Multi-scale Convergence in PDEs and Application to a High-Contrast Optimal Control problem

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Recent advances on control theory of PDE systems
ICTS, Bangalore, India
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Outline

1. Oscillating Domains

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Pillar type oscillating domain with strong contrasting composites

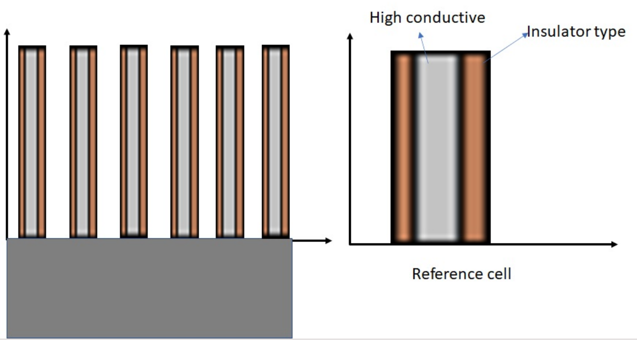


Figure: Pillar type oscillating domain

Reference: A. K. Nandakumaran and A. Sufian, *Strong contrasting diffusivity in general oscillating domains: Homogenization of optimal control problems*, *Journal of Differential Equations*, 291(2021) 57-89. https://doi.org/10.1016/j.jde.2021.04.031

More general oscillating domain with strong contrasting composites

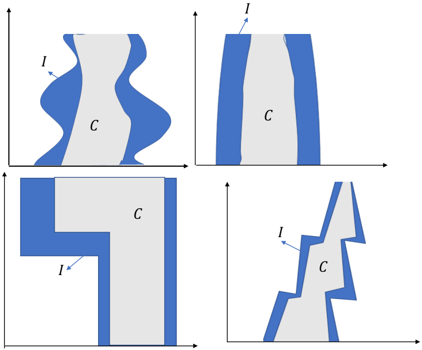
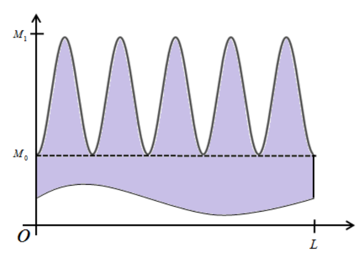


Figure: Typical example of reference cells

Domains with Oscillating Boundary; Sample Model Domains



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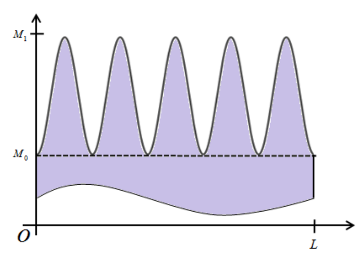

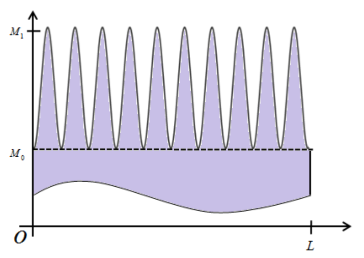

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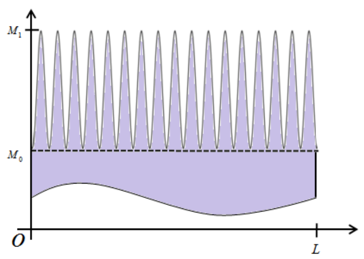

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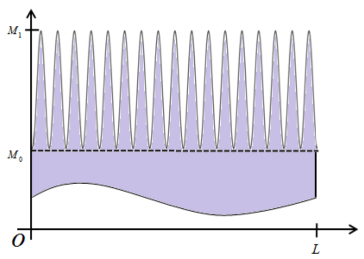

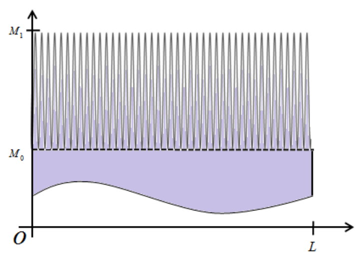

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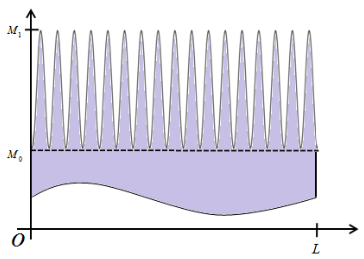

