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Semi-Abelian gauge theories, non-invertible symmetry, and string tensions beyond N-ality

Mendel Nguyen,[1] **Yuya Tanizaki,**[2] **Mithat Ünsal**[1]

[1]*Department of Physics, North Carolina State University, Raleigh, NC 27607, USA*

[2]*Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*

E-mail: mendelnguyen@gmail.com, yuya.tanizaki@yukawa.kyoto-u.ac.jp, unsal.mithat@gmail.com

Abstract: We study a 3d lattice gauge theory with gauge group $\mathrm{U}(1)^{N-1} \rtimes \mathrm{S}_N$, which is obtained by gauging the S_N global symmetry of a pure $\mathrm{U}(1)^{N-1}$ gauge theory, and we call it the semi-Abelian gauge theory. We compute mass gaps and string tensions for both theories using the monopole-gas description. We find that the effective potential receives equal contributions at leading order from monopoles associated with the entire $\mathrm{SU}(N)$ root system. Even though the center symmetry of the semi-Abelian gauge theory is given by $\mathbb{Z}_N$, we observe that the string tensions do not obey the N-ality rule and carry more detailed information on the representations of the gauge group. We find that this refinement is due to the presence of non-invertible topological lines as a remnant of $\mathrm{U}(1)^{N-1}$ one-form symmetry in the original Abelian lattice theory. When adding charged particles corresponding to W-bosons, such non-invertible symmetries are explicitly broken so that the N-ality rule should emerge in the deep infrared regime.

Contents

1 Introduction

In all calculable, confining $\mathrm{SU}(N)$ gauge theories in continuum, such as the Polyakov model on $\mathbb{R}^3$ [1, 2], the Seiberg–Witten model on $\mathbb{R}^4$ [3], and deformed Yang–Mills and adjoint QCD on $\mathbb{R}^3\times S^1$ [4, 5], the gauge dynamics Abelianize to $\mathrm{U}(1)^{N-1}$ at long distances. While these models have taught us much about confinement, they have several features that we do not expect of the dynamics of *non-Abelian* confinement. One particularly salient feature common to all of these models is the complete Higgsing of the S_N subgroup of the $\mathrm{SU}(N)$ gauge group. This Higgsing of S_N pervades the physics of these theories: it always gives rise to multiple masses for the dual photons and generically to multiple fundamental string tensions [6–8] (see Ref. [9] for a case in which fundamental string tensions remain equal).

The characterization of string tensions is an especially important point of difference between Abelianizing and non-Abelianizing confining gauge theories. Indeed, it is well-known that at asymptotically large distances, the string tensions of confining gauge theories that do not undergo Abelianization should be solely characterized by center symmetry. This is not the case in the Abelianizing theories mentioned above, at least within the low-energy effective field theory, where string tensions are dictated by charges under $\mathrm{U}(1)^{N-1}$ rather than N-ality [6–8].

Rather curiously, however, it has been observed in numerical experiments that the dynamics of non-Abelian confinement admit an intermediate distance scale [10–17] where the string tensions are not solely characterized by center symmetry either: they carry more detailed information on the representations of the gauge group. This naturally suggests that we should try to construct an intermediate theory between Abelian and non-Abelian worlds. As mentioned above, an understanding of unbroken S_N should be an important clue in this direction.

Thus, the purpose of this work is to construct a 3-dimensional lattice model in which these considerations can be addressed quite explicitly. We call it the *semi-Abelian gauge theory*. To define it, we begin with a pure Abelian lattice model (henceforth to be referred to as the 'Abelian model') with gauge group $\mathrm{U}(1)^{N-1}$ such that the permutation group S_N is present as a global symmetry. The semi-Abelian theory is then obtained by gauging this S_N, and the gauge group becomes

$$G_{\text{gauge}} = \mathrm{U}(1)^{N-1} \rtimes \mathrm{S}_N. \tag{1.1}$$

As pure gauge theories are *not* usually equipped with non-Abelian global symmetries, the global or local S_N symmetry of these models has some rather interesting consequences.

We first show that in both models, the mass gap is generated by Polyakov's mechanism whereby the proliferation of lattice monopole-instantons results in Debye screening. Crucially, the unbroken (or un-Higgsed) S_N symmetry implies that the effective potential receives equal contributions from the monopoles associated with the entire $\mathrm{SU}(N)$ root system, and that the $N-1$ dual photons have exactly the same mass. (As the gauge groups of both theories can be realized as subgroups of $\mathrm{SU}(N)$, we find it useful to use the language of $\mathrm{SU}(N)$ representations.) This feature sharply contrasts with the mass generation in the Polyakov model on $\mathbb{R}^3$ or in deformed Yang–Mills on $\mathbb{R}^3 \times S^1$, where the effective potential is sourced at leading order only by the monopoles associated with the (affine) simple roots and the $N-1$ dual photon masses are not degenerate.

After studying these properties of local operators, we move on to study properties of test electric particles, which can be described by the behavior of Wilson loops. In 3 spacetime dimensions, the Coulomb potential is already log-confining, but due to the mass gap generated by the monopole-instantons, the interparticle potential becomes linear-confining. Using the dual formulation of the Wilson loop, we give a semi-classical formula for the string tensions within a reasonable ansatz. We then find that there are infinitely many string tensions. In particular, the semi-Abelian theory furnishes a unique fundamental string tension.

This, however, raises a puzzle about the string tensions. In order to study the spectral properties of the confining forces for test quarks, we would like to have some symmetry that acts nontrivially on the Wilson loops. One is the well-known center symmetry, which has been recently axiomatized in the framework of higher-form symmetry [18, 19]. Before gauging S_N, our model has a $\mathrm{U}(1)^{N-1}$ 1-form symmetry, which provides sufficiently strong selection rules to support infinitely many string tensions. But after gauging S_N, the center of the gauge group becomes tiny,

$$Z(G_{\text{gauge}}) = \mathbb{Z}_N, \tag{1.2}$$

so the 1-form symmetry group becomes $\mathbb{Z}_N$, as it is in $\mathrm{SU}(N)$ Yang–Mills. And as we know from Yang–Mills, the $\mathbb{Z}_N$ 1-form symmetry can only explain the N-ality behavior of the string tensions at asymptotically large distances. However, the list of string tensions for the semi-Abelian gauge theory turns out to be unchanged by the gauging of S_N. Thus, as in the case of $\mathrm{SU}(N)$ Yang–Mills at intermediate distances, the string tensions of the semi-Abelian theory cannot be dictated by N-ality alone.

We find a resolution to this puzzle in a not-so-obvious but important symmetry of the semi-Abelian theory, a *non-invertible symmetry*. Indeed, after the generalization of symmetry to higher-form symmetry, it has been recognized that the most essential feature of the conservation law is the existence of topological defects, at least in the context of relativistic quantum field theories (QFTs). In other words, as long as one keeps intact the existence of topological defect operators, one may make up a new kind of symmetry by replacing or weakening other features of generalized global symmetry in Ref. [18] (e.g. higher-group symmetry [19–26]). Non-invertible symmetries are also generated by topological defects, but their fusion rules do not conform to group multiplication. As the name suggests, a non-invertible symmetry transformation need not have an inverse, which of course never occurs if the transformations form a group. The notion of non-invertible symmetry is still in its infancy, and it seems that its mathematical formulation has been so far established only in 2-dimensional spacetimes. Nevertheless, the utility of such topological operators in probing quantum systems has been elucidated in several recent studies, as the notion of symmetry itself tends to be broadened [27–35]. In that context, the new symmetry goes by various names, such as non-invertible symmetry, categorical symmetry, etc. Here, we would like to emphasize that the non-invertible symmetry clarifies an important feature of our 3-dimensional semi-Abelian gauge theory. Thanks to the simplicity of the model, the symmetry considerations we propose can be checked against concrete calculation.

We construct a generator of continuous non-invertible symmetry, and compute its commutation relations with several Wilson loops. By looking at its eigenvalues, we show that we can distinguish different string tensions even if they correspond to representations of the same N-ality. We also discuss conditions where such extra selection rules by noninvertible symmetry are lost by the addition of dynamical electric particles, and we compare them with the standard string-breaking arguments to check that they are consistent. Finally, as an application, we discuss an example where the non-invertible symmetry is explicitly broken to a discrete sub-symmetry, so that even though the number of string tensions becomes finite, there still remain some string tensions beyond N-ality.

2 3d $\mathrm{U}(1)^{N-1}$ lattice gauge theory with S_N global symmetry

There are two basic models that we study in this paper.

- $\mathrm{U}(1)^{N-1}$ theory with global non-Abelian discrete symmetry S_N
- $\mathrm{U}(1)^{N-1} \rtimes \mathrm{S}_N$ semi-Abelian gauge theory

The second one can be obtained by gauging the S_N 0-form symmetry of the first theory, and the main purpose of this paper is to understand its properties. To this end, we must first understand the properties of the Abelian theory, and this is the goal of this section.

2.1 Description of the $\mathrm{U}(1)^{N-1}$ lattice gauge theory

The $\mathrm{U}(1)^{N-1}$ lattice gauge model with the S_N global symmetry can be realized either by a standard Wilson-type formulation [36] or by a Villain-type formulation [37]. Since they provide somewhat complementary perspectives, we end up working with both.

2.1.1 Villain formulation

To give the Villain formulation, we consider a link field $\boldsymbol{A}_\ell$ valued in $\mathbb{R}^{N-1}$ and a plaquette field $\boldsymbol{n}_p$ valued in the root lattice $\Gamma_{\mathrm{r}} \subset \mathbb{R}^{N-1}$ of $\mathrm{SU}(N)$. We take for the action

$$S = \frac{1}{4\pi e^2} \sum_p (\boldsymbol{F}_p + 2\pi \boldsymbol{n}_p)^2 \tag{2.1}$$

where $\boldsymbol{F}_p = (\mathrm{d}\boldsymbol{A})_p$ is the field-strength and e is the gauge coupling. The partition function is given by

$$Z = \sum_{\{\boldsymbol{n}_p \in \Gamma_{\mathrm{r}}\}} \int_{\mathbb{R}^{N-1}} [\mathrm{d}\boldsymbol{A}_\ell]\, \mathrm{e}^{-S} \,. \tag{2.2}$$

This theory is invariant under the 0-form gauge symmetry

$$\boldsymbol{A}_\ell \to \boldsymbol{A}_\ell + (\mathrm{d}\boldsymbol{\lambda})_\ell \,, \qquad \boldsymbol{\lambda}_s \in \mathbb{R}^{N-1}, \tag{2.3}$$

and the 1-form gauge symmetry

$$\boldsymbol{A}_\ell \to \boldsymbol{A}_\ell + 2\pi \boldsymbol{\beta}_\ell \,, \qquad \boldsymbol{n}_p \to \boldsymbol{n}_p - (\mathrm{d}\boldsymbol{\beta})_p \,, \qquad \boldsymbol{\beta}_\ell \in \Gamma_{\mathrm{r}} \,. \tag{2.4}$$

In view of the fact that $\mathbb{R}^{N-1}/(2\pi\Gamma_{\mathrm{r}}) \simeq \mathrm{U}(1)^{N-1}$, we see that this indeed defines a $\mathrm{U}(1)^{N-1}$ gauge model.

One way to understand this Villain-type formulation [1, 38] is to imagine that we had begun with pure $\mathbb{R}^{N-1}$ gauge theory,

$$Z = \int_{\mathbb{R}^{N-1}} [\mathrm{d}\boldsymbol{A}_\ell] \exp\left(-\frac{1}{4\pi e^2} \sum_p \boldsymbol{F}_p^2 \right), \tag{2.5}$$

and then considered gauging the discrete subgroup $2\pi\Gamma_{\mathrm{r}}$ of the $\mathbb{R}^{N-1}$ 1-form center symmetry group, which acts according to

$$\boldsymbol{A}_\ell \to \boldsymbol{A}_\ell + \boldsymbol{\theta}_\ell \,, \qquad \boldsymbol{\theta}_\ell \in \mathbb{R}^{N-1} \,, \qquad (\mathrm{d}\boldsymbol{\theta})_p = 0 \,. \tag{2.6}$$

The simplest way to do that is to introduce the discrete Γ_{r}-valued plaquette field $\boldsymbol{n}_p$, and then demand that the local transformations (2.4) be gauge redundancies. Minimal coupling to the field $\boldsymbol{n}_p$ would then produce the action (2.1) and the partition function (2.2).

