Exploration Sheet: Proofs by Coloring

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Introduction

Coloring is a powerful tool in combinatorics, often used to partition sets and prove impossibility results. In this sheet, you will explore several classic and creative problems that use coloring arguments.

1. The Chessboard Problem

a) Domino Tiling: An 8×8 chessboard can be covered by 2×1 dominoes in 224×9012 ways.
b) Mutilated Chessboard: What happens if you remove two diagonally opposite corners? Can you still cover the board with dominoes?
Explore: Use coloring to argue why it is or isn't possible.

2. T-Tetrominoes and Chessboards

Can a 10×10 chessboard be covered by 25 T-tetrominoes? (A T-tetromino is a shape made of four squares in the shape of a 'T'.)
Explore: Try coloring the board in a way that helps you prove your answer.

3. Beetles on a Chessboard

A beetle sits on each square of a 9×9 chessboard. At a signal, each beetle crawls diagonally onto a neighboring square. Some squares may end up with multiple beetles; some may be empty.
Question: What is the minimal possible number of free (empty) squares after the move?
Hint: Try coloring the board and tracking beetle movements.

4. Coloring the Plane: Rectangles

Every point of the plane is colored either red or blue.
Show: There must exist a rectangle whose four vertices are all the same color.
Extension: Can you generalize this result?

5. Coloring the Plane: Distances

Suppose every point in the plane is colored red or blue.
Show: For any distance d, one of the two colors contains two points that are exactly d apart.

6. Knight's Tour on Rectangular Boards

Claim: There is no closed knight's tour on a $4 \times n$ chessboard.
Explore: Use coloring or parity arguments to justify this impossibility.

7. Matrix Product Sums

Each entry of a 25×25 matrix is either $+1$ or -1. Let a_i be the product of all elements in the ith row, and b_j the product in the jth column.
Prove: $a_1 + a_2 + \cdots + a_{25} + b_1 + b_2 + \cdots + b_{25} \neq 0$.

8. Coloring in Space

Every point in space is colored red or blue.
Show: Among all unit squares in space, there is at least one with three red vertices or one with all four vertices blue.

Reflection

For each problem, try to:

- Experiment with different colorings.
- Look for patterns or invariants.
- Justify your answers with clear arguments.