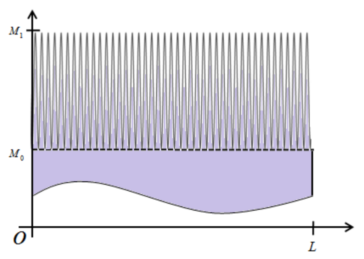

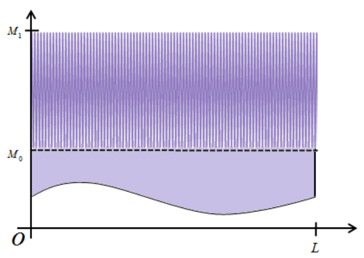

Some References

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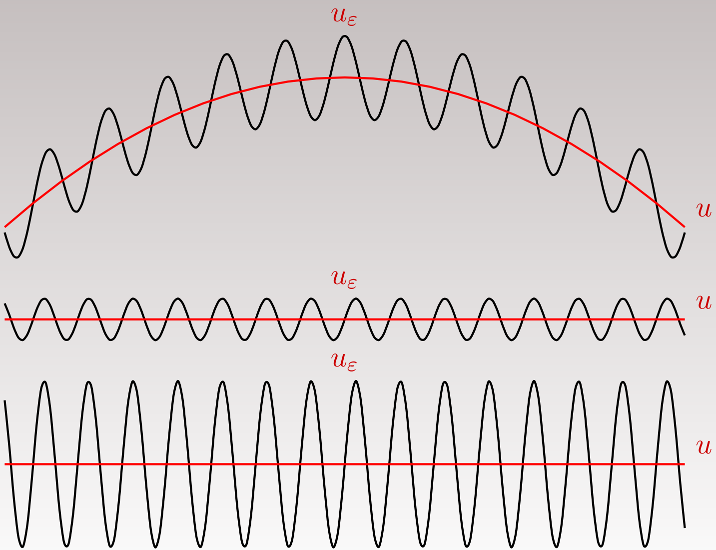
1. S. Aiyappan, A. K. Nandakumaran and Ravi Prakash, *Generalization of Unfolding Operator for Highly Oscillating Smooth Boundary Domains and Homogenization*, Calculus of Variations and PDE (2019) 57-86. https://doi.org/10.1007/s00526-018-1354-6.

2. S Aiyappan, A. K. Nandakumaran and Ravi Prakash, *Semi-linear optimal control problem on a smooth oscillating domain*, Communications in Contemporary Mathematics, 1-26 (2019). DOI: 10.1142/S0219199719500299

Rapidly oscillating functions



Concentration phenomena, Vanishing energies

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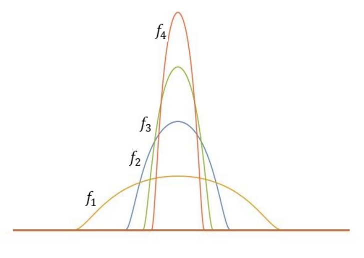
- The lack of strong convergence can be due to the concentration phenomena, vanishing energies

$$f_n(x) = \begin{cases} n \exp\left(\frac{1}{1-n^2x^2}\right), \text{if } |x| \leq 1/n \\ 0, \text{otherwise} \end{cases} \quad ; \quad f_n(x) = f(x-n)$$