Global symmetries: Let us now discuss the global symmetries of this model. First, as already noted above, there is a $\mathrm{U}(1)^{N-1}$ 1-form center symmetry (2.6), where the group is $\mathrm{U}(1)^{N-1}$ rather than $\mathbb{R}^{N-1}$ thanks to the 1-form gauge structure (2.4).

Importantly, the theory has a discrete non-Abelian 0-form global symmetry

$$\boldsymbol{A}_\ell \to \Pi \boldsymbol{A}_\ell \,, \qquad \boldsymbol{n}_p \to \Pi \boldsymbol{n}_p, \tag{2.7}$$

under $\mathrm{O}(N-1)$ transformations Π that preserve the root lattice Γ_r. Such transformations constitute the automorphism group of the $\mathrm{SU}(N)$ root system, and therefore the symmetry group here is

$$G^{[0]}_{\text{global}} = \begin{cases} \mathrm{S}_N \rtimes \mathbb{Z}_2, & (N>2), \\ S_2 \simeq \mathbb{Z}_2, & (N=2). \end{cases} \tag{2.8}$$

The S_N corresponds to the Weyl group of $\mathrm{SU}(N)$, which is generated by the reflections in the hyperplanes orthogonal to the roots

$$\boldsymbol{A}_\ell \to \boldsymbol{A}_\ell - \boldsymbol{\alpha}(\boldsymbol{\alpha}\cdot\boldsymbol{A}_\ell)\,, \qquad \boldsymbol{n}_p \to \boldsymbol{n}_p - \boldsymbol{\alpha}(\boldsymbol{\alpha}\cdot\boldsymbol{n}_p)\,, \qquad \boldsymbol{\alpha}\in\Phi, \tag{2.9}$$

where Φ is the set of roots for $\mathrm{SU}(N)$. The pair $(\boldsymbol{A}_\ell, \boldsymbol{n}_p)$ thus transforms in the standard representation D_{std} of S_N, which is the $(N-1)$-dimensional irreducible representation. Meanwhile, the $\mathbb{Z}_2$ is simply generated by the reflection

$$\boldsymbol{A}_\ell \to -\boldsymbol{A}_\ell \,, \qquad \boldsymbol{n}_p \to -\boldsymbol{n}_p, \tag{2.10}$$

which we may think of as charge conjugation. We note that, for $N=2$, these two operations are identical.

Note that the existence of the non-Abelian global symmetry (2.8) is somewhat unusual for a pure gauge theory. In general, pure gauge theories without matter fields, either Abelian or non-Abelian, do not possess *non-Abelian* global symmetries. In the $\mathrm{U}(1)^{N-1}$ gauge theory we are considering, this symmetry is present. The gauging of the permutation group S_N will generate a genuinely non-Abelian gauge theory, which we shall investigate.

The basic observables we are concerned with are the Wilson loops, which are here given by

$$W_{\boldsymbol{w}}(C) = \exp\left(\mathrm{i}\int_C \boldsymbol{w}\cdot\boldsymbol{A}\right), \tag{2.11}$$

with $\boldsymbol{w}$ in the weight lattice Γ_w of $\mathrm{SU}(N)$. Note that it is invariance under the 1-form gauge transformations (2.4) that requires the electric charge to be a weight. The Wilson lines transform under the 0-form discrete symmetry (2.7) as

$$W_{\boldsymbol{w}}(C) \to W_{\Pi^{-1}\boldsymbol{w}}(C), \tag{2.12}$$

and under the 1-form center symmetry (2.6) as

$$W_{\boldsymbol{w}}(C) \to W_{\boldsymbol{w}}(C)\exp\left(\mathrm{i}\int_C \boldsymbol{w}\cdot\boldsymbol{\theta}\right). \tag{2.13}$$

2.1.2 Wilson formulation

To construct the $\mathrm{U}(1)^{N-1}$ lattice gauge theory in the Wilson formulation, we consider N gauge fields $a_\ell^1, ..., a_\ell^N$ and a Lagrange multiplier v_ℓ which is an integer-valued link-field. The dynamics is determined by the action

$$S_{\mathrm{W}} = \beta \sum_p \sum_{i=1}^{N} (1 - \cos f_p^i) - \mathrm{i} \sum_\ell \sum_{i=1}^{N} v_\ell a_\ell^i, \tag{2.14}$$

where the $f_p^i = (\mathrm{d}a^i)_p$ are the field-strengths. In the partition function, we integrate over $a_\ell^i \in [0, 2\pi]$ and sum over $v_\ell \in \mathbb{Z}$:

$$Z = \sum_{\{v_\ell \in \mathbb{Z}\}} \int_0^{2\pi} [\mathrm{d}a_\ell^i] \mathrm{e}^{-S_{\mathrm{W}}}. \tag{2.15}$$

In particular, summation over v_ℓ in the partition function produces the constraint

$$\sum_{i=1}^{N} a_\ell^i = 0 \mod 2\pi, \tag{2.16}$$

so that only $N-1$ of the photons are physical.[1]

One nice thing about this formulation is that the S_N symmetry is manifest; it acts simply by permuting the N photons:

$$(a_\ell^1, ..., a_\ell^N) \to (a_\ell^{P(1)}, ..., a_\ell^{P(N)}), \qquad P \in \mathrm{S}_N. \tag{2.18}$$

For now, we shall prefer to work with the Villain form over the Wilson one, because the former enjoys exact dualities which allow us to analyze the dynamics most simply. Nevertheless, the two formulations are equivalent at weak coupling, as we demonstrate in App. B. Later on, in Sec. 3 where we gauge the S_N global symmetry, we will find the Wilson form more convenient.

2.2 Mass gap and spectrum

In this subsection, we discuss the mass gap of the lattice Abelian gauge theory with the S_N symmetry.

First, as we shall review in Sec. 2.2.1, we note that the Villain form is *exactly* dual to a multi-component Coulomb gas; that is, the partition function can rewritten in the form:

$$Z = \sum_{\{\boldsymbol{q}(\tilde{x}) \in \Gamma_{\mathrm{r}}\}} \exp\left(-\frac{\pi}{e^2} \sum_{\tilde{x}, \tilde{x}'} v(\tilde{x} - \tilde{x}') \boldsymbol{q}(\tilde{x}) \cdot \boldsymbol{q}(\tilde{x}') \right), \tag{2.19}$$

[1] We could integrate out v_ℓ and any one of the photon fields. Then after some simple field redefinitions, we would obtain the action

$$S_{\mathrm{W}} = \beta \sum_p \sum_{i=1}^{N} (1 - \cos(\boldsymbol{\nu}_i \cdot \boldsymbol{f}_p)), \tag{2.17}$$

where the $\boldsymbol{\nu}_i$ are the weights of the defining representation of $\mathrm{SU}(N)$ and $\boldsymbol{f}_p$ is the field-strength of an $(N-1)$-component Abelian gauge field $\boldsymbol{a}_\ell$.

where $v(\tilde{x})$ is the lattice Coulomb potential, and $\boldsymbol{q}(\tilde{x})$ is a Γ_{r}-valued field on the dual lattice. Here, one can interpret a configuration $\{\boldsymbol{q}(\tilde{x})\}$ as a configuration of magnetic monopoles, where the charge of the monopole at $\tilde{x}$ is $\boldsymbol{q}(\tilde{x})$. As is familiar, the proliferation of monopoles in the Euclidean description of the vacuum results in Debye screening, and hence, the correlation length remains finite for any nonzero value of the coupling [1]. While this is more or less self-evident, we can go further and obtain the long-distance effective field theory:

$$Z = \int \mathcal{D}\boldsymbol{\sigma} \exp\left(-\frac{e^2}{2\pi}\int \mathrm{d}^3x \left\{\frac{1}{2}(\nabla\boldsymbol{\sigma})^2 + M^2 \sum_{\boldsymbol{\alpha}\in\Phi^+}\left(1-\cos(\boldsymbol{\alpha}\cdot\boldsymbol{\sigma})\right)\right\}\right), \tag{2.20}$$

where the dual photon field $\boldsymbol{\sigma}$ is a $2\pi\Gamma_{\mathrm{w}}$-periodic scalar, Φ^+ is a set of positive roots for $SU(N)$, and $M^2 \propto \mathrm{e}^{-\mathrm{const.}/e^2}/e^2$. This effective description shows very clearly the presence of a nonzero mass gap. It will be derived in Sec. 2.2.2.

We can immediately observe that the $N-1$ dual photons must have exactly the same mass. The degeneracy is a consequence of the S_N global symmetry inherited from the microscopic theory. To see this, note that the dual photons $\boldsymbol{\sigma}$ transform in the standard representation D_{std} of S_N:

$$\boldsymbol{\sigma} \to D_{\mathrm{std}}(P)\boldsymbol{\sigma}\,, \qquad P\in \mathrm{S}_N. \tag{2.21}$$

The mass matrix for the dual photons,

$$(M^2_{\boldsymbol{\sigma}})^{ij} = M^2 \sum_{\boldsymbol{\alpha}\in\Phi^+} \alpha^i\alpha^j, \tag{2.22}$$

is also invariant under the S_N transformation,

$$D_{\mathrm{std}}(P)M^2_{\boldsymbol{\sigma}}D^{-1}_{\mathrm{std}}(P) = M^2_{\boldsymbol{\sigma}}, \qquad P\in \mathrm{S}_N. \tag{2.23}$$

Since D_{std} is irreducible, it follows from Schur's lemma that $M^2_{\boldsymbol{\sigma}}$ must be proportional to the identity matrix. The mass gap is thus the $(N-1)$-fold degenerate eigenvalue of $M^2_{\boldsymbol{\sigma}}$. By taking the trace of $M^2_{\boldsymbol{\sigma}}$ and using $\boldsymbol{\alpha}^2=2$, one easily finds the mass gap to be

$$M_{\mathrm{gap}} = \sqrt{N}M \propto \frac{\sqrt{N}}{e}\mathrm{e}^{-\mathrm{const.}/e^2}. \tag{2.24}$$

2.2.1 Multi-component Coulomb gas representation of the Villain form

Here we show that the Villain form (2.2) of our theory is exactly dual to multi-component Coulomb gas (2.19), using standard techniques in Abelian lattice gauge theory [2, 39–41]. We derive the equivalence very briefly here, but the detailed derivation for the single-component $U(1)$ gauge theory is reviewed in App. A.2.

We first note that the Poisson summation formula can be generalized on the weight and root lattices to give

$$\sum_{\boldsymbol{n}_p\in\Gamma_{\mathrm{r}}} \exp\left(-\frac{1}{4\pi e^2}(\boldsymbol{F}_p+2\pi\boldsymbol{n}_p)^2\right) = \sum_{\boldsymbol{k}_p\in\Gamma_{\mathrm{w}}} \exp\left(-\pi e^2\boldsymbol{k}_p^2 + \mathrm{i}\boldsymbol{k}_p\cdot\boldsymbol{F}_p\right) \tag{2.25}$$

up to an overall coefficient. By performing the $\boldsymbol{A}_\ell$ integration exactly, we obtain the constraint $(\mathrm{d}^\dagger \boldsymbol{k})_\ell = 0$, which can be easily solved by setting

$$* \boldsymbol{k} = \mathrm{d}\boldsymbol{m}, \tag{2.26}$$

where $\boldsymbol{m}_{\tilde{x}}$ is a Γ_{w}-valued scalar field on the dual lattice.[2] After this replacement, the partition function becomes

$$Z = \sum_{\{\boldsymbol{m}_{\tilde{x}} \in \Gamma_{\mathrm{w}}\}} \exp\left(-\pi e^2 \sum_{\tilde{\ell}} (\mathrm{d}\boldsymbol{m})^2_{\tilde{\ell}} \right). \text{[3]} \tag{2.27}$$

We now wish to replace $\boldsymbol{m}(\tilde{x})$ by a continuous field; it can be done by using the Poisson summation formula once more in the form

$$\sum_{\boldsymbol{m}(\tilde{x}) \in \Gamma_{\mathrm{w}}} \delta(\boldsymbol{\sigma}(\tilde{x}) - 2\pi \boldsymbol{m}(\tilde{x})) = \sum_{\boldsymbol{q}(\tilde{x}) \in \Gamma_{\mathrm{r}}} \exp(\mathrm{i}\, \boldsymbol{q}(\tilde{x}) \cdot \boldsymbol{\sigma}(\tilde{x})), \tag{2.28}$$

and the result is

$$Z = \int [\mathrm{d}\boldsymbol{\sigma}(\tilde{x})] \sum_{\{\boldsymbol{q}(\tilde{x}) \in \Gamma_{\mathrm{r}}\}} \exp\left(-\frac{e^2}{4\pi} \sum_{\tilde{x}} (\partial^-_\mu \boldsymbol{\sigma}(\tilde{x}))^2 + \mathrm{i} \sum_{\tilde{x}} \boldsymbol{q}(\tilde{x}) \cdot \boldsymbol{\sigma}(\tilde{x}) \right). \tag{2.29}$$

After performing the Gaussian integration over $\boldsymbol{\sigma}$, we arrive at the multi-component Coulomb gas representation (2.19).

