Celebrating the Science of Giorgio Parisi

Planar Diagrams

and

the large N limit of theories with

Matrix degrees of freedom

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Planar Diagrams

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- In this talk we will outline the foundational role played by this paper in the development of the large N limit of matrix models and non-Abelian gauge theories.

- We will briefly review i) the historical context of this paper; ii) the main results and iii) its conceptual impact on the development of string theory and conclude with a possible answer to the mean field of a U(N) gauge theory at large N.

Non-Abelian Gauge theories in the early 1970s

Non-Abelian gauge theories came to the center stage of high energy physics with:

1. The discovery of the unification of electromagnetic and weak forces;
2. The discovery of scaling in high energy deep inelastic scattering and discovery of asymptotic freedom of non-Abelian gauge theories;
3. The deeply mysterious fact that quarks are permanently confined in hadrons.

Continued…

Lattice gauge theory, dual models (string theory)

- K. G. Wilson proposed lattice gauge theory (1973). “Violate Lorentz invariance but not gauge invariance”

He calculated the area law for the `Wilson loop’ in the strong coupling   expansion demonstrating quark confinement. The loop is tiled by string bits

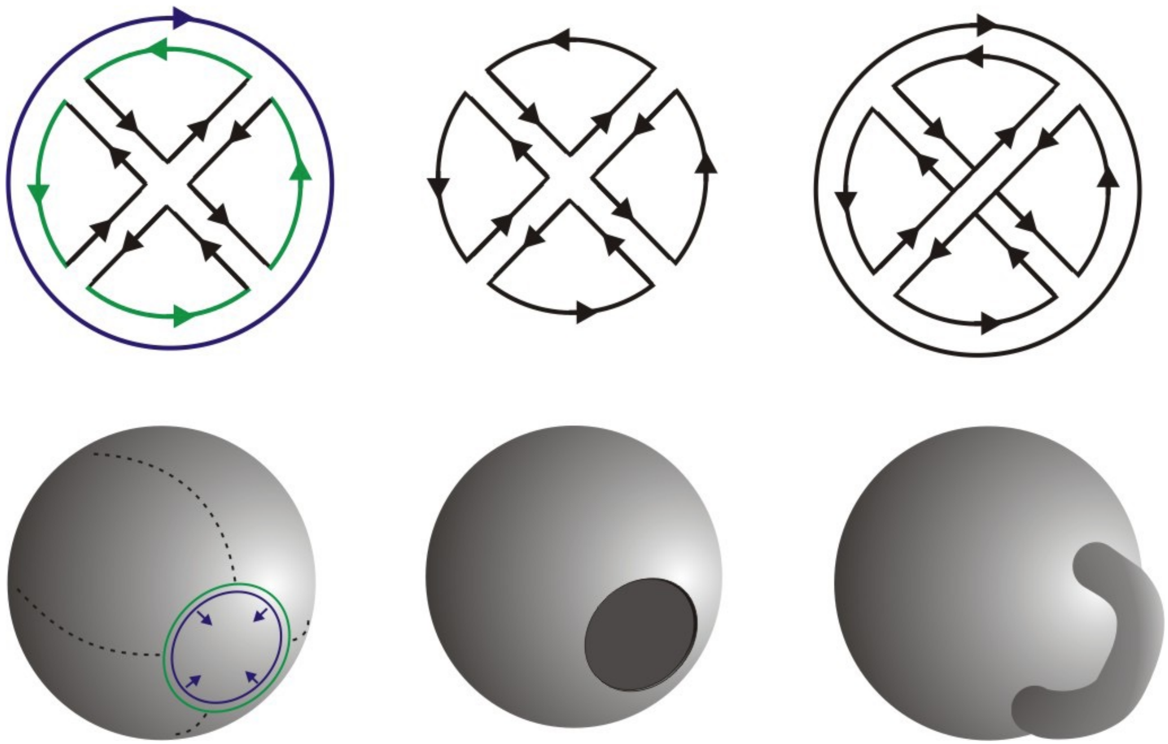
- There were parallel developments from the 1960s trying to understand the strong interactions in the subject that was called dual models and now is called string theory.
- Moving strings trace out 2-dim world sheets.
- This led to the development of string like Feynman diagrams in gauge theories by G. ‘t Hooft (1974) which resemble duality diagrams of the dual resonance model (Sakita and Virasoro-1970).

