight) basis of } V.$$

5.5 To consider

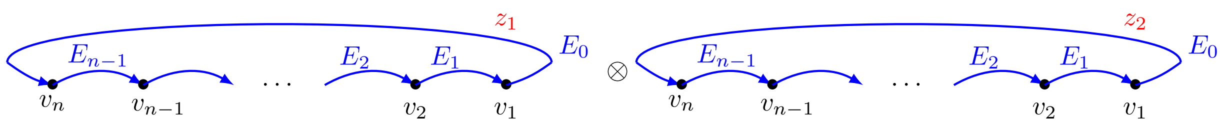
(1) For generic $z_1, z_2 \in \mathbb{C}^\times$, the $\mathbf{U}'$-modules $V(z_1) \otimes V(z_2)$ and $V_(z_2) \otimes V(z_1)$ are irreducible. So

$$\check{R}_{VV}(z_1, z_2) \quad \text{is unique up to multiplication by a constant.}$$

(2) The quantum group $\mathring{\mathbf{U}}$ of the finite diemsnional Lie algebra is a subalgebra of $\mathbf{U}'$ and so

$$\check{R}_{VV} \text{ is an element of } \mathcal{Z}_2 = \mathrm{End}_{\mathring{\mathbf{U}}}(V^{\otimes 2}).$$

For example, if $\mathbf{U}'$ is type $A_{n-1}^{(1)}$ and $V = L(\varepsilon_1)$ and, pictorially, $V(z_1) \otimes V(z_2)$ is

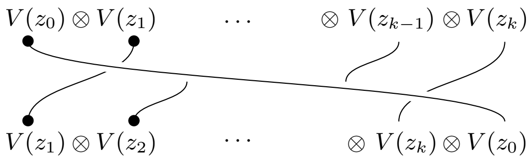


Then $\check{R}_{VV}(z_1, z_2) \in \mathcal{Z}_2$ is an element of the Iwahori-Hecke algebra. A computation gives

$$\check{R}_{VV}(z_1, z_2) = T_1 - (t - t^{-1}) \frac{1}{1 - z_2 z_1^{-1}}.$$

(It happens that, in this case, this formula for $\check{R}_{VV}(z_1, z_2)$ coincides with a formula for the intertwiner in the representation theory of the double affine Hecke algebra and Macdonald polynomials).

(3) The morphisms



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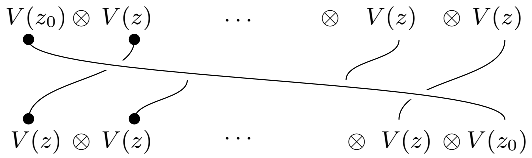
and



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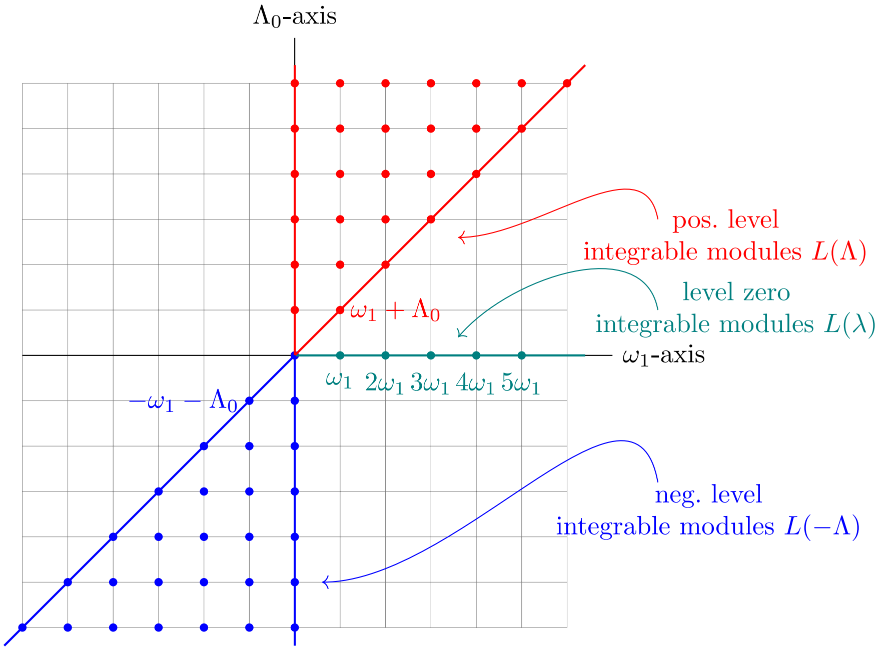
are monodromy matrices used to make transfer matrices.

Write these as elements of the Iwahori-Hecke algebra (in type A) and the BMW algebra (in other classical types).

6 Conclusion

Let us return to the picture of the points indexing integrable **U**-modules where the height of the dot is the level of the corresponding modules.



If M and N are **U**-modules with

$$M \text{ of level } k \text{ and } N \text{ of level } \ell \qquad \text{then} \qquad M \otimes N \text{ has level } k+\ell.$$

This indicates that

$$\text{if} \quad \mathcal{C} = (\text{category of level 0 modules with good conditions})$$

then $\mathcal{C}$ is a tensor category. Various choices for $\mathcal{C}$ (depending on which coniditions are in the "good conditions") are intricately fascinating. A good way to study $\mathcal{C}$ is by studying its actions on other categories. If

$$\mathcal{D} = (\text{category of level } k \text{ modules with some nice conditions})$$

then $\otimes$ will give an action of $\mathcal{C}$ on $\mathcal{D}$. There are many wonderful things to discover here.