2.2.2 Long-distance effective theory

We now wish to pass to the long-distance effective description (2.20) [2, 39–41]. To this end, we first split the Green function $v(\tilde{x}) = \Delta^{-1}$ in (2.19) into two parts by adding and subtracting $(\Delta + M_{\mathrm{PV}}^2)^{-1}$:

$$\Delta^{-1} = \underbrace{\Delta^{-1} - (\Delta + M_{\mathrm{PV}}^2)^{-1}}_{\equiv u_{M_{\mathrm{PV}}}} + \underbrace{(\Delta + M_{\mathrm{PV}}^2)^{-1}}_{\equiv w_{M_{\mathrm{PV}}}}. \tag{2.30}$$

where $u_{M_{\mathrm{PV}}}(\tilde{x})$ is the Green function of the Pauli–Villars regulated Laplacian $\Delta_{M_{\mathrm{PV}}} \equiv \Delta(1 + \Delta/M_{\mathrm{PV}}^2)$, and $w_{M_{\mathrm{PV}}}(\tilde{x})$ is the Yukawa Green function. Since $w_{M_{\mathrm{PV}}}(\tilde{x} - \tilde{x}')$ decays exponentially fast, we can take $w_{M_{\mathrm{PV}}}(\tilde{x}) = w_{M_{\mathrm{PV}}}(0)\delta_{\tilde{x},0}$. Furthermore, it is straightforward to show that $w_{M_{\mathrm{PV}}}(0) = v(0) - \mathcal{O}(1/M_{\mathrm{PV}}) \approx 0.253 - \mathcal{O}(1/M_{\mathrm{PV}})$ [42].

We can now rewrite the partition function as

$$Z = \sum_{\{\boldsymbol{q}(\tilde{x}) \in \Gamma_{\mathrm{r}}\}} \exp\left(-\frac{\pi}{e^2} \sum_{\tilde{x}, \tilde{x}'} u_{M_{\mathrm{PV}}}(\tilde{x} - \tilde{x}') \boldsymbol{q}(\tilde{x}) \cdot \boldsymbol{q}(\tilde{x}') - \frac{1}{2} I \sum_{\tilde{x}} \boldsymbol{q}(\tilde{x})^2 \right), \tag{2.31}$$

[2]For clarity, we ignore the effect of nontrivial spacetime topology; see App. A.2.

[3]This representation may be thought of as a "Γ_{w}-ferromagnet" by analogy with the corresponding expression with $\mathbb{Z}$ in place of Γ_{w}. The Γ_{w}-ferromagnet representation is exactly dual to the Coulomb gas representation (2.19). While the latter converges rapidly at weak coupling $e^2 \to 0$, the former converges rapidly at strong coupling $e^2 \to \infty$.

where $I \equiv 2\pi v(0)/e^2$. To go further, we introduce a Gaussian integral over the dual photon field $\boldsymbol{\sigma}(\tilde{x})$:

$$Z = \int [\mathrm{d}\boldsymbol{\sigma}(\tilde{x})] \mathrm{e}^{-\frac{e^2}{4\pi}\sum_{\tilde{x}} \boldsymbol{\sigma}(\tilde{x})\Delta_{M_{\mathrm{PV}}}\boldsymbol{\sigma}(\tilde{x})} \sum_{\{\boldsymbol{q}(\tilde{x})\in\Gamma_{\mathrm{r}}\}} \mathrm{e}^{\mathrm{i}\sum_{\tilde{x}}\boldsymbol{q}(\tilde{x})\boldsymbol{\sigma}(\tilde{x})} \mathrm{e}^{-\frac{1}{2}I\sum_{\tilde{x}}\boldsymbol{q}(\tilde{x})^2}. \tag{2.32}$$

At this point, we want to perform a cluster expansion of the partition function. For weak coupling, I is large, and so e^{-I} is exponentially small. Thus, at leading order in semi-classics, one can restrict the summation over $\boldsymbol{q}(\tilde{x})$ to $\{0\}\cup\Phi = \{0, \pm\boldsymbol{\beta}_1, ..., \pm\boldsymbol{\beta}_K\}$, where the $\boldsymbol{\beta}_k$ are the positive roots of $\mathrm{SU}(N)$ and $K \equiv N(N-1)/2$. Since $\boldsymbol{\beta}_k^2 = 2$ is independent of k, all the monopoles whose charges are roots have the same minimal action I, and we therefore have $N(N-1)$ degenerate saddles at leading order in semi-classics. Performing the summation over $\boldsymbol{q}(\tilde{x})$ with this restriction then yields

$$\prod_{\tilde{x}} \sum_{\boldsymbol{q}(\tilde{x})\in\{0\}\cup\Phi} \mathrm{e}^{\mathrm{i}\boldsymbol{q}(\tilde{x})\boldsymbol{\sigma}(\tilde{x})} \mathrm{e}^{-\frac{1}{2}I\boldsymbol{q}(\tilde{x})^2} \approx \exp\left(2\mathrm{e}^{-I}\sum_{\tilde{x}}\sum_{k=1}^{K}\cos(\boldsymbol{\beta}_k\cdot\boldsymbol{\sigma}(\tilde{x}))\right). \tag{2.33}$$

where we introduced terms involving coincident operators in order to sum the expansion in closed form. Inserting this into (2.32), we get

$$Z = \int [\mathrm{d}\boldsymbol{\sigma}(\tilde{x})] \exp\left(-\frac{e^2}{4\pi}\sum_{\tilde{x}}\boldsymbol{\sigma}(\tilde{x})\Delta_{M_{\mathrm{PV}}}\boldsymbol{\sigma}(\tilde{x}) + 2\mathrm{e}^{-I}\sum_{\tilde{x}}\sum_{\boldsymbol{\alpha}\in\Phi^+}\cos(\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}(\tilde{x}))\right), \tag{2.34}$$

which, upon taking the continuum limit, coincides with (2.20). We note that, in this derivation, we have neglected the effect of the spacetime topology, and thus the periodicity of the dual photon $\sigma(\tilde{x})$ is undetermined. Had we taken it into account, we would have identified $\boldsymbol{\sigma}$ as a $2\pi\Gamma_{\mathrm{w}}$-periodic scalar. We elaborate on this subtlety in Appendix A.2.

The fact that the sum over monopoles goes over all positive roots $\boldsymbol{\alpha}\in\Phi^+$ and that all monopoles associated with these roots have the same action distinguishes our $\mathrm{U}(1)^{N-1}$ lattice gauge theory with S_N symmetry from Yang–Mills adjoint Higgs systems which exhibit dynamical Abelianization $\mathrm{SU}(N)\to\mathrm{U}(1)^{N-1}$.[4] In the latter, if the adjoint Higgs are algebra-valued, as in Polyakov model [1], the sum over monopoles at leading order in semi-classics is restricted to the $N-1$ simple roots $\boldsymbol{\alpha}\in\Delta$. If the adjoint Higgs are group-valued, such as in deformed Yang–Mills, then the N roots in the affine root system contribute at leading order in semi-classics [4]. There are monopoles associated with non-simple roots as well, but these are higher action and do not contribute at leading order. These monopoles split into $\mathbb{Z}_N$-orbits with hierarchical fugacities $\mathrm{e}^{-S_0} \gg \mathrm{e}^{-2S_0} \gg \cdots \gg \mathrm{e}^{-(N-1)S_0}$. In our construction, S_N permutation symmetry guarantees that all N^2-N monopoles associated with the roots have the same action. In theories like the Polyakov model and Seiberg–Witten theory, S_N is part of the gauge structure of the microscopic theory, but it

[4] It is also worth nothing that in $\mathcal{N}=4$ $\mathrm{SU}(N)$ super Yang–Mills theory softly broken down to $\mathcal{N}=1^*$ on $\mathbb{R}^3\times S^1$ as well, it is necessary to sum over monopoles associated with non-simple roots in order to capture the ground state properties correctly [43]. This data is encoded in an elliptic superpotential, but the S_N symmetry is still Higgsed in generic vacua.

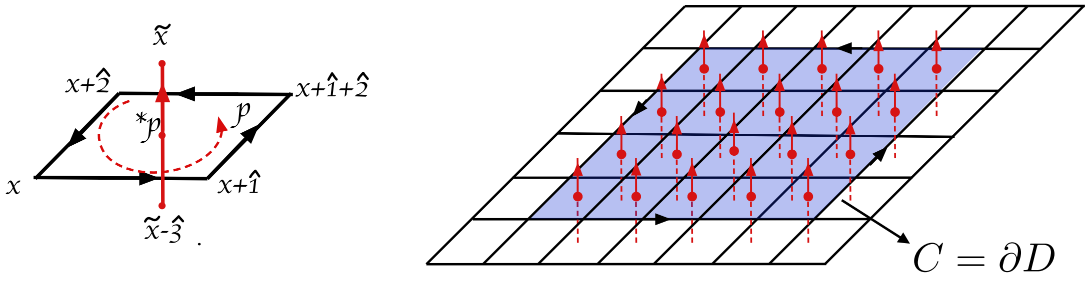

Figure 1. Left: Dual of the plaquette p is a link $*p$ on the dual lattice intersecting p as shown. Right: D is the shaded region bounded by the curve C. The Poincare dual $[D]$ is a bump 1-form function on the dual lattice that is 1 on the red links $*p$ on the dual lattice, and zero everywhere else.

is spontaneously broken by the vacuum expectation value of the Higgs field which imposes an ordering on the eigenvalues of the adjoint Higgs. These models therefore exhibit $\mathcal{O}(N)$ different types of fundamental string tensions. We will see how the string tensions behave in our $\mathrm{U}(1)^{N-1}$ Abelian model in the following subsection.

2.3 Wilson loops and string tensions

In this subsection, we show that the Abelian gauge model confines and we approximately determine the string tensions.

We begin by showing how Wilson loops are computed in the long-distance effective theory [2]. We consider the Wilson loop with the electric charge $\boldsymbol{w} \in \Gamma_{\mathrm{w}}$,

$$W_{\boldsymbol{w}}(C) = \exp\left(\mathrm{i}\int_C \boldsymbol{w}\cdot\boldsymbol{A}\right). \tag{2.35}$$

For our purposes, it will suffice to take C to be a contractible loop, so that it is the boundary of a 2-dimensional surface D. Then we can write

$$W_{\boldsymbol{w}}(C) = \exp\left(\mathrm{i}\int_D \boldsymbol{w}\cdot\boldsymbol{F}\right) = \exp\left(\mathrm{i}\sum_p [D]_{*p}(\boldsymbol{w}\cdot\boldsymbol{F}_p)\right). \tag{2.36}$$

Here we have introduced the Poincaré dual $[D]$ of D; it is a bump 1-form on the dual lattice (see Fig. 1) given by

$$[D]_{*p} = \begin{cases} 1, & \text{if } p\subset D, \\ 0, & \text{otherwise}. \end{cases} \tag{2.37}$$

Let us repeat the derivation of the dual theory, now with the insertion of a single Wilson loop. Using the Poisson summation (2.25), the path-integral weight becomes

$$\exp\left(-\pi e^2 \boldsymbol{k}_p^2 + \mathrm{i}\boldsymbol{k}_p\cdot\boldsymbol{F}_p + \mathrm{i}[D]_{*p}(\boldsymbol{w}\cdot\boldsymbol{F}_p)\right), \tag{2.38}$$

where the last term comes from the Wilson loop. The integration over $\boldsymbol{A}$ produces the constraint, $\mathrm{d}^\dagger(\boldsymbol{k} + *(\boldsymbol{w}[D])) = 0$, which can be solved by

$$*\,\boldsymbol{k} = -\boldsymbol{w}[D] + \mathrm{d}\boldsymbol{m}, \tag{2.39}$$

instead of (2.26). At this point, the rest of the derivation proceeds exactly as before, and we obtain

$$\langle W_{\boldsymbol{w}}(C)\rangle = \int \mathcal{D}\boldsymbol{\sigma} \exp\left(-\frac{e^2}{2\pi}\int \mathrm{d}^3x \left\{\frac{1}{2}\left|\mathrm{d}\boldsymbol{\sigma} - 2\pi\boldsymbol{w}[D]\right|^2 + M^2 \sum_{\boldsymbol{\alpha}\in\Phi^+}\left(1-\cos(\boldsymbol{\alpha}\cdot\boldsymbol{\sigma})\right)\right\}\right). \tag{2.40}$$

This expression shows that the Wilson loop is realized as a defect operator in the dual formulation. That is, it is evaluated by removing the loop C from the spacetime and restricting the path integral to configurations satisfying $\oint_{S^1}\mathrm{d}\boldsymbol{\sigma} = 2\pi\boldsymbol{w}$ for small loops S^1 that link with the loop C. Taking the ratio with the unconstrained path integral, we obtain the expectation value of the Wilson loop.