A Planar Diagram Theory for the strong interactions
G. ’ t Hooft - 1974

Abstract: “A gauge theory with colour gauge group U(N) and quarks having a colour index running from one to N is considered in the limit $N \to \infty$, g^2N fixed …. the topological structure of the perturbation series in 1/N is identical to that of the dual models, such that the number 1/N corresponds to the dual coupling constant. …. A mathematical framework is proposed to link these concepts of planar diagrams with the functional integrals of Gervais, Sakita (1973) and Mandelstam (1973) for the dual string.”

In summary U(N) gauge theories are `related in some way' with string theory and this relationship seems manifest by reorganising perturbation theory in powers of $(1/N)^{\chi}$ and $\lambda = g^2N$ held fixed as $N \to \infty$. χ is the Euler character of the Feynman diagram.

Topology of Feynman Diagrams – an example



χ = (2-2H-B), H=handles, B=boundaries

N^2 (sphere), N (sphere with a hole),

N^0 = 1 (sphere + handle)

Topological expansion – in general

As an example, consider general vacuum to vacuum graph of gluons:

- L = number of index loops, P = number of gluon propagators and V = number of vertices
- Then the graph if proportional to N^(L-P+V)
- But L-P+V = F - E+V = χ = Euler Character = 2-2H

F= number of faces, E= number of edges, V= number of vertices,

H= number of handles on S^2 and the genus g=2-2H.

Hence the vacuum energy can be organized as a genus expansion:

$\ln Z(\lambda, N) = N^2 E_0 (\lambda) + N^0 E_1 (\lambda) + N^{-2} E_2 (\lambda) + \ldots + N^{2-2H} E_H (\lambda) + \ldots$

How does one sum the planar graphs i.e., calculate E_0 (λ)?

- BIPZ → **Abstract.** We investigate the planar approximation to field theory through the limit of a large internal symmetry group. This yields an alternative and powerful method to count planar diagrams. Results are presented for cubic and quartic vertices, some of which appear to be new. Quantum mechanics treated in this approximation is shown to be equivalent to a free Fermi gas system.

- BIPZ solved matrix models in 0-dim and 1-dim and made a connection with the old subject of Random Matrix Theory developed by Wishart in statistics and Wigner in nuclear physics. In doing so they established some general principles of the large N limit for more complicated systems.

- One can calculate **E_0 (λ)** exactly in d=0 and d=1!

Matrix models in $d=0$

$$Z = \int [dM]\, e^{-NS(M)} \quad , \quad M \text{ is a } N\times N \text{ matrix}$$

$M = M^\dagger$ and $S(U^\dagger M U) = S(M)$, $U^\dagger U = \underline{1\!\!1}$

The measure $[dM]$ is easily derived from the metric, $ds^2 = \mathrm{tr}(dM^\dagger dM)$, $(dM)_{ij} = dM_{ij}$

Polar decomposition : $M = U^\dagger D U$,

$D_{ij} = \delta_{ij}\lambda_j$, $\lambda_j \in \mathbb{R}^1$

Since $S(M) = S(U^\dagger M U)$

$$[dM] = \prod_i d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 \quad , \text{ upto a constant.}$$

$$Z = \int \prod_{i=1}^{N} d\lambda_i \; e^{-NS(\{\lambda_i\}) + \sum_{i<j} \ln(\lambda_i - \lambda_j)^2}$$

e.g. $S(M) = \frac{1}{2}\mathrm{tr} M^2 + g\, \mathrm{tr} M^4$

Then the effective action is

$$S_{eff}(\lambda) = \left[\frac{1}{N}\sum_{i=1}^{N} \left(\frac{\lambda_i^2}{2} + g\,\lambda_i^4 \right) - \frac{1}{N^2}\sum_{i<j} \ln(\lambda_i - \lambda_j)^2 \right]$$

The next step was to rewrite the eff action in terms of a 'hydrodynamic' variable :

$$u(\lambda) = \frac{1}{N}\sum_i \delta(\lambda-\lambda_i), \quad \int d\lambda\, u(\lambda) = 1, \quad u(\lambda) \geq 0$$

$$S_{eff}[u] = \left[\int d\lambda\, u(\lambda)\left(\frac{\lambda^2}{2} + g\lambda^4\right) - \int d\lambda\, d\lambda'\, u(\lambda) \ln(\lambda-\lambda')^2\, u(\lambda') \right]$$

$$Z(u) \sim \int [du]\, e^{-N^2 S_{eff}[u]}$$

Saddle point $\frac{\delta}{\delta u(\lambda)} S_{eff}[u] = 0$

subject to $u(\lambda) \geq 0$ and $\int d\lambda\, u(\lambda) = 1$

$$\frac{\lambda}{2} + 2g\lambda^3 = \int_{-2a}^{2a} d\mu\, \frac{u(\mu)}{\lambda-\mu}, \quad |\lambda| \leq 2a$$

