Puzzles in Number Theory and Algebra

Pre-Lecture Worksheet

Part 1: The Chinese Remainder Theorem

Find the smallest positive integer $x > 1$ that satisfies the given conditions. If no solution exists, explain why.

1. $x \equiv 1 \pmod{2}$ and $x \equiv 1 \pmod{3}$.

2. $x \equiv 2 \pmod{5}$ and $x \equiv 3 \pmod{6}$.

3. $x \equiv 1 \pmod{4}$ and $x \equiv 0 \pmod{8}$.

4. $x \equiv 1 \pmod{4}$, $x \equiv 2 \pmod{5}$, and $x \equiv 3 \pmod{7}$.

A Challenge Problem

The following puzzle explores a surprising application of the Chinese Remainder Theorem.

a. Can you find two relatively prime integers, a and b, such that their product minus one is the product of two consecutive integers? That is, $ab - 1 = t(t + 1)$ for some integer t.

b. Can you find three *pairwise* relatively prime integers, a, b, and c, such that $abc - 1 = t(t + 1)$ for some integer t?

c. (For further thought) Can you explore the above theme for *any positive integer*, k of your choice? That is, can you always find k pairwise relatively prime integers $a_1, a_2, \ldots, a_k$ that satisfy

$$\left(\prod_{i=1}^{k} a_i\right) - 1 = t(t + 1) \text{ where } t \text{ is some natural number.}$$

Part 2: Lagrange Interpolation

5. Find the unique polynomial of degree 1 that passes through the points $(0, 0)$ and $(1, 1)$.

6. Consider finding a polynomial of degree 2 that passes through the points $(1, 3)$ and $(2, 3)$.

(a) Find one such polynomial.

(b) Can you find a different one? How many do you think exist?

7. Find the *unique* polynomial of degree 2 that passes through all three of the following points: $(1, 3)$, $(2, 3)$, and $(3, 5)$.