We are now in a position to approximately determine the string tensions. It will suffice to compute the functional integral in (2.40) in the classical approximation. For convenience, let us take the loop C as well as D to lie in the $z \equiv x_3 = 0$ plane. If we take the loop C to be so large that D essentially fills the $z = 0$ plane, then the action density localizes around $z = 0$. As a result, the area law decay for the Wilson loop $W_{\boldsymbol{w}}(C)$ is observed,

$$\langle W_{\boldsymbol{w}}(C)\rangle \sim \exp\Big(-T_{\boldsymbol{w}}\,\mathrm{Area}(D)\Big), \tag{2.41}$$

and its string tension is given by the minimal action density:

$$T_{\boldsymbol{w}} = \min_{\boldsymbol{\sigma}(z)} \frac{e^2}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}z \left\{\frac{1}{2}\left(\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}z}\right)^2 + M^2\sum_{\boldsymbol{\alpha}\in\Phi^+}\Big(1-\cos(\boldsymbol{\alpha}\cdot\boldsymbol{\sigma})\Big)\right\}, \tag{2.42}$$

with the boundary condition

$$\boldsymbol{\sigma}(-\infty) = 0, \qquad \boldsymbol{\sigma}(+\infty) = 2\pi\boldsymbol{w}. \tag{2.43}$$

To proceed, we take as a plausible ansatz

$$\boldsymbol{\sigma}(x) = \boldsymbol{w}\sigma(z), \tag{2.44}$$

and then we can evaluate the string tension analytically within this ansatz. Substituting (2.44) into (2.42), we obtain

$$\begin{aligned} T_{\boldsymbol{w}} &= \min_{\sigma(z)} \frac{e^2}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}z \left\{\frac{\boldsymbol{w}^2}{2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 + M^2\sum_{\boldsymbol{\alpha}\in\Phi^+}\Big(1-\cos(\boldsymbol{\alpha}\cdot\boldsymbol{w}\sigma)\Big)\right\} \\ &= \frac{e^2 M}{\pi}\int_0^{2\pi}\mathrm{d}\sigma\sqrt{\frac{\boldsymbol{w}^2}{2}\sum_{\boldsymbol{\alpha}\in\Phi^+}\Big(1-\cos(\boldsymbol{\alpha}\cdot\boldsymbol{w}\sigma)\Big)}, \end{aligned} \tag{2.45}$$

which is the Bogomol'nyi–Prasad–Sommerfield (BPS) bound [44, 45]. Although this is just an upper bound for the actual string tension, we assume that it gives a reasonable estimate.

2.3.1 Explicit evaluation of string tensions

Using the formula (2.45), let us evaluate string tensions explicitly for several cases. Here, we take $\boldsymbol{w} = \boldsymbol{\mu}_1$, $2\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$, which correspond to the highest weights of the fundamental, symmetric, and anti-symmetric representations of $\mathrm{SU}(N)$, respectively. We will also comment on the case $\boldsymbol{w} = \boldsymbol{\alpha} \in \Gamma_{\mathrm{r}}$, corresponding to the adjoint representation of $\mathrm{SU}(N)$.

Let us start with $\boldsymbol{w} = \boldsymbol{\mu_1}$, which is the highest weight of the fundamental representation of $\mathrm{SU}(N)$. We obtain

$$\begin{aligned} T_{\boldsymbol{\mu}_1} &= \frac{e^2 M}{\pi}\frac{N-1}{\sqrt{2N}}\int_0^{2\pi} \mathrm{d}\sigma \sqrt{1-\cos(\sigma)} \\ &= \frac{4e^2 M}{\pi}\frac{N-1}{\sqrt{N}}. \end{aligned} \tag{2.46}$$

We will use this quantity as the unit for other string tensions.

We next consider $\boldsymbol{w} = 2\boldsymbol{\mu}_1$, the highest weight of the $\mathrm{SU}(N)$ two-index symmetric representation. Since σ wraps S^1 twice, we find that

$$T_{2\boldsymbol{\mu}_1} = 2T_{\boldsymbol{\mu}_1}. \tag{2.47}$$

Thus, the symmetric string tension is twice the fundamental one, which suggests that the symmetric string can be interpreted as the sum of two independent fundamental strings. The multi-string ansatz is also a candidate, which may give a reasonable approximation of confining strings [46], so we will compare it with (2.44) for other strings, too.

We consider the two-index anti-symmetric string, $\boldsymbol{w} = \boldsymbol{\mu}_2$. We then find

$$\begin{aligned} T_{\boldsymbol{\mu}_2} &= \frac{e^2 M}{\pi}\frac{\sqrt{2}(N-2)}{\sqrt{N}}\int_0^{2\pi} \mathrm{d}\sigma \sqrt{1-\cos(\sigma)} \\ &= \frac{8e^2 M}{\pi}\frac{N-2}{\sqrt{N}} \\ &= \frac{2(N-2)}{N-1}T_{\boldsymbol{\mu_1}}. \end{aligned} \tag{2.48}$$

For $N=3$, $\boldsymbol{\mu}_2$ gives the conjugate representation of $\boldsymbol{\mu}_1$, and in this case we indeed find that $T_{\boldsymbol{\mu}_2} = T_{\boldsymbol{\mu}_1}$. For $N > 3$, we find $T_{\boldsymbol{\mu}_1} < T_{\boldsymbol{\mu}_2} < T_{2\boldsymbol{\mu}_1} = 2T_{\boldsymbol{\mu_1}}$, and $T_{\boldsymbol{\mu_2}} \approx 2T_{\boldsymbol{\mu}_1}$ for $N \gg 1$. Since $\boldsymbol{\mu}_2 = \boldsymbol{\mu}_1 + (\boldsymbol{\mu}_1 - \boldsymbol{\alpha}_1)$, we can understand this upper bound $2T_{\boldsymbol{\mu}_1}$ as a sum of two-independent fundamental strings $2T_{\boldsymbol{\mu}_1} = T_{\boldsymbol{\mu}_1} + T_{\boldsymbol{\mu}_1 - \boldsymbol{\alpha}_1}$. Our calculation shows that, for the anti-symmetric string, the ansatz (2.44) gives a more severe upper bound for $T_{\boldsymbol{\mu}_2}$.

Lastly, let us consider the adjoint string $\boldsymbol{w} = \boldsymbol{\alpha} \in \Gamma_{\mathrm{r}}$. Applying the formula (2.45) within the ansatz (2.44), we obtain it as

$$\begin{aligned} T_{\boldsymbol{\alpha}} &= \frac{e^2 M}{\pi}\int_0^{2\pi} \mathrm{d}\sigma \sqrt{(1-\cos(2\sigma)) + 2(N-2)(1-\cos(\sigma))} \\ &= \frac{4e^2 M}{\pi}\left(\sqrt{N} + \frac{(N-2)}{\sqrt{2}}\cosh^{-1}\sqrt{\frac{N}{N-2}}\right) \\ &= \frac{1}{N-1}\left(N + \frac{(N-2)\sqrt{N}}{\sqrt{2}}\cosh^{-1}\sqrt{\frac{N}{N-2}}\right)T_{\boldsymbol{\mu_1}}. \end{aligned} \tag{2.49}$$

According to this formula, the adjoint string tension satisfies that $T_{\boldsymbol{\alpha}} \geq 2T_{\boldsymbol{\mu}_1}$, which turns out to be only slightly larger than $2T_{\boldsymbol{\mu}_1}$. In the case of the adjoint string, however, it turns out that we can do a little bit better. Let us explicitly take $\boldsymbol{w} = \boldsymbol{\alpha}_1$, and we can consider a double-string ansatz, in which the adjoint string consists of two fundamental strings, $\boldsymbol{\mu}_1$ and $\boldsymbol{\alpha}_1 - \boldsymbol{\mu}_1$. Up to permutation, $\boldsymbol{\mu}_1$ and $\boldsymbol{\alpha}_1 - \boldsymbol{\mu}_1$ are related by complex conjugation, and thus we find

$$T_{\boldsymbol{\alpha}} = T_{\boldsymbol{\mu}_1} + T_{\boldsymbol{\alpha}_1 - \boldsymbol{\mu}_1} = 2T_{\boldsymbol{\mu}_1}. \tag{2.50}$$

Therefore, for the adjoint string, the two-independent-string ansatz is slightly better than (2.44), unlike the case of anti-symmetric string.

3 Semi-Abelian theory

In this section, we consider the $\mathrm{U}(1) \rtimes \mathrm{S}_N$ gauge theory by gauging the S_N global symmetry of the $\mathrm{U}(1)^{N-1}$ gauge theory considered in Sec. 2. We call it the semi-Abelian gauge theory.

3.1 Gauging of the S_N global symmetry

In relativistic quantum field theories, global symmetry is generated by a set of codim-1 defects, which are topological and obey the group-multiplication law [18]. When the global symmetry is discrete, we can gauge it by summing up all possible networks of such codim-1 defects. This procedure may be obstructed by anomalies, which are characterized by a topological action in one higher dimension [47–49]. We also note that the gauging procedure admits the freedom to add a topological phase to each network configuration of the topological defects as long as it is consistent with locality and unitarity.

In this section, we gauge the S_N symmetry of the $\mathrm{U}(1)^{N-1}$ lattice gauge theory in Sec. 2. Assuming that its low-energy description enjoys emergent Lorentz symmetry due to the cubic lattice rotational invariance, the gauging procedure of S_N should fit into the above general discussion. Absence of the S_N anomaly is guaranteed by the explicit construction of the lattice gauge theory. As extra topological terms, there are Dijkgraaf-Witten (DW) terms [50] characterized by $H^3(B\mathrm{S}_N, \mathrm{U}(1))$, which are nontrivial for all $N \geq 2$.[5] In this paper, we limit ourselves to discuss the case without the 3d S_N DW term.

Let us now construct the semi-Abelian gauge theory on a cubic lattice, and the simplest way to proceed is to gauge S_N in the Wilson formulation (2.14). For concreteness, it is convenient to realize the semi-Abelian gauge symmetry by $N \times N$ matrices. We can realize

[5] Although detailed information is not relevant for us as we neglect the nontrivial DW twist, let us give its full information for completeness, which may be useful for possible extensions. By the universal coefficient theorem, we obtain $H^d(B\mathrm{S}_N, \mathrm{U}(1)) \simeq H^{d+1}(B\mathrm{S}_N, \mathbb{Z}) \simeq H_d(B\mathrm{S}_N, \mathbb{Z})$ because they only have the torsion part. We can find in literatures that the list of the 3d DW twist is given as

N	2	3	4	5	6	$\cdots$
$H^3(B\mathrm{S}_N, \mathrm{U}(1))$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$	$(\mathbb{Z}_2)^2 \oplus \mathbb{Z}_{12}$	$\cdots$

(3.1)

For $N \geq 6$, this group cohomology stabilizes and $H^3(B\mathrm{S}_N, \mathrm{U}(1)) \simeq (\mathbb{Z}_2)^2 \oplus \mathbb{Z}_{12}$, i.e. we can add three distinct DW terms, two of which give the (± 1) phases and the another one gives the phases $\exp\left(\frac{2\pi \mathrm{i}}{12} n\right)$, in the path integral of S_N gauge fields.

the element of $\mathrm{U}(1)^{N-1} \rtimes \mathrm{S}_N$ inside $\mathrm{SU}(N)$ as

$$P \cdot C \in \mathrm{SU}(N), \tag{3.2}$$

where $C = \mathrm{diag}(\mathrm{e}^{\mathrm{i}a_1}, \cdots, \mathrm{e}^{\mathrm{i}a_N})$ with $\det(C) = 1$ describes the Cartan components, and $P \in \mathrm{S}_N$ is the $N \times N$ matrix representation of a Weyl reflection, which is realized as a signed permutation matrix. The group multiplication law is given as

$$(P_1 \cdot C_1)(P_2 \cdot C_2) = \underbrace{(P_1 P_2)}_{\in \mathrm{S}_N} \cdot \underbrace{((P_2^{-1} C_1 P_2) C_2)}_{\in \mathrm{U}(1)^{N-1}}, \tag{3.3}$$

and this expression elucidates the semi-direct structure in a concrete manner. Letting $(P_\ell \cdot C_\ell) \in \mathrm{SU}(N)$ denote the link variable, the gauge-invariant plaquettes for the Lagrangian consist of the two terms:

$$\mathrm{Re}\left(\beta_1 \mathrm{tr}\left[\mathbf{1}_N - \prod_{\ell \subset \partial p} (P_\ell \cdot C_\ell) \right] + \beta_2 \mathrm{tr}\left[\mathbf{1}_N - \prod_{\ell \subset \partial p} P_\ell \right] \right). \tag{3.4}$$

The 2nd term is the gauge-invariant kinetic term only for S_N. By sending $\beta_2 \to +\infty$, we can impose the flatness condition on the S_N gauge field,

$$\prod_{\ell \subset \partial p} P_\ell = \mathbf{1}_N \in \mathrm{S}_N. \tag{3.5}$$

As it is this limit that fits into the general discussion given above for the continuum description, let us then work with this flatness condition (3.5) on the S_N link variables. We can readily check that the Lagrangian (3.4) is equal to the Wilson action (2.14) when the S_N gauge fields are trivial, i.e. $P_\ell = 1$ for all the links ℓ, by putting $\beta_1 = \beta$. Therefore, we obtain the $\mathrm{U}(1)^{N-1} \rtimes \mathrm{S}_N$ gauge theory out of the $\mathrm{U}(1)^{N-1}$ pure gauge theory.