Solution :

$$u(\lambda) = \frac{1}{\pi}\left(\frac{1}{2} + 4g a^2 + 2g\lambda^2\right)\sqrt{4a^2 - \lambda^2}$$

$$a^2 = \frac{1}{24g}\left[(1+48g)^{1/2} - 1\right]$$

$$= 1 - 12g + 2(12g)^2 - 5(12g)^3 + \dots$$

$g = 0$ corresponds to Wigner's semi-circle law

Greens Functions

$$G_{2p}(g) = \left\langle \frac{\operatorname{tr}}{N} M^{2p} \right\rangle = \int_{-2a}^{2a} d\lambda\, u(\lambda)\, \lambda^{2p}$$

$$= \frac{(2p)!}{p!\,(p+2)!}\, a^{2p}\left[2p + 2 - pa^2\right]$$

Factorization

$$\boxed{\left\langle \frac{\operatorname{tr}}{N} M^{2p} \cdot \frac{\operatorname{tr}}{N} M^{2q} \right\rangle = \left\langle \frac{\operatorname{tr}}{N} M^{2p} \right\rangle \left\langle \frac{\operatorname{tr}}{N} M^{2q} \right\rangle + o\left(\frac{1}{N^2}\right)}$$

Evident since there exists $u(\lambda)$.

Matrix models in d = 1

The matrix M_{ij} is time dependent

$M(t)$, $t \in \mathbb{R}^1$.

$$L = \frac{1}{2} \operatorname{tr} \dot{M}^2 - V(M) \,, \qquad V(U^\dagger M U) = V(M)$$

Solve the corresponding Schrödinger eqn.

$$H\Psi = E\Psi$$

$$H = -\frac{1}{2}\nabla_M^2 + V(M)$$

∇_M^2 is the Laplacian corresponding to the metric : $ds^2 = \operatorname{tr} dM dM^\dagger$

$V(M) = \frac{1}{2}\operatorname{tr} M^2 + g \operatorname{tr} M^4 \;\ldots$ (single trace)

Once again make a change of variables:

$$M = U^\dagger D U \,, \quad D_{ij} = \delta_{ij}\lambda_j$$

$$-\nabla^2 = -\frac{1}{\Delta(\lambda)} \sum_{i=1}^{N} \frac{\partial^2}{\partial \lambda_i^2} \Delta(\lambda) + \sum_{i<j} \frac{L_{ij} L_{ij}}{(\lambda_i - \lambda_j)^2}$$

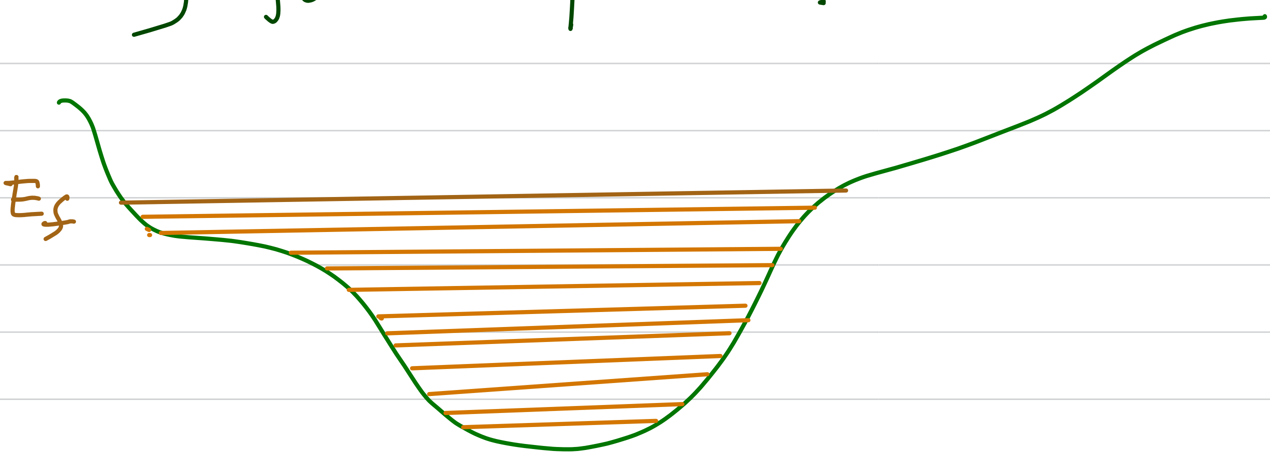
where $\Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j)$