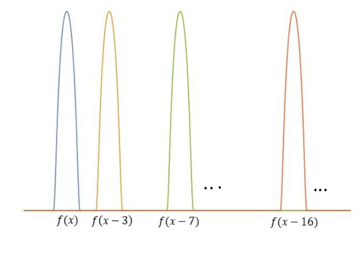
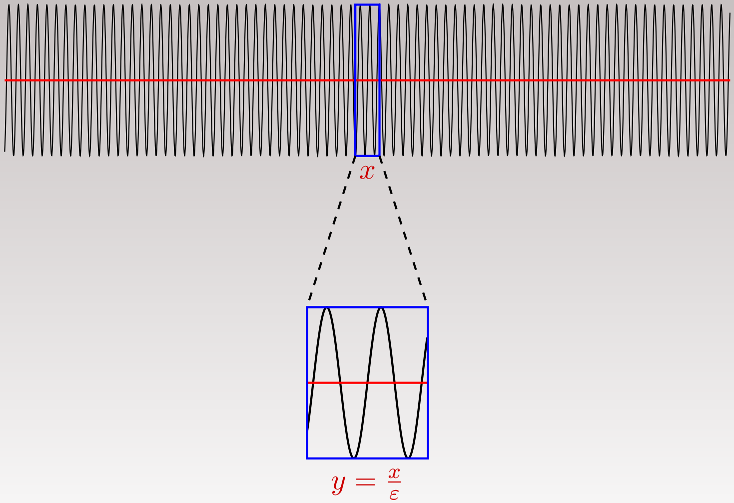


Figure: mollifiers; vanishing-functions

Oscillating functions and Scaling



Strong and Weak Convergence

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  - it does not give the convergence in energy or norm;
  - how to pass the limit in the product (u_n, v_n), nonlinearity;
  - hence difficulties in dealing with the problems of interest.

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Definition (two-scale convergence)

A sequence of functions $\{v_\varepsilon\}$ in $L^2(\Omega)$ is said to two-scale converge to a limit $v \in L^2(\Omega \times Y)$ (denoted as $v_\varepsilon \xrightarrow{2s} v$) if

$$\int_\Omega v_\varepsilon \phi\left(x, \frac{x}{\varepsilon}\right)\, dx \to \int_\Omega \int_Y v(x,y)\phi(x,y)\, dy\, dx$$

for all $\phi \in L^2[\Omega; C_{\text{per}}(Y)]$.

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- it can be proved that any purely oscillatory function $f^\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right)$ obtained from a periodic function f converges in two-scale to $f(y)$;

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Also, if v_ε is bounded in $H^1(\Omega)$, then v is independent of y and is in $H^1(\Omega)$, and there exists a $v_1 \in L^2[\Omega; H^1_{per}(Y)]$ such that, up to a subsequence, ∇v_ε two-scale converges to $\nabla v + \nabla_y v_1$.

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- Let $\{u_\varepsilon\}$ two-scale converges to u and $\bar{u}$ is the weak limit. Then

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Definition

We say u_ε strongly two-scale converges to $u = u(x,y)$ in $L^2(\Omega)$, denoted by $u_\varepsilon \overset{strong-2s}{\rightharpoonup} u$ if $u_\varepsilon \overset{2s}{\rightharpoonup} u$ and $\lim_{\varepsilon\to 0} ||u_\varepsilon||_{L^2(\Omega)} = ||u||_{L^2(\Omega\times Y)}$.

Passage to limit in the product

- For smooth ψ, $\psi\left(x, \frac{x}{\varepsilon}\right)$, converges strong two-scale to $\psi(x,y)$. In fact,

$$\psi\left(x, \frac{x}{\varepsilon}\right)\psi\left(x, \frac{x}{\varepsilon}\right) \to \int_Y \psi(x,y)\psi(x,y).$$

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Theorem

Let u_ε and v_ε be two sequences in $L^2(\Omega)$ such that u_ε strongly two-scale converges to u in $L^2(\Omega)$ and v_ε two-scale converges to v in $L^2(\Omega)$, then the product

$$u_\varepsilon(x)v_\varepsilon(x) \rightharpoonup \int_Y u(x,y)v(x,y)dy$$

in $\mathcal{D}'(\Omega)$, that is in distribution. Further, if $u \in L^2(\Omega; C_\#(Y))$, then

$$\lim_{\varepsilon\to 0}\left\|u_\varepsilon(x) - u\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)} = 0.$$

Unfolding Operators

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- In other words, unfold the second hidden scale in the given sequence. This is done via the notion of scale decomposition of $\mathbb{R}^n$.
- Finally, understand the topology of two-scale convergence.