Because of the flatness condition (3.5), the local dynamics should not be much affected by the gauging of S_N, and all the interesting things in the deep infrared are about the global aspects of the theory. To be more precise, let us assume that we prepare a sufficiently large torus T^3 for the spacetime and that we are interested in computing the correlation functions inside an open ball $B^3 \subset T^3$, which has a trivial topology. Using the flatness condition, we may perform a local S_N gauge transformation so that the S_N gauge fields P_ℓ are fixed to 1 inside B^3. Hence, correlation functions should be identical with those of the $\mathrm{U}(1)^{N-1}$ theory in Sec. 2.1 as it has a nonzero mass gap, as long as we neglect exponentially small corrections that vanish in the thermodynamic limit. In this sense, the gauging of S_N is locally trivial.

More physically, by sending the parameter $\beta_2 \to \infty$ in the Lagrangian, the magnetic monopoles for the S_N gauge group are energetically forbidden. As a result, the S_N gauge fields are deconfined; i.e. the Wilson loops of S_N gauge fields obey the perimeter law at any length scale. Now, assume that we wish to compute the correlation functions of local operators. For the sake of exposition, consider a two-point function of the $\mathrm{U}(1)^{N-1}$ gauge theory

$$\langle O_1(x_1) O_2(x_2) \rangle_{\mathrm{U}(1)^{N-1}}, \tag{3.6}$$

though it is straightforward to extend the discussion to general n-point functions. As the S_N global symmetry is not spontaneously broken in the $\mathrm{U}(1)^{N-1}$ theory, this correlation function has a non-zero expectation value in the thermodynamic limit only if $O_1(x_1)O_2(x_2)$ contains an S_N-singlet component. Thus, we may assume that $O_1(x_1)O_2(x_2)$ is S_N singlet without loss of generality. Here, we note that the operator O_1, O_2 can be S_N non-singlet, but they have to be mutually conjugate representations. By introducing an S_N Wilson line $W^{(\mathrm{S}_N)}(x_1, x_2)$ connecting x_1 and x_2, we can construct the S_N gauge invariant operator $O_1(x_1)W^{(\mathrm{S}_N)}(x_1, x_2)O_2(x_2)$. Since the S_N gauge field is deconfined, we find that $\left\langle O_1(x_1)W^{(\mathrm{S}_N)}(x_1, x_2)O_2(x_2)\right\rangle_{\mathrm{U}(1)^{N-1}\rtimes \mathrm{S}_N}$ is independent of the continuous deformation of the path connecting x_1 and x_2. In particular, by choosing a trivial path which does not go around any nontrivial cycles of the spacetime, we obtain

$$\left\langle O_1(x_1)W^{(\mathrm{S}_N)}(x_1, x_2)O_2(x_2)\right\rangle_{\mathrm{U}(1)^{N-1}\rtimes \mathrm{S}_N} = \langle O_1(x_1)O_2(x_2)\rangle_{\mathrm{U}(1)^{N-1}} \tag{3.7}$$

in the thermodynamic limit. Any local correlation function of the $\mathrm{U}(1)^{N-1}$ theory can be recovered in the semi-Abelian gauge theory.

3.2 $\mathbb{Z}_N$ center symmetry

In this and the following subsections, we discuss properties of the (electric) Wilson loops in order to identify the string tensions from the viewpoint of the symmetry. We pay especial attention to the 1-form symmetry, or center symmetry, of the $\mathrm{U}(1)^{N-1} \rtimes \mathrm{S}_N$ gauge theory in this subsection.

The 1-form symmetry is generated by codim-2 topological defects, whose fusion rule obeys group multiplication [18]. In general, when we consider a pure gauge theory with a gauge group G_{gauge}, the theory enjoys a 1-form symmetry with the symmetry group $Z(G_{\text{gauge}})$, which is the center of G_{gauge}. Since this acts as $Z(G_{\text{gauge}})$ phase rotations of Wilson loops, this has been historically called the center symmetry.

In our case, the gauge group is $G_{\text{gauge}} = \mathrm{U}(1)^{N-1} \rtimes \mathrm{S}_N$, and thus

$$Z(G_{\text{gauge}}) = \mathbb{Z}_N. \tag{3.8}$$

To see this, it is convenient to use the embedding of $\mathrm{U}(1)^{N-1} \rtimes \mathrm{S}_N \subset \mathrm{SU}(N)$ used above, and to consider the defining representation of the latter. Using Schur's lemma, one sees that the $N \times N$ matrix representation of center elements must be proportional to the identity matrix. Such matrices are included only in $\mathrm{U}(1)^{N-1}$, which is the same as the Cartan factor of $\mathrm{SU}(N)$, and thus the center elements of $U(1)^{N-1} \rtimes \mathrm{S}_N$ are the same with those of $\mathrm{SU}(N)$. Before gauging S_N, the 1-form symmetry group is given by $Z(\mathrm{U}(1)^{N-1}) = \mathrm{U}(1)^{N-1}$ since the theory is an Abelian gauge theory without any electric matter fields. Therefore, in view of the 1-form symmetry, one might be led to claim that the semi-Abelian gauge theory should be more similar to $\mathrm{SU}(N)$ gauge theories than the $\mathrm{U}(1)^{N-1}$ theory.

This, however, raises the following puzzle about the string tensions. In the $\mathrm{U}(1)^{N-1}$ theory, there are infinitely many different string tensions depending on the representations of Wilson loops, which are characterized by charges of the $\mathrm{U}(1)^{N-1}$ 1-form symmetry. As we have seen in the previous subsection, the local dynamics is not affected by gauging S_N

because we can locally set the S_N gauge field to be zero by gauge transformations. As long as we measure string tensions using large and contractible Wilson loops, the same discussion from before should apply here, and thus, there have to be infinitely many different string tensions also for the $U(1)^{N-1} \rtimes S_N$ gauge theory. But this seems rather unnatural, because the $\mathbb{Z}_N$ 1-form symmetry is too weak to give selection rules for these string tensions. To put the question another way:

> *How is the presence of infinitely many different string tensions compatible with the finite center symmetry?*

To make things more concrete, let us construct the generator of $\mathbb{Z}_N^{[1]}$ out of $(U(1)^{[1]})^{N-1}$ generators of the $U(1)^{N-1}$ gauge theory. In terms of the dual photon field $\boldsymbol{\sigma} \in \mathbb{R}^{N-1}/2\pi\Gamma_w$ from the monopole gas description in Sec. 2.2.1, the generators of $(U(1)^{[1]})^{N-1}$ are given by

$$U_\theta^{(k)}(C) = \exp\left(\mathrm{i}\frac{\theta}{2\pi}\int_C \boldsymbol{\alpha}_k \cdot \mathrm{d}\boldsymbol{\sigma}\right), \qquad (k = 1, \dots, N-1), \tag{3.9}$$

where the transformation parameter θ is 2π periodic due to the $2\pi\Gamma_{\mathrm{w}}$-periodicity of $\boldsymbol{\sigma}$. After gauging S_N, these operators are no longer gauge invariant because the dual photon field $\boldsymbol{\sigma}$ transforms under the standard representation of the Weyl permutations S_N. Nevertheless, the generator of $\mathbb{Z}_N^{[1]}$ can be constructed as

$$U_n(C) = \prod_{k=1}^{N-1} U^{(k)}_{\frac{2\pi}{N}kn}(C) = \exp\left(\mathrm{i}\frac{n}{N}\int_C (\boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2 + \cdots + (N-1)\boldsymbol{\alpha}_{N-1}) \cdot \mathrm{d}\boldsymbol{\sigma}\right). \tag{3.10}$$

Thanks to the periodicity of $\boldsymbol{\sigma}$, $U_n(C)$ is invariant under S_N transformations and thus remains a good operator for the semi-Abelian gauge theory. Moreover, we have the group multiplication law

$$U_n(C)U_m(C) = U_{n+m \bmod N}(C) \tag{3.11}$$

so $U_1(C)$ only generates the $\mathbb{Z}_N$ subgroup of $U(1)^{N-1}$. Other generic combinations, $\prod_k U^{(k)}_{\theta_k}$, cannot satisfy the S_N invariance, and they drop out from the possible generators of the 1-form symmetry.

3.3 String tensions beyond N-ality, and noninvertible topological lines

Let us explicitly check whether or not the string tensions of the semi-Abelian gauge theory obey the standard N-ality rule. Using the embedding $U(1)^{N-1} \rtimes S_N \subset SU(N)$, we construct the Wilson loops with the $SU(N)$ gauge field first, and then we restrict it to the $U(1)^{N-1} \rtimes S_N$ gauge field. As we can locally eliminate the S_N gauge field, we may restrict the $SU(N)$ gauge field to its diagonal component in a naive way, as long as the Wilson loop is contractible. The N-ality of the obtained Wilson loop is the same as that of the Wilson loop with $SU(N)$ gauge fields.

Let us write

$$W_k(C) = \exp\left(\mathrm{i}\Big(\boldsymbol{\mu}_1 - \sum_{j=1}^{k-1}\boldsymbol{\alpha}_j\Big) \cdot \int_C \boldsymbol{A}\right), \quad (k = 1, \dots, N), \tag{3.12}$$

so that, for example, the fundamental Wilson loop is given by

$$W_{\rm fd}(C) = W_1(C) + W_2(C) + \cdots + W_N(C). \tag{3.13}$$

Each Wilson loop W_i in $W_{\rm fd}$ has the same string tension, and for large loops C it obeys the area law:

$$\left\langle W_{\rm fd}(C) \right\rangle \sim \exp(-T_{\boldsymbol{\mu}_1}\mathrm{Area}). \tag{3.14}$$

Under the 1-form symmetry,

$$\mathbb{Z}_N^{[1]} : W_{\rm fd} \to \mathrm{e}^{2\pi \mathrm{i}/N} W_{\rm fd}, \tag{3.15}$$

or, more precisely,

$$\langle U(C_1) W_{\rm fd}(C_2)\rangle = \exp\left(\frac{2\pi\mathrm{i}}{N}\mathrm{Link}(C_1, C_2)\right)\langle W_{\rm fd}(C_2)\rangle. \tag{3.16}$$

In order to determine whether string tensions are controlled by the 1-form symmetry, let us consider the adjoint Wilson loop,

$$W_{\rm ad}(C) = \sum_{i\neq j} W_i(C) W_j^*(C) = |W_{\rm fd}|^2 - N. \tag{3.17}$$

This has trivial N-ality, but we can readily check that its string tension is *not* zero using the result of Sec. 2.3:

$$\left\langle W_{\rm ad}(C) \right\rangle \sim \exp(-T_{\boldsymbol{\alpha}}\mathrm{Area}), \tag{3.18}$$

with $T_{\boldsymbol{\alpha}} \simeq 2T_{\boldsymbol{\mu}_1}$. This example clearly tells us that the string tensions of the semi-Abelian gauge theory carry detailed data of its gauge-group representations, which cannot be captured by the $\mathbb{Z}_N^{[1]}$ symmetry.