If we assume that the wave function is $U(N)$ invariant, then

$$H\Psi = E\Psi, \quad \Psi(\lambda_1, \dots \lambda_N) \text{ is symmetric}$$

$$\Rightarrow \quad -\frac{1}{2}\frac{1}{\Delta}\sum_{i=1}^{N}\left[(\Delta\Psi) + V(\lambda_i)\Psi\right] = E\Psi$$

$\Delta(\lambda)\Psi = \Phi_F(\lambda_1 \dots, \lambda_N)$ is anti-symmetric and we have a exactly soluble many fermion problem!



Ground state energy:

$$N^2 E_{gnd} = \int \frac{d\lambda\, dP}{2\pi}\, u(\lambda, P)\, h(\lambda, P)$$

$$u(\lambda,p) = \Theta\big(E_f - h(\lambda,p)\big)\,, \quad \int \frac{d\lambda\, dp}{2\pi}\, u(\lambda,p) = 1$$

is the phase space density of fermions

$h(\lambda,p) = \frac{1}{2}p^2 + V(\lambda)$ is the single particle hamiltonian.

Comment

- If the potential has multi-trace operators: e.g. $\left(\frac{\mathrm{tr}}{N}M^2\right)^2 \longrightarrow \frac{\mathrm{tr}}{N}M^2 \left\langle \frac{\mathrm{tr}}{N}M^2 \right\rangle + o\!\left(\frac{1}{N^2}\right)$ and we still have a many fermion problem + consistency conditions.

What do the d=0 and d=1 matrix models teach us?

1) Importance of recasting the problem in terms of U(N) invariants. In this simple example the invariants are the functions of the eigenvalues. This is a very general lesson as we will see a bit later, and it is the source of non-locality of the effective action.

2) The large N expansion written in terms of U(N) invariant variables becomes a semi-classical expansion in 1/N, with $N^2 \sim \frac{1}{\hbar}$ and the path integral over the density/phase space density is evaluated to leading order in N by a `classical solution'. This was the first example of what people later called the large N `master field'.

3) Expectation values of U(N) invariant operators factorize, which is equivalent to point 2).

BIPZ inspired further explorations: Unitary matrix models

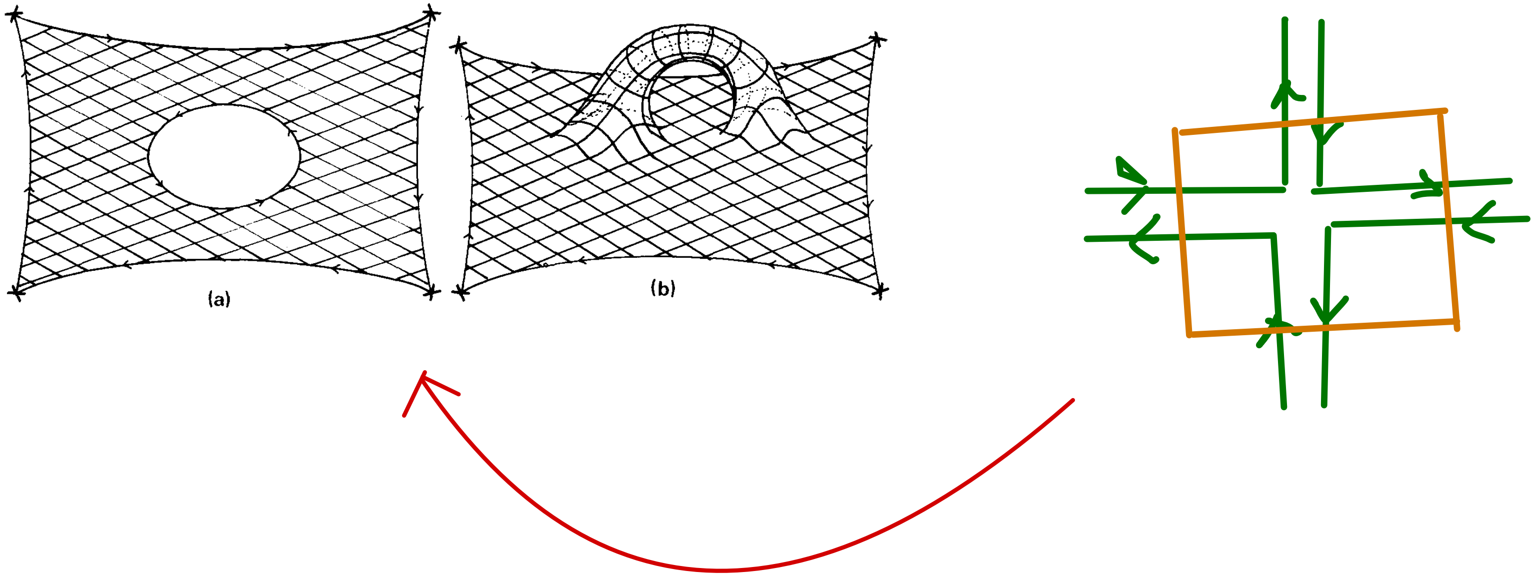
- The partition function of the U(N) gauge theory in 2-dim with free boundary conditions reduces to the single unitary matrix integral: $Z = z^{V},$ where $z = \int dU \exp -\beta(\mathrm{tr}U + \mathrm{tr}U^{+})$ and V is the volume of 2-dim Euclidean space. (SRW and Gross-Witten). The solution exhibits a 3rd order large N phase transition (1979).