Unfolding Method: scale decomposition

- Let $Y = [0,1)^n$, $Y_k = Y + k$, $k \in \mathbb{Z}^n$, then $\mathbb{R}^n = \biguplus_{k\in\mathbb{Z}^n} Y_k$.
- For any $x \in \mathbb{R}^n$, we can write $x = N(x) + R(x)$, where $N(x)$ and $R(x)$ are the integer and fractional parts, respectively.

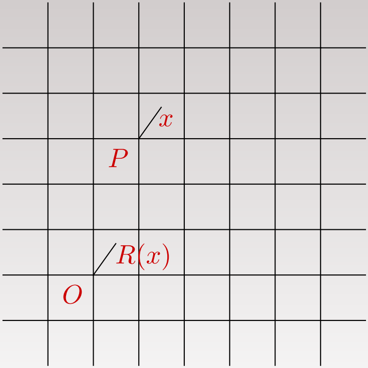

Figure: $P = N(x)$

Unfolding Method: scale decomposition

- In fact, we can use any two independent vectors to decompose $\mathbb{R}^n$.

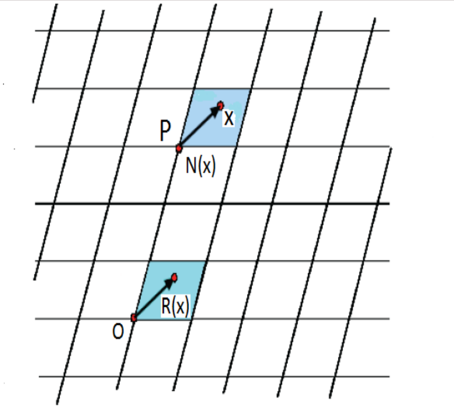

Fig. 1(a)

Unfolding Method: scale decomposition

- Also decompose $\mathbb{R}^n$ with $\varepsilon - cells$ as $\mathbb{R}^n = \biguplus_{k\in\mathbb{Z}^n} \varepsilon Y_k$, where $\varepsilon Y_k = \varepsilon Y + \varepsilon k$. For any $\varepsilon > 0$, we may write $x = \varepsilon \left[N(\frac{x}{\varepsilon}) + R(\frac{x}{\varepsilon}) \right]$ for any $x \in \mathbb{R}^n$.

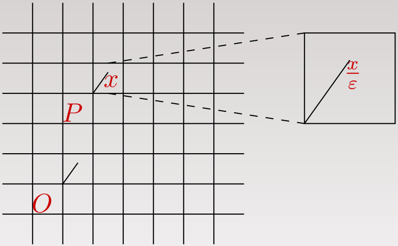

Figure: $P = \varepsilon N\left(\frac{x}{\varepsilon}\right)$ and $x = \varepsilon\left[N\left(\frac{x}{\varepsilon}\right) + R\left(\frac{x}{\varepsilon}\right)\right]$

Unfolding Operator

- Two-scale composition function: Define $S_\varepsilon : \mathbb{R}^n \times Y \to \mathbb{R}^n$ as

$$S_\varepsilon(x,y) = \varepsilon N\left(\frac{x}{\varepsilon}\right) + \varepsilon y.$$

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- Clearly $S_\varepsilon(x,y) = x + \varepsilon(y - R(\frac{x}{\varepsilon})) \to x$ uniformly in $\mathbb{R}^n \times Y$.

Definition (Unfolding Operator)

Let $u \in L^1(\mathbb{R}^n)$. The ε-unfolding of u is defined as

$$T^\varepsilon(u)(x,y) = uoS_\varepsilon(x,y) = u\left(\varepsilon N\left(\frac{x}{\varepsilon}\right) + \varepsilon y\right) \quad (1)$$

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Theorem

Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$, then $T_\varepsilon(u_\varepsilon)$ converges to $u(x,y)$ weakly in $L^2(\Omega \times Y)$ if and only if $u_\varepsilon \overset{2-s}{\rightharpoonup} u$.