Something new is needed to explain the nonobservance of the N-ality rule, and this is where the non-invertible topological lines come in. We can easily construct such an operator by summing over all the S_N conjugates of $U_\theta^{(1)}(C)$:

$$\begin{aligned}\mathcal{T}_\theta(C) &\equiv \frac{1}{N!}\sum_{P\in \mathrm{S}_N} P U_\theta^{(1)}(C) P^{-1} \\ &= \frac{1}{N(N-1)}\sum_{\boldsymbol{\alpha}\in\Phi}\exp\left(\mathrm{i}\frac{\theta}{2\pi}\int_C \boldsymbol{\alpha}\cdot \mathrm{d}\boldsymbol{\sigma}\right).\end{aligned} \tag{3.19}$$

Since this operator is S_N singlet, it can be a physical operator of the S_N-gauged theory. Since each operator in the sum is topological, so too is $\mathcal{T}_\theta(C)$. Therefore, these S_N-invariant operators share important features of the 1-form symmetry generators. However, the group multiplication law is not satisfied for $\mathcal{T}_\theta(C)$, as one can easily check:

$$\mathcal{T}_\theta(C)\mathcal{T}_{\theta'}(C) \neq \mathcal{T}_{\theta+\theta'}(C). \tag{3.20}$$

Because of the violation of the group multiplication property, we cannot regard $\mathcal{T}_\theta(C)$ as a generator of an ordinary 1-form symmetry in contrast with (3.11). Instead, it is a generator of non-invertible symmetry.

Let us consider a component of the Wilson loop that corresponds to the weight $\boldsymbol{w} \in \Gamma_{\mathrm{w}}$. Its eigenvalue of $\mathcal{T}_\theta$ is given by

$$\frac{1}{N(N-1)} \sum_{\boldsymbol{\alpha}\in\Phi} \exp\left(\mathrm{i}\theta\boldsymbol{\alpha}\cdot\boldsymbol{w}\right). \tag{3.21}$$

As a consequence, the fundamental Wilson loop transforms as

$$W_{\mathrm{fd}} \mapsto \frac{1}{N}\big(N-2+2\cos(\theta)\big)W_{\mathrm{fd}}. \tag{3.22}$$

More importantly, the adjoint Wilson loop also transforms nontrivially as

$$W_{\mathrm{ad}} \mapsto \frac{(N-2)(N-3)+4(N-2)\cos(\theta)+2\cos(2\theta)}{N(N-1)}W_{\mathrm{ad}}. \tag{3.23}$$

This elucidates that we can detect the detailed information of the Wilson loop beyond N-ality by using non-invertible topological line operators $\mathcal{T}_\theta$.

As another example, we can detect the difference between the symmetric and anti-symmetric two-index representations, W_{sym} and W_{asym}, of $\mathrm{SU}(N)$, whose highest weights are given by $2\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$, respectively. We have to note, however, that W_{sym} is not an eigen-operator of $\mathcal{T}_\theta$, because the two-index symmetric representation of $\mathrm{SU}(N)$ decomposes into two irreducible representations of $\mathrm{U}(1)^{N-1}\rtimes \mathrm{S}_N$. Since $\boldsymbol{\mu}_2 = 2\boldsymbol{\mu}_1 - \boldsymbol{\alpha}_1$ and $(2\boldsymbol{\mu}_1)\cdot\boldsymbol{\alpha}_1 = 2$, we can identify that the list of charges in the anti-symmetric representation is included inside that of the symmetric one exactly once, and thus the correct eigen-operator is $W_{\mathrm{sym}} - W_{\mathrm{asym}}$. Indeed, one can check that

$$\Big\langle (W_{\mathrm{sym}} - W_{\mathrm{asym}})(C)\Big\rangle \sim \exp(-2T_{\boldsymbol{\mu}_1}\mathrm{Area}), \qquad \Big\langle W_{\mathrm{asym}}(C))\Big\rangle \sim \exp(-T_{\boldsymbol{\mu}_2}\mathrm{Area}), \tag{3.24}$$

with $T_{\boldsymbol{\mu}_2} < 2T_{\boldsymbol{\mu}_1}$, as we have discussed in Sec. 2.3. We find

$$(W_{\mathrm{sym}} - W_{\mathrm{asym}}) \mapsto \frac{N-2+2\cos(2\theta)}{N}(W_{\mathrm{sym}} - W_{\mathrm{asym}}), \tag{3.25}$$

and

$$W_{\mathrm{asym}} \mapsto \frac{(N-2)(N-3)+2+4(N-2)\cos(\theta)}{N(N-1)}W_{\mathrm{asym}}. \tag{3.26}$$

This gives another explicit demonstration of the fact that one can distinguish different string tensions for representations of the same N-ality by computing commutation relations with the topological operators $\mathcal{T}_\theta$.

3.4 Effect of dynamical electric particles

In the previous section, we discussed the behavior of string tensions for the pure semi-Abelian gauge theory. String tensions do not obey the N-ality rule, and the presence of non-invertible topological lines explain why they carry more detailed information. In this section, we discuss what will happen for string tensions once dynamical electric charges are added.

Once electric charges are incorporated as dynamical excitations, their pair creation can break confining strings if it is energetically favorable. If the fundamental electric charge is added, we expect that all the confining strings can be broken and all Wilson loops will obey the perimeter law. If the adjoint charge is added instead, we expect that the string tensions should obey the N-ality rule, because the adjoint Wilson loop would then obey the perimeter law. Can we justify these expectations from the viewpoint of topological lines?

For this purpose, we need to identify which line operators cease to be topological once the dynamical electric charges are included. If a line has a nontrivial commutation relation with the Wilson loop corresponding to the dynamical excitations, then it is no longer topological after introducing dynamical charges [33]. This is because the corresponding Wilson loop can end on charged local operators, so that the linking number is no longer well-defined; i.e. the topological invariance of the symmetry operator is lost.

Let us add dynamical adjoint particles, and then let us determine whether or not $\mathcal{T}_\theta$ is topological. Since the eigenvalue of W_{adj} has to be 1 for the topological operators, $\mathcal{T}_\theta$ is topological only if

$$\frac{(N-2)(N-3)+4(N-2)\cos(\theta)+2\cos(2\theta)}{N(N-1)}=1. \tag{3.27}$$

This is solved only by $\theta = 0 \bmod 2\pi$, and thus only the trivial one $\mathcal{T}_{\theta=0}=1$ is topological. This implies that the non-invertible symmetry ceases to be an exact symmetry once adjoint matter field is added. On the other hand, the $\mathbb{Z}_N$ 1-form symmetry is kept intact because the generator acts trivially on W_{ad}. In this case, the string tensions obey the N-ality rule at least if the Wilson loops are sufficiently large, which is consistent with the observation for the 3d $\mathrm{SU}(N)$ Yang-Mills theory.

As a nontrivial exercise, we can add dynamical particles corresponding to $n\Gamma_{\mathrm{r}}$ with $n>1$, instead of W_{ad}. Then, the non-invertible line $\mathcal{T}_\theta$ is topological if

$$\frac{(N-2)(N-3)+4(N-2)\cos(n\theta)+2\cos(2n\theta)}{N(N-1)}=1, \tag{3.28}$$

and it has nontrivial solutions, $\theta\in(2\pi/n)\mathbb{Z}$. Therefore, the continuous part of the non-invertible symmetry $\mathcal{T}_\theta$ is explicitly broken by dynamical electric charges $n\Gamma_{\mathrm{r}}$, but the discrete part $\mathcal{T}_{\theta=2\pi k/n}$, $k=1,\dots,n$ still generates a good non-invertible symmetry. As a result, Wilson lines distinguished by $\mathcal{T}_{\theta=2\pi k/n}$ can have different string tensions even if they share the same N-ality.

4 Summary and Discussions

In this paper, we have studied the properties of the semi-Abelian gauge theory in 3 spacetime dimensions, where the gauge group is $G_{\mathrm{gauge}}=\mathrm{U}(1)^{N-1}\rtimes \mathrm{S}_N$. As we have imposed the flatness condition on the S_N gauge field, we can locally eliminate it completely, so the spectral properties of the mass gap, string tensions can be calculated as the $\mathrm{U}(1)^{N-1}$ theory. We have seen that the mass gap is generated by the Polyakov mechanism as a consequence of the Debye screening with monopole-instantons. We can classify their magnetic charges using the $\mathrm{SU}(N)$ representation, and all the monopoles for the roots give equally dominant contributions to the effective potential. This point is very different from the Polyakov model or QCD(adj) with an S^1 compactification, where only the monopoles associated with the simple roots play the dominant role, and it comes from the S_N invariance of our model.

Using the dual formulation, we also computed various string tensions, and we found that there are infinitely many different string tensions. When the S_N symmetry is not gauged, this can be explained very naturally in the context of the 1-form symmetry, because the center of $\mathrm{U}(1)^{N-1}$ is $\mathrm{U}(1)^{N-1}$ itself, and thus the 1-form symmetry group is large enough to explain the selection rules between infinitely many confining strings.

A puzzle arises, however, after gauging S_N, because the center symmetry is just $Z(G_{\mathrm{gauge}}) = \mathbb{Z}_N$. This is because most of the elements of $\mathrm{U}(1)^{N-1}$ do not commute with the permutations, and the permutation invariance requires that the center elements be proportional to the identity matrix. Thus, the 1-form symmetry of semi-Abelian gauge theory is as small as that of $\mathrm{SU}(N)$ Yang–Mills theory, where the string tensions are characterized by N-ality alone. Therefore, for the semi-Abelian theory, there is a clear discrepancy between the actual behavior of the string tensions and the natural expectation from $\mathbb{Z}_N$ center symmetry.

We find that the discrepancy is resolved by recognizing the presence of noninvertible symmetry. We constructed the topological line operators $\mathcal{T}_\theta$ out of the $U(1)^{N-1}$ 1-form symmetry generators, which remain well-defined and topological after gauging S_N but do not satisfy the group multiplication law. Though these operators are noninvertible, their commutation relations with the Wilson lines are able to distinguish representations with the same N-ality. Thus, we have demonstrated the utility of an extended notion of symmetry in a 3d toy example of gauge theories.

We should mention that the formal development of non-invertible symmetry is still an important task. In the case of higher-form or higher-group symmetry, their formalization not only provided the rigorous definition and generalization of the center symmetry, but also gave new tools to analyze interacting QFTs, such as generalizations of anomaly matching [51–63]. It would be very nice if this repertoire of useful techniques could be enhanced to include non-invertible symmetry.

Lastly, let us describe some speculation. As we stated in the introduction, a similar behavior regarding the N-ality rule has been observed in simulations of $\mathrm{SU}(N)$ Yang–Mills on the lattice: there is an intermediate distance scale where the quark-antiquark potential exhibits linear confinement but its string tension depends on the details of the gauge-group representation. Though it is widely believed that the string tension becomes solely dictated by N-ality once the quark-antiquark separation becomes sufficiently large, it is logically possible that 'sufficiently large' is parametrically larger than the strong length scale Λ^{-1} at which confinement sets in. For instance, viewing N as a parameter, it may very well be that the N-ality rule sets in at a distance scale $h(N)\Lambda^{-1}$, where $h(N) \to \infty$ as $N \to \infty$. We think it would be an intriguing possibility if, even in pure Yang–Mills, some approximate notion of non-invertible symmetry could be used to explain the behavior of string tensions beyond N-ality at these intermediate distances.

A more striking example may be QCD with fundamental or two-index matter fields, where the 1-form $\mathbb{Z}_N$ center symmetry is either completely or partially lost, or Yang–Mills theories with simply-connected gauge groups without a center, such as G_2. Even in the case where center symmetry is completely lost, we believe that an approximate non-invertible symmetry may give a precise meaning to confinement of arbitrary test charges, and potentially provide the long sought definition of confinement in such theories.

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A Review of Abelian duality on the lattice

A.1 Differential forms on the lattice

There is, on the lattice, a close analog of the notion of differential forms, and it is especially convenient for treating Abelian lattice gauge theories. Here we give a somewhat informal introduction to this formalism, which we use throughout this article. See [38] for a more systematic discussion.

We begin with a bit of lattice geometry. Consider a d-dimensional cubic lattice Λ^d. Such a lattice contains "r-cells" $c^{(r)}$ for each $r = 0, 1, ..., d$. Thus, the 0-cells are the sites s, the 1-cells the links ℓ, the 2-cells the plaquettes p, the 3-cells the cubes c, and so on, and everything is oriented. For example, for a link $\ell = (x; \widehat{\mu})$, the oppositely oriented link is given by $-\ell \equiv (x + \widehat{\mu}; -\widehat{\mu})$, and these are to be viewed as distinct objects despite corresponding to the same (undirected) edge.[6] By convention, whenever we write a sum or product over r-cells, we do not double count r-cells which only differ by orientation.