- The same is true for the unitary matrix quantum mechanics (SRW). There is a 3rd order phase transition when the fermi level reaches the top of the periodic potential (1979).

- Literature: “ The Large N limit in Quantum Field Theory and Statistical Mechanics: From Spin systems to 2-dim Gravity” edited by E. Brezin and SRW; “Instantons and Large N” a textbook by M. Marino.

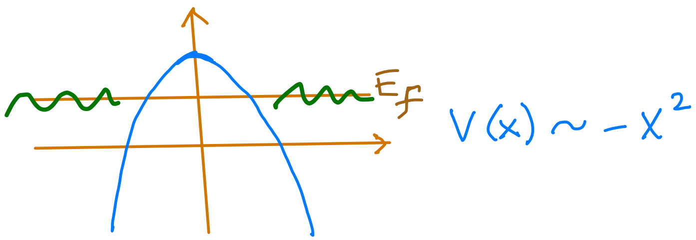
Matrix models and 2-dim quantum gravity

- Around 1990 the BIPZ model was used in defining a theory of 2-dim gravity. (Kazakov, Brezin, Douglas, Shenker, Gross, Migdal…Marinari, Parisi)
- The Feynman diagrams on a genus g surface in the ‘t Hooft double line notation correspond to a tiling of the genus g surface in terms of dual polygons. By adjusting the couplings one can take a continuum limit of the `triangulation’ of the surface and define a continuum theory of fluctuating surfaces: 2-dim quantum gravity.