Pillar type oscillating domain with strong contrasting composites

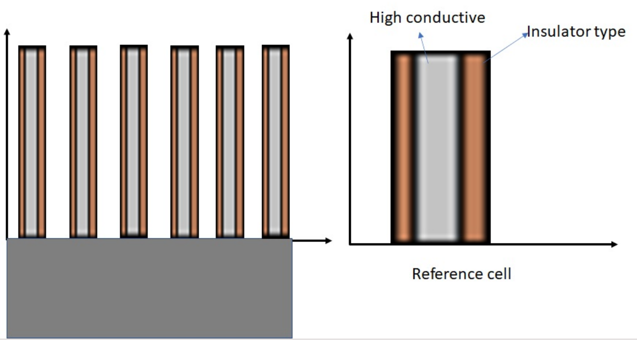

Figure: Pillar type oscillating domain

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Variational Problem

- Consider the ε dependent variational problem, for all $\phi \in H^1(\Omega_\varepsilon)$, where $f \in L^2(\Omega)$:

$$\left\{ \begin{array}{l} \text{find } u_\varepsilon \in H^1(\Omega_\varepsilon) \text{ such that} \\ \displaystyle\int_{\Omega_\varepsilon} \left(\chi_{\Omega^-} + \chi_{C_\varepsilon} + \varepsilon^2 \chi_{I_\varepsilon}\right) \nabla u_\varepsilon \nabla \phi + \int_{\Omega_\varepsilon} u_\varepsilon \phi = \int_{\Omega_\varepsilon} f\phi, \end{array} \right. \tag{2}$$

- There is no uniform ellipticity leading to non-uniform estimates.

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• Instead of Laplacian, one can consider more general elliptic operators.

• One can also consider α_ε^2 instead of the coefficient ε^2 and limiting problem may depend on the limit of $\frac{\alpha_\varepsilon}{\varepsilon}$.

Estimates

- We get the estimates in a standard way as

$$\|\chi_{C_\varepsilon^+}\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon^+)} + \varepsilon\|\chi_{I_\varepsilon^+}\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon^+)} + \|\nabla u_\varepsilon\|_{L^2(\Omega^-)} + \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leqslant \|f\|_{L^2(\Omega_\varepsilon)} \tag{3}$$

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• By the properties of unfolding operator and weak compactness of $H^1(\Omega^-)$ and $L^2(\Omega^u)$ there exist

$u^- \in H^1(\Omega^-), \quad u_0(x, y_1) \in L^2(\Omega^u),$
$\eta(x, y_1) = (\eta_1, \eta_2), \quad z(x, y_1) = (z_1, z_2) \in (L^2(\Omega^u))^2$ such that, weakly

• By the properties of unfolding operator and weak compactness of $H^1(\Omega^-)$ and $L^2(\Omega^u)$ there exist

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- $T^\varepsilon(u_\varepsilon^+) \rightharpoonup u_0(x,y_1)$ in $L^2(\Omega^u)$
- $T^\varepsilon(\chi_{C_\varepsilon^+}(\nabla u_\varepsilon)) = T^\varepsilon_{\mathrm{C}}(\nabla u_\varepsilon) \rightharpoonup \chi_C(y_1,x_2)(\eta_1,\eta_2)$ in $(L^2(\Omega^u_{\mathrm{C}}))^2$

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- $T^\varepsilon(\varepsilon \chi_{I_\varepsilon^+} \nabla u_\varepsilon) \rightharpoonup \chi_I(y_1, x_2) z(x, y_1) = \chi_I(y_1, x_2)(z_1, z_2)$ in $(L^2(\Omega^u))^2$.

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We need to identify $u_0, \eta_1, \eta_2, z_1, z_2$ and get properties enjoyed by these functions. This is the technical aspects.

Limit problem

The limit problem in variational form is

$$\begin{cases} \text{find } u = (u^+, u^-) \in H(\Omega) \text{ such that} \\ \displaystyle\int_{\Omega^+} |Y_C(x_2)| \frac{\partial u^+}{\partial x_2} \frac{\partial \phi}{\partial x_2} + \int_{\Omega^+} \alpha(x) u^+ \phi + \int_{\Omega^-} u^- \phi \\ \qquad\qquad \displaystyle + \int_{\Omega^-} \nabla u^- \nabla \phi = \int_{\Omega^+} \alpha(x) f \phi + \int_{\Omega^-} f \phi, \end{cases}$$

for all $\phi \in H(\Omega)$.