The 'boundary operator' ∂ takes an r-cell into the (oriented) sum of the $(r-1)$-cells that constitute its boundary. For example, the boundary operator on a plaquette $p = (x; \widehat{\mu}, \widehat{\nu})$ yields

$$\partial(x; \widehat{\mu}, \widehat{\nu}) = (x; \widehat{\mu}) + (x + \widehat{\mu}; \widehat{\nu}) - (x + \widehat{\nu}; \widehat{\mu}) - (x; \widehat{\nu}). \tag{A.1}$$

Importantly, the boundary operator is nilpotent, $\partial^2 = 0$. By a slight abuse of notation, we write

$$c^{(r-1)} \subset \partial c^{(r)} \tag{A.2}$$

if the r-cell $c^{(r)}$ contains in its boundary the $(r-1)$-cell $c^{(r-1)}$. We thus have (tautologically)

$$\partial c^{(r)} = \sum_{c^{(r-1)} \subset \partial c^{(r)}} c^{(r-1)}. \tag{A.3}$$

[6] In the case of sites, an opportunity for confusion may arise. In this notation, the sites s and $-s$ correspond to the same point x, say, but are equipped with opposite orientations.

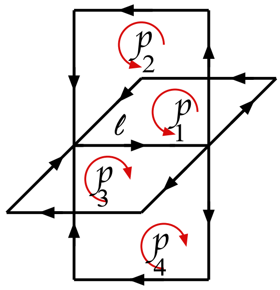

Figure 2. Coboundary operator on a link in a 3d lattice: $\delta\ell = p_1 + p_2 + p_3 + p_4$

We also have a kind of dual to the boundary operator, the 'coboundary operator' δ. It takes an r-cell into the sum of the $(r+1)$-cells that each contains $c^{(r)}$ in its boundary,

$$\delta c^{(r)} = \sum_{\partial c^{(r+1)} \supset c^{(r)}} c^{(r+1)}. \tag{A.4}$$

For example, for a link $\ell = (x; \widehat{1})$ in a three-dimensional lattice, we have

$$\delta(x;\widehat{1}) = (x;\widehat{1},\widehat{3}) + (x;\widehat{1},\widehat{2}) + (x-\widehat{3};\widehat{3},\widehat{1}) + (x-\widehat{2};\widehat{2},\widehat{1}). \tag{A.5}$$

See Fig. 2. It is easy to show that $\delta^2 = 0$.

There is another lattice $\tilde{\Lambda}^d$, the 'dual lattice', that is naturally associated with the primary lattice Λ^d. The points of $\tilde{\Lambda}^d$ are given by $\tilde{x} = x + \frac{1}{2}(\widehat{1} + \cdots + \widehat{d})$, with x any point of Λ^d. These lattices are connected by an operator $*$, which takes r-cells in the primary lattice into $(d-r)$-cells in the dual lattice and vice versa; it is defined as follows: for an r-cell $c^{(r)}$ in the primary lattice, $*c^{(r)}$ is the unique $(d-r)$-cell in the dual lattice such that $c^{(r)}$ and $*c^{(r)}$ intersect transversally, and such that the orientation of the ordered pair $(c^{(r)}, *c^{(r)})$ is positive. For example, for a plaquette $p = (x;\widehat{1},\widehat{2})$ in a 3d lattice Λ^3, we have $*p = (\tilde{x} - \widehat{3}; \widehat{3})$. On r-cells, we have

$$*^2 = (-)^{r(d-r)}. \tag{A.6}$$

We can now define differential forms on the lattice. An r-form ω is simply a gadget that assigns a value $\omega_{c^{(r)}}$ to each r-cell $c^{(r)}$, and it extends as a linear map. To compare with more conventional lattice field theory notation, consider for example a 1-form θ. We may write its value on a link $\ell = (x; \widehat{\mu})$ as

$$\theta_\ell = \theta_{x,\mu} \equiv \theta_\mu(x). \tag{A.7}$$

We define the 'exterior differential' operator d to take r-forms to $(r+1)$-forms according to the formula

$$(\mathrm{d}\omega)_{c^{(r+1)}} \equiv \sum_{c^{(r)} \subset \partial c^{(r+1)}} \omega_{c^{(r)}} = \omega_{\partial c^{(r+1)}}. \tag{A.8}$$

To again compare with more conventional notation, we note that the differential $\mathrm{d}\theta$ of the 1-form θ on a plaquette $p = (x; \mu, \widehat{\nu})$ is given by

$$(\mathrm{d}\theta)_p = \theta_\mu(x) + \theta_\nu(x+\mu) - \theta_\mu(x+\nu) - \theta_\nu(x). \tag{A.9}$$

We also define the dual $\mathrm{d}^\dagger$ of the exterior differential, the 'codifferential', which takes r-forms to $(r-1)$-forms, according to the formula

$$(\mathrm{d}^\dagger\omega)_{c^{(r-1)}} \equiv \sum_{c^{(r)}\subset\delta c^{(r-1)}} \omega_{c^{(r)}} = \omega_{\delta c^{(r-1)}}. \tag{A.10}$$

It is easy to see that $\mathrm{d}^2 = (\mathrm{d}^\dagger)^2 = 0$. The star operator $*$ takes r-forms on the primary lattice to $(d-r)$-forms on the dual lattice, and vice versa, according to the formulae

$$(*\omega)_{\tilde{c}^{(d-r)}} = \omega_{*\tilde{c}^{(d-r)}}\,, \qquad (*\tilde{\omega})_{c^{(d-r)}} = \tilde{\omega}_{*c^{(d-r)}}. \tag{A.11}$$

It is easy to show that on r-forms, we have

$$*^2 = (-)^{r(d-r)}. \tag{A.12}$$

One of the more useful features of lattice form notation is that it enables us to 'integrate by parts' mindlessly. That is, we have the formula

$$\sum_{c^{(r)}}(\mathrm{d}\omega)_{c^{(r)}}\eta_{c^{(r)}} = \sum_{c^{(r-1)}} \omega_{c^{(r-1)}}(\mathrm{d}^\dagger\eta)_{c^{(r-1)}}. \tag{A.13}$$

Actually, it is this partial integration formula that justifies calling $\mathrm{d}^\dagger$ the dual of d. To illustrate the utility of the notation, let us prove (A.13):

$$\text{L.H.S.} \equiv \sum_{c^{(r)}} \sum_{c^{(r-1)}\subset\partial c^{(r)}} \omega_{c^{(r-1)}}\eta_{c^{(r)}} = \sum_{c^{(r-1)}} \sum_{c^{(r)}\subset\delta c^{(r-1)}} \omega_{c^{(r-1)}}\eta_{c^{(r)}} \equiv \text{R.H.S.} \tag{A.14}$$

Finally, let us discuss the lattice analog of the 'Hodge decomposition'. As in the continuum, we define the Laplacian on forms by $\Delta = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d}$. In particular, on 0-forms φ, we have

$$(\Delta\varphi)(x) = \sum_{\mu=1}^{d}\{2\varphi(x) - \varphi(x+\mu) - \varphi(x-\mu)\}. \tag{A.15}$$

Forms annihilated by Δ are called 'harmonic'. It is simple to show that harmonic forms are annihilated by both d and $\mathrm{d}^\dagger$. The Hodge decomposition is the statement that any r-form α can be written uniquely as

$$\alpha = \mathrm{d}\beta + \mathrm{d}^\dagger\gamma + \eta, \tag{A.16}$$

where η is harmonic. We will not prove this here.

A.2 3d compact QED on the lattice

Here we discuss the dual representation of 3d U(1) lattice gauge theory following the presentation of Ref. [41] (see also [39, 40]). We also give some attention to global issues involving the spacetime topology. We note that, although we restrict our presentation to three dimensions, many techniques used here are also applicable in four-dimensional spacetime lattices, where interesting phase diagrams have been expected through electromagnetic dualities [64–68].

We start from the Wilson formulation of the U(1) lattice gauge theory in $d = 3$ spacetime dimensions:

$$\exp(-S) = \exp\left(\beta \sum_p (\cos f_p - 1)\right) = \prod_p \mathrm{e}^{\beta(\cos f_p - 1)}. \tag{A.17}$$

Since the action is periodic in f_p, we can expand $\exp(-S)$ as a Fourier series:

$$\mathrm{e}^{\beta(\cos f_p - 1)} = \sum_{k_p \in \mathbb{Z}} \mathrm{e}^{\mathrm{i}k_p f_p} I_{k_p}(\beta)\mathrm{e}^{-\beta}, \tag{A.18}$$

where $I_{k_p}(\beta)$ is the modified Bessel function of the first kind of order k_p. This representation is useful because it allows us to integrate over the link fields in the Abelianized theory in a straightforward manner.

The partition function can be rewritten as

$$Z = \int_0^{2\pi} [\mathrm{d}a_\ell] \exp(-S) = \sum_{\{k_p \in \mathbb{Z}\}} \int_0^{2\pi} [\mathrm{d}a_\ell] \exp\left(\mathrm{i} \sum_p k_p f_p\right) \prod_p I_{k_p}(\beta)\mathrm{e}^{-\beta}. \tag{A.19}$$

From this expression, in the weak coupling limit $\beta \gg 1$, we can obtain the Villain form. Using the asymptotic expansion, $\mathrm{e}^{-\beta} I_{k_p}(\beta) \sim \frac{1}{\sqrt{2\pi\beta}} \mathrm{e}^{-\frac{1}{2\beta}k_p^2}$, we can rewrite the summation over k_p by Poisson summation formula,

$$\sum_{k_p=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}k_p f_p} \frac{1}{\sqrt{2\pi\beta}} e^{-\frac{1}{2\beta}k_p^2} = \sum_{n_p=-\infty}^{+\infty} \mathrm{e}^{-\frac{\beta}{2}(f_p - 2\pi n_p)^2} \tag{A.20}$$

Here, n_p can be viewed as the flux passing through the corresponding surface p. The total flux passing through the surface of the cube centered at $\tilde{x}$ is

$$\oint_{\text{faces, } \tilde{x}} n_p = q(\tilde{x}), \tag{A.21}$$

which is just the magnetic charge located at $\tilde{x}$. In the following, we concentrate only on this weak-coupling limit that is exactly equivalent to the Villain formulation.

Dual formulation, from Λ^3 to $\tilde{\Lambda}^3$: In order to obtain the dual representation of the Villain form, we perform the exact integration over a_ℓ before the summation over k_p in (A.19). As $\sum_p k_p (\mathrm{d}a)_p = \sum_\ell (\mathrm{d}^\dagger k)_\ell a_\ell$, the exact integration over a_ℓ enforces the constraint,

$$(\mathrm{d}^\dagger k)_\ell = 0. \tag{A.22}$$

As a result, the partition function can be written as a constrained sum over the k_p:

$$Z = \sum_{\{k_p \in \mathbb{Z}\}} \left\{ \prod_\ell \delta_{(\mathrm{d}^\dagger k)_\ell, 0} \right\} \left\{ \prod_p \mathrm{e}^{-\frac{1}{2\beta}k_p^2} \right\}. \tag{A.23}$$

To construct the dual formulation of the theory, it is useful to turn the constrained sum into an unconstrained sum. To this end, we consider the decomposition of $(*k)_{\tilde{\ell}}$ as

$$(*k)_{\tilde{\ell}} = (\mathrm{d}m)_{\tilde{\ell}} + \tilde{a}_{\tilde{\ell}}, \tag{A.24}$$

where $m_{\tilde{s}}$ is an integer-valued scalar field, and $\tilde{a}_{\tilde{\ell}}$ is an integer-valued link field on the dual lattice. Since k_p satisfies the constraint (A.22), $\tilde{a}$ can be regarded as the flat connection. In computing the partition function, we may make the replacement, $k_p \to (*\mathrm{d}m)_p + (*\tilde{a})_p$, and the constrained sum over $\{k_p \in \mathbb{Z}\}$ becomes an unconstrained sum over $\{m_{\tilde{s}} \in \mathbb{Z}\}$ and $[\tilde{a}] \in H^1(\tilde{\Lambda}, \mathbb{Z})$. As a result, the partition function in the weak-coupling limit takes the simple form,

$$Z = \sum_{\tilde{a}_{\tilde{\ell}} \in H^1} \sum_{\{m_{\tilde{s}} \in \mathbb{Z}\}} \exp\left(-\frac{1}{2\beta} \sum_{\tilde{\ell}} (\mathrm{d}m + \tilde{a})^2_{\tilde{\ell}}\right), \tag{A.25}$$

after using the duality relation (A.24). This model is sometimes called the $\mathbb{Z}$-ferromagnet when $\tilde{a} = 0$.