C=1 Matrix model

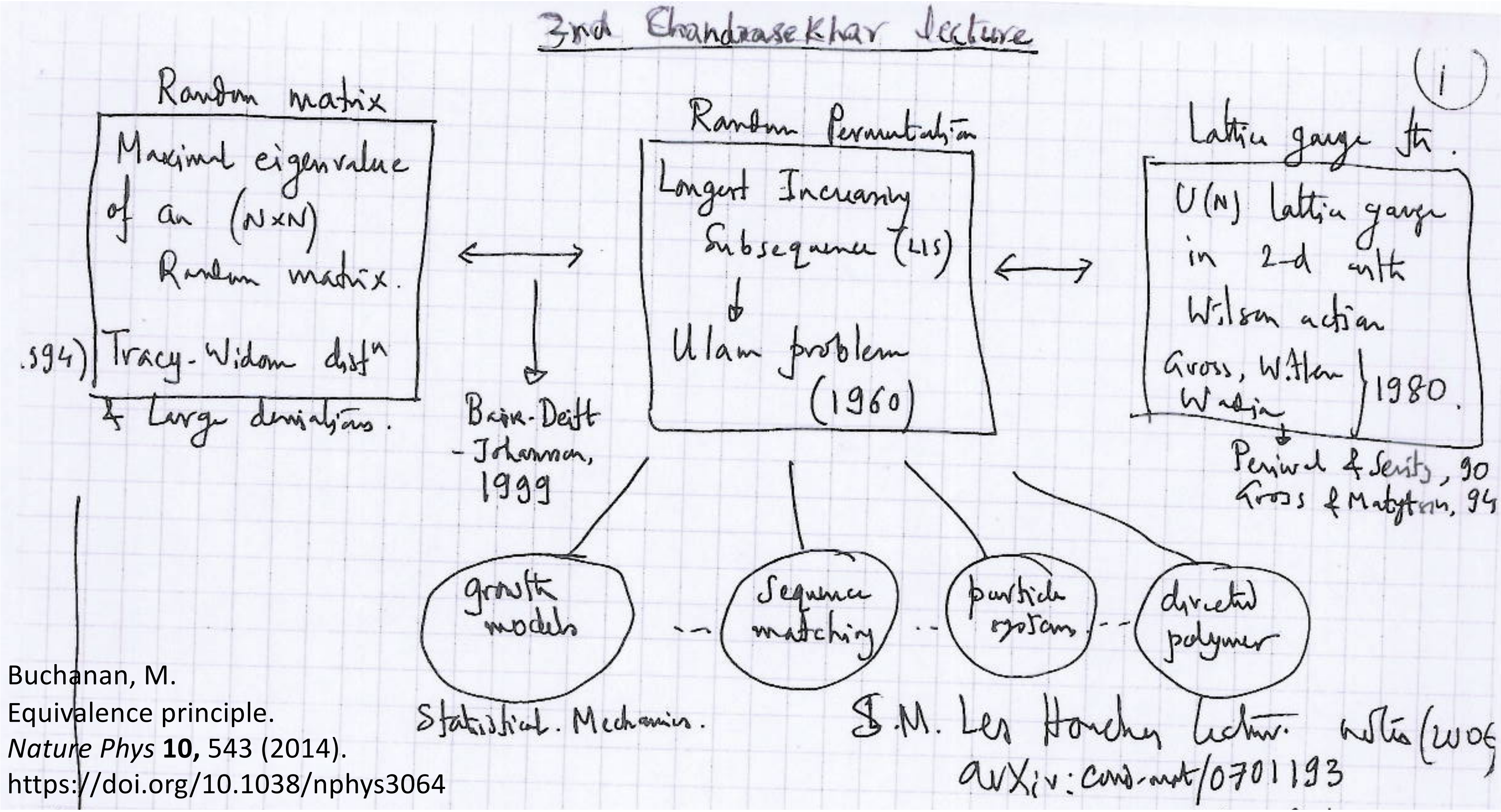
- In particular, c=1 matter coupled to 2-dim gravity is described by the d=1 BIPZ matrix model described by the matrix M(t) and a potential V(M).
- When the `bare' fermi level E_f reaches the maximum of the potential one can define a scaling limit which magnifies the maximum of the potential $V(\lambda)$ of the fermions and effectively works with $V(\lambda) = -\lambda^2$.



- The small oscillations of the fermi surface are described by a boson field $\phi(t, x)$ which satisfies the following equation in 1+1 dim. The space coordinate `x' is a function of λ. This equation is the same as that of a massless boson in a 1+1 dim string theory with a linear dilaton background. (A.M. Sengupta, SRW; Gross, Klebanov; Das, Jevicki).

$$\partial_+\partial_-\phi = \frac{\pi}{2N}\left[\partial_+\{\rho_f^2(\partial_+\phi)^2\} - \partial_-\{\rho_f^2(\partial_-\phi)^2\}\right]$$

Many applications of RMT … a page from a talk at ICTS by Satya Majumdar



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Equivalence principle.
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The `master field’ of large N gauge theories?

- We began our talk with planar diagrams and learnt from the work of BIPZ that one needs to find a set of equations that would compute the sum of planar diagrams in higher dim gauge theories. The `answer’ to this question came from a very unexpected direction and is deeply seated in:

- Ideas of duality in string theory and SUSY field theory(Kikkawa, Yamasaki; Font, Quevedo, Lust; Schwarz, Sen; Witten; Seiberg, Witten and others).

- The discovery of D-branes which are new degrees of freedom in gravity/string (Polchinski); and the model of black holes engineered from these degrees of freedom (Strominger and Vafa) that accounts for black hole entropy and equates it to Boltzmann’s formula. Slightly exciting these black holes one can study Hawking radiation and derive Hawking’s formulas from string theory (A. Dhar, G. Mandal, SRW; S. Das, S. Mathur).

The `master field’ of large N gauge theories…contd

- Putting all these insights together in 1996 Juan Maldacena proposed the AdS/CFT correspondence in which at large N and large λ, the `master field’ is a solution of Einstein’s equations of general relativity in 1 higher dimension. Asymptotically the space-time is AdS.
- There have been many tests of this correspondence over the past 25 years…and many new discoveries on the way…
- Here too matrix models have been useful e.g., the derivation of the Gopakumar-Vafa duality using matrix models, see Marcos Marino Les