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for all $\phi \in H(\Omega)$.Here $\alpha(x) = \left(|Y(x_2)| - \int_{Y_I(x_2)} \xi dy_1 \right)$,where

$$\begin{cases} \xi(x_2, \cdot) \in V^{x_2} \\ \displaystyle\int_{Y(x_2)} \frac{\partial \xi(x_2, y_1)}{\partial y_1} \frac{\partial w(y_1)}{\partial y_1} + \int_{Y(x_2)} \xi(x_2, y_1) w(y_1) = \int_{Y(x_2)} w(y_1), \end{cases}$$

for all $w \in V^{x_2}$.

Convergence Theorem

Theorem

We have the following Convergences: as $\varepsilon \to 0$

$$u_\varepsilon^- \rightharpoonup u^- \text{ weakly in } H^1(\Omega^-),$$

$$\begin{cases} \widetilde{u_\varepsilon^+} \rightharpoonup |Y(x_2)|u^+ + \displaystyle\int_{Y_I(x_2)} (f - u^+)\xi(x_2, y_1)dy_1 \\ \chi_{C_\varepsilon}^+ \widetilde{\dfrac{\partial u_\varepsilon^+}{\partial x_1}} \rightharpoonup 0, \quad \chi_{C_\varepsilon}^+ \widetilde{\dfrac{\partial u_\varepsilon^+}{\partial x_2}} \rightharpoonup |Y_C(x_2)| \dfrac{\partial u^+}{\partial x_2} \\ \varepsilon \chi_{I_\varepsilon}^+ \widetilde{\dfrac{\partial u_\varepsilon^+}{\partial x_1}} \rightharpoonup (f - u^+) \displaystyle\int_{Y_I(x_2)} \dfrac{\partial \xi}{\partial y_1} dy_1, \quad \varepsilon \chi_{I_\varepsilon}^+ \widetilde{\dfrac{\partial u_\varepsilon^+}{\partial x_2}} \rightharpoonup 0 \\ \text{weakly in } L^2(\Omega^+) \end{cases}$$

Optimal Control Problems

Control on C_ε

For $\theta_\varepsilon \in L^2(C_\varepsilon)$ consider the cost functional

$$J_\varepsilon(u_\varepsilon, \theta_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} |u_\varepsilon - u_d|^2 + \frac{\beta}{2}\int_{C_\varepsilon} |\theta_\varepsilon|^2$$

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$$\begin{cases} \text{find } u_\varepsilon \in H^1(\Omega_\varepsilon) \text{ such that} \\ \displaystyle\int_{\Omega_\varepsilon} \left(\chi_{\Omega^-} + \chi_{C_\varepsilon} + \varepsilon^2 \chi_{I_\varepsilon}\right) \nabla u_\varepsilon \nabla \phi + u_\varepsilon \phi = \int_{\Omega_\varepsilon} f\phi + \int_{\Omega_\varepsilon} \chi_{C_\varepsilon} \theta_\varepsilon \phi, \\ \quad \text{for all } \phi \in H^1(\Omega_\varepsilon). \end{cases}$$

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The optimal control problem is to find $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \in H^1(\Omega_\varepsilon) \times L^2(C_\varepsilon)$ such that

$$J_\varepsilon(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) = \inf\{J_\varepsilon(u_\varepsilon, \theta_\varepsilon)\}. \quad (4)$$

Two-scale limit control problem

For controls $\theta \in L^2(\Omega^+)$, consider the following L^2 cost functional

$$J(u, u_1, \theta) = \frac{1}{2}\int_{\Omega^u} |u^+ + u_1 - u_d|^2 + \frac{1}{2}\int_{\Omega^-} |u^- - u_d|^2 + \frac{\beta}{2}\int_{\Omega^u_C} |\theta|^2,$$

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where $(u, u_1) \in H(\Omega) \times V(\Omega)$ satisfies the micro-macro system

$$\begin{cases} \text{find } (u, u_1) \in H(\Omega) \times V(\Omega) \text{ such that} \\ \displaystyle\int_{\Omega^u_C} \frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^u_I} \frac{\partial u_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} (u^+ + u_1)(\phi + \phi_1) + \int_{\Omega^-} \nabla u^- \nabla \phi \\ \displaystyle\qquad + \int_{\Omega^-} u^- \phi = \int_{\Omega^u} (f + \chi_C(y_1, x_2)\theta)(\phi + \phi_1) + \int_{\Omega^-} f\phi, \end{cases}$$

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Optimal control problem: find $(\bar{u},\bar{u}_1,\bar{\theta}) \in H(\Omega) \times V(\Omega) \times L^2(\Omega^+)$

$$J(\bar{u},\bar{u}_1,\bar{\theta}) = \inf\{J(u,u_1,\theta)\}.$$

Separation of scales - Homogenized system

Scale separated cost functional:

$$J(u,\theta) = \frac{1}{2}\int_{\Omega^+}\int_{Y(x_2)} \left|(1-\xi)u^+ + f\xi - u_d\right|^2 + \frac{1}{2}\int_{\Omega^-} |u^- - u_d|^2 + \frac{\beta}{2}\int_{\Omega^+} |Y_{\mathrm{C}}(x_2)||\theta|^2$$

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Scale separated limit state equation:

$$\begin{cases} \text{find } u \in H(\Omega), \text{ such that,} \\ \displaystyle\int_{\Omega^+} |Y_C(x_2)| \frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^+} \alpha(x)u^+\phi + \int_{\Omega^-} u\phi + \int_{\Omega^-} \nabla u^- \nabla \phi \\ \displaystyle\qquad = \int_{\Omega^+} \alpha(x) f\phi + \int_{\Omega^-} f\phi + \int_{\Omega^+} |Y_C(x_2)|\theta\phi, \\ \text{for all } \phi \in H(\Omega). \end{cases}$$

Control on I_ε

For $\theta_\varepsilon \in L^2(I_\varepsilon)$, consider the following L^2-cost functional

$$J_\varepsilon(u_\varepsilon, \theta_\varepsilon) = \frac{1}{2}\int_{\Omega_\varepsilon} |u_\varepsilon - u_d|^2 + \frac{\beta}{2}\int_{\mathrm{I}_\varepsilon} |\theta_\varepsilon|^2,$$

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where u_ε is the unique solution of the following variational problem: for $f \in L^2(\Omega)$

$$\begin{cases} \text{find } u_\varepsilon \in H^1(\Omega_\varepsilon) \text{ such that} \\ \displaystyle\int_{\Omega_\varepsilon} \left(\chi_{\Omega^-} + \chi_{\mathrm{C}_\varepsilon} + \varepsilon^2 \chi_{\mathrm{I}_\varepsilon}\right) \nabla u_\varepsilon \nabla \phi + u_\varepsilon \phi = \int_{\Omega_\varepsilon} f\phi + \int_{\Omega_\varepsilon} \chi_{\mathrm{I}_\varepsilon} \theta_\varepsilon \phi, \end{cases} \tag{5}$$

for all $\phi \in H^1(\Omega_\varepsilon)$.

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for all $\phi \in H^1(\Omega_\varepsilon)$. The optimal control problem is to find $(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) \in H^1(\Omega_\varepsilon) \times L^2(I_\varepsilon)$ such that

$$J_\varepsilon(\bar{u}_\varepsilon, \bar{\theta}_\varepsilon) = \inf\{J_\varepsilon(u_\varepsilon, \theta_\varepsilon) : (u_\varepsilon, \theta_\varepsilon) \text{ satisfies } (5)\}. \tag{6}$$

Two-scale limit optimal control problem

- For the source term $f \in L^2(\Omega)$ and control $(\theta, \theta_1) \in L^2(\Omega^+) \times L^2(\Omega_{\mathrm{I}}^u)$ (or one can think $\theta_1 \in L^2(\Omega^u)$ with $\theta_1 = 0$ a.e. in Ω_{C}^u), the limit L^2-cost functional is

$$J(u, u_1, \theta, \theta_1) = \frac{1}{2} \int_{\Omega^u} (u^+ + u_1 - u_d)^2 + \int_{\Omega^-} (u^- - u_d)^2 + \frac{\beta}{2} \int_{\Omega_{\mathrm{I}}^u} (\theta + \theta_1)^2$$

- Here $(u, u_1) \in H(\Omega) \times V(\Omega)$ satisfies

$$\int_{\Omega^u_{\mathrm{C}}} \frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega^u_{\mathrm{I}}} \frac{\partial u_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} (u^+ + u_1)(\phi + \phi_1)$$
$$+ \int_{\Omega^-} (\nabla u^- \nabla \phi + u\phi) = \int_{\Omega^u} (f + \chi_{\mathrm{I}}(y_1, x_2)(\theta + \theta_1))(\phi + \phi_1) + \int_{\Omega^-} f\phi,$$

for all $(\phi, \phi_1) \in H(\Omega) \times V(\Omega)$.

- Here $(u, u_1) \in H(\Omega) \times V(\Omega)$ satisfies

$$\int_{\Omega_C^u} \frac{\partial u^+}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \int_{\Omega_I^u} \frac{\partial u_1}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} (u^+ + u_1)(\phi + \phi_1)$$

$$+ \int_{\Omega^-} (\nabla u^- \nabla \phi + u\phi) = \int_{\Omega^u} (f + \chi_I(y_1, x_2)(\theta + \theta_1))(\phi + \phi_1) + \int_{\Omega^-} f\phi,$$

for all $(\phi, \phi_1) \in H(\Omega) \times V(\Omega)$.

- Now the optimal control problem is to find $(\bar{u}, \bar{u}_1, \bar{\theta}, \bar{\theta}_1) \in H(\Omega) \times V(\Omega) \times L^2(\Omega^+) \times L^2(\Omega_I^u)$ such that

$$J(\bar{u}, \bar{u}_1, \bar{\theta}, \bar{\theta}_1) = \inf\{J(u, u_1, \theta, \theta_1)\}.$$

Partial scale separation

- A complete scale separation is not available

Reduced cost functional: The L^2-cost functional reduces to

$$J(u, u_{11}, \theta, \theta_1) = \frac{1}{2}\int_{\Omega^+}\int_{Y(x_2)} ((1-\xi)u^+ + \xi f + u_{11} - u_d)^2 + \int_{\Omega^-}(u^- - u_d)^2 + \frac{\beta}{2}\int_{\Omega^+}\int_{Y(x_2)}(\theta+\theta_1)^2$$

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Reduced state equation: $(\bar{u}, \bar{u}_{11}) \in H(\Omega) \times V(\Omega)$ satisfies

$$\begin{cases} \int_{\Omega^+} |Y_C(x_2)| \frac{\partial u^+}{\partial x_2}\frac{\partial \phi^+}{\partial x_2} + \int_{\Omega^+}\alpha(x)u^+\phi^+ + \int_{\Omega^-}\nabla u^- \nabla \phi^- + \int_{\Omega^-} u^-\phi \\ \quad = \int_{\Omega^+}\int_{Y(x_2)} ((1-\xi)f + (1-\xi)(\theta+\theta_1))\phi^+ + \int_{\Omega^-} f\phi^-, \\ \int_{\Omega^u}\frac{\partial u_{11}}{\partial y_1}\frac{\partial \phi_1}{\partial y_1} + \int_{\Omega^u} u_{11}\phi_1 = \int_{\Omega^u}(\theta+\theta_1)\phi_1, \end{cases}$$

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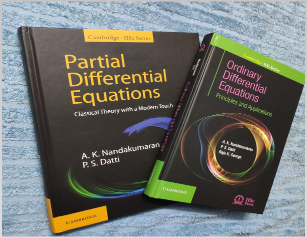

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