Two more steps are needed to convert $\mathbb{Z}$-ferromagnet representation to a continuum QFT. First, convert the sum into an integration over a continuous variable. Using the Poisson resummation identity repeatedly through the lattice $\tilde{\Lambda}_3$,

$$\prod_{\tilde{x} \in \tilde{\Lambda}_3} \sum_{m(\tilde{x})=-\infty}^{\infty} \delta(\sigma(\tilde{x}) - 2\pi m(\tilde{x})) = \prod_{\tilde{x} \in \tilde{\Lambda}_3} \sum_{q(\tilde{x})=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\, q(\tilde{x})\sigma(\tilde{x})}, \tag{A.26}$$

we immediately obtain the partition function as an infinite dimensional integral,

$$Z = \sum_{\tilde{a}_{\tilde{\ell}} \in H^1} \int [\mathrm{d}\sigma] \sum_{\{q(\tilde{x})\}} \exp\left(-\frac{1}{8\pi^2\beta} \sum_{\tilde{x}} (\partial^-_\mu \sigma(\tilde{x}) + 2\pi \tilde{a}_\mu(\tilde{x}))^2 + \mathrm{i} \sum_{\tilde{x}} q(\tilde{x})\sigma(\tilde{x})\right), \tag{A.27}$$

where $q(\tilde{x}) \in \mathbb{Z}$ has an interpretation as the magnetic charge of a monopole-instanton at position $\tilde{x} \in \tilde{\Lambda}_3$. The kinetic term of this expression clarifies that $\tilde{a}$ plays the role of the gauge fields for the discrete shift symmetry $\sigma \mapsto \sigma + 2\pi$, and thus the dual photon field σ is 2π-periodic scalar. Having made this point, for simplicity of notation, we neglect the effect of nontrivial topology, and set $\tilde{a} = 0$.

The Gaussian integration over σ can be done exactly to produce the Coulomb gas representation for the magnetic monopoles:

$$Z = \sum_{\{q(\tilde{x})\}} \exp\left(\frac{1}{2\beta} \sum_{\tilde{x}, \tilde{x}'} (-4\pi^2\beta^2) q(\tilde{x}) v(\tilde{x} - \tilde{x}') q(\tilde{x}')\right), \tag{A.28}$$

where $v(\tilde{x} - \tilde{x}')$ is the three dimensional Coulomb interaction formulated on the lattice (lattice Green function), formally given by $v(\tilde{x}) = \Delta^{-1}$. Let us split this Green function into two parts by adding and subtracting $(\Delta + M_{\mathrm{PV}}^2)^{-1}$,

$$\begin{aligned} \Delta^{-1} &= \Delta^{-1} - (\Delta + M_{\mathrm{PV}}^2)^{-1} + (\Delta + M_{\mathrm{PV}}^2)^{-1} \\ &= \Delta^{-1}(1 + \Delta/M_{\mathrm{PV}}^2)^{-1} + (\Delta + M_{\mathrm{PV}}^2)^{-1} \\ &= u_{M_{\mathrm{PV}}}(\tilde{x}) + w_{M_{\mathrm{PV}}}(\tilde{x}), \end{aligned} \tag{A.29}$$

where $u_{M_{\mathrm{PV}}}(\tilde{x})$ is the Green function of the Pauli-Villars (PV) regulated Laplacian $\Delta_{M_{\mathrm{PV}}} \equiv \Delta(1 + \Delta/M_{\mathrm{PV}}^2)$, and $w_{M_{\mathrm{PV}}}(\tilde{x})$ is the Yukawa Green function.

With this decomposition, we reintroduce the scalar field σ for the PV regulated propagator $u_{M_{\rm PV}}(\tilde{x})$, and then we obtain the partition function as

$$Z=\int[\mathrm{d}\sigma]\sum_{\{q(\tilde{x})\}}\mathrm{e}^{\sum_{\tilde{x}}\left(-\frac{1}{8\pi^2\beta}\sigma(\tilde{x})\Delta_{M_{\rm PV}}\sigma(\tilde{x})+\mathrm{i}\,q(\tilde{x})\sigma(\tilde{x})\right)-2\pi^2\beta\sum_{\tilde{x},\tilde{x}'}q(\tilde{x})w_{M_{\rm PV}}(\tilde{x}-\tilde{x}')q(\tilde{x}')}. \tag{A.30}$$

Since $w_{M_{\rm PV}}(\tilde{x}-\tilde{x}')$ is a massive propagator with the Pauli-Villars mass $M_{\rm PV}$, it is exponentially damping if $|\tilde{x}-\tilde{x}'|\gtrsim 1/M_{\rm PV}$. Therefore, we can take $w_{M_{\rm PV}}(\tilde{x}-\tilde{x}')=w_{M_{\rm PV}}(0)\delta_{\tilde{x}-\tilde{x}',0}$, where $w_{M_{\rm PV}}(0)=v(0)-\mathcal{O}(1/M_{\rm PV})\approx 0.253-\mathcal{O}(1/M_{\rm PV})$. In this limit, the partition function simplifies into

$$Z=\int[\mathrm{d}\sigma]\mathrm{e}^{-\frac{1}{8\pi^2\beta}\sum_{\tilde{x}}\sigma(\tilde{x})\Delta_{M_{\rm PV}}\sigma(\tilde{x})}\sum_{\{q(\tilde{x})\}}\prod_{\tilde{x}}\mathrm{e}^{-2\pi^2\beta v(0)(q(\tilde{x}))^2+\mathrm{i}q(\tilde{x})\sigma(\tilde{x})}, \tag{A.31}$$

In (A.31), $2\pi^2\beta v(0)(q(\tilde{x}))^2$ has an interpretation as the action of the configurations with magnetic charge $q(\tilde{x})$, and let us denote the minimal action as $S_0=2\pi^2\beta v(0)$.

We now perform the dilute-gas approximation as the leading-order semiclassical approximation. We only take into account the minimal effect of the monopole-instantons corresponding to $q(\tilde{x})=\pm 1$ seriously, and higher-order effects in e^{-S_0} are regarded as unimportant. As a result, we may approximate the q-summation as

$$\sum_q \mathrm{e}^{-S_0 q^2+\mathrm{i}q\sigma(\tilde{x})}=\exp\left(2\mathrm{e}^{-S_0}\cos(\sigma(\tilde{x}))\right)+\mathcal{O}(\mathrm{e}^{-2S_0}). \tag{A.32}$$

Substituting this expression into (A.31), we obtain the local Lagrangian for dual photon fields

$$Z=\int[\mathrm{d}\sigma]\exp\left(-\frac{1}{8\pi^2\beta}\sum_{\tilde{x}}\sigma(\tilde{x})\Delta_{M_{\rm PV}}\sigma(\tilde{x})+2\mathrm{e}^{-S_0}\sum_{\tilde{x}}\cos(\sigma(\tilde{x}))\right) \tag{A.33}$$

B Wilson to Villain at weak coupling

As mentioned in Sec. 2.1, semi-Abelian $\mathrm{U}(1)^{N-1}$ gauge theory may also be formulated in the Wilson formulation by taking the action

$$S_{\rm W}=\beta\sum_p\sum_{i=1}^N(1-\cos f_p^i)-\mathrm{i}\sum_\ell\sum_{i=1}^N v_\ell a_\ell^i, \tag{B.1}$$

where $a_\ell^i\in[0,2\pi]$ are U(1) gauge fields, the $f_p^i=(\mathrm{d}a^i)_p$ are the corresponding field strengths, and $v_\ell\in\mathbb{Z}$ is a Lagrange multiplier. This expression has manifest S_N global symmetry. The aim of this section is to demonstrate the weak-coupling equivalence of this and the Villain-type formulation (2.1).

The first step is to use (A.18) for $\cos(f_p^i)$ for $i=1,\ldots,N$ with the weak-coupling approximation, and then to apply (A.20) for $i=1,\ldots,N-1$. We obtain a new action,

$$S_1=\frac{\beta}{2}\sum_p\sum_{i=1}^{N-1}(f_p^i+2\pi n_p^i)^2-\mathrm{i}\sum_\ell\sum_{i=1}^{N-1}v_\ell a_\ell^i+\frac{1}{2\beta}\sum_p k_p^2-\mathrm{i}\sum_p k_p f_p^N-\mathrm{i}\sum_\ell v_l a_\ell^N, \tag{B.2}$$

and we have introduced integer-valued plaquette-fields n_p^i $(i = 1, ..., N-1)$ and k_p, over which we must perform the summation in the partition function. The exact integration over a_ℓ^N gives the constraint,

$$v = -\mathrm{d}^\dagger k, \tag{B.3}$$

and then the action becomes

$$\begin{aligned} S_2 &= \frac{\beta}{2}\sum_p \sum_{i=1}^{N-1} (f_p^i + 2\pi n_p^i)^2 + \mathrm{i}\sum_\ell \sum_{i=1}^{N-1} (\mathrm{d}^\dagger k)_\ell a_\ell^i + \frac{1}{2\beta}\sum_p k_p^2 \\ &= \frac{\beta}{2}\sum_p \sum_{i=1}^{N-1} (f_p^i + 2\pi n_p^i)^2 + \mathrm{i}\sum_\ell \sum_{i=1}^{N-1} k_p f_p^i + \frac{1}{2\beta}\sum_p k_p^2. \end{aligned} \tag{B.4}$$

Applying the Poisson summation formula in terms of k_p, we get

$$S_3 = \frac{\beta}{2}\sum_p \sum_{i=1}^{N-1} (f_p^i + 2\pi n_p^i)^2 + \frac{\beta}{2}\sum_p \left(\sum_{i=1}^{N-1} f_p^i + 2\pi n_p\right)^2, \tag{B.5}$$

where a new integer-valued plaquette-field n_p has taken the place of k_p. This is already a Villain form of the Wilson action with which we started, but it is not yet the one for which we are aiming. A few more steps are needed.

For convenience, let us rewrite this last action in the form

$$S_3 = \frac{\beta}{2}\sum_{i=1}^{N-1} (f_p^i + b_p^i)^2 + \frac{\beta}{2}\left(\sum_{i=1}^{N-1} f_p^i + b_p\right)^2 \tag{B.6}$$

by defining $b_p^i \equiv 2\pi n_p^i$, $b_p \equiv 2\pi n_p$. Completing the square then yields

$$S_3 = \frac{\beta}{2}\sum_p \sum_{i,j=1}^{N-1} D^{ij}(f_p^i + b_p^i + w_p/N)(f_p^j + b_p^j + w_p/N) + \frac{\beta}{2}\sum_p w_p^2 \tag{B.7}$$

where we have defined

$$w_p \equiv b_p - \sum_{i=1}^{N-1} b_p^i\,, \qquad D^{ij} \equiv 1 + \delta^{ij} \tag{B.8}$$

At this point, we realize that if we are only interested in the weak coupling regime, then since the fluctuations in w_p are gapped and discrete, we are entitled to set $w_p = 0$. Making this step leaves us with the action

$$S_4 = \frac{\beta}{2}\sum_p \sum_{i,j=1}^{N-1} D^{ij}(f_p^i + b_p^i)(f_p^j + b_p^j) \tag{B.9}$$

Now we note that the unimodular matrix M given by

$$M^{i,j} = \delta^{i,j} - \delta^{i+1,j} \tag{B.10}$$

satisfies

$$M^{\mathrm{t}} D M = C \tag{B.11}$$

where C is the Cartan matrix of SU(N):

$$C^{ij} = \boldsymbol{\alpha}^i \cdot \boldsymbol{\alpha}^j \tag{B.12}$$

It follows that we can make the field redefinitions:

$$a^i_\ell \to \sum_{j=1}^{N-1} M^{ij} A^j_\ell\,, \qquad b^i_p \to \sum_{j=1}^{N-1} M^{ij} B^j_p \tag{B.13}$$

where, because of the unimodularity of M, the new fields live in the same domains, i.e. $A^i_\ell \in [0, 2\pi]$, $B^i_p \in 2\pi\mathbb{Z}$. We obtain finally:

$$S = \frac{\beta}{2} \sum_p \sum_{i,j=1}^{N-1} C^{ij} (F^i_p + B^i_p)(F^j_p + B^j_p) \tag{B.14}$$

which is equivalent to (2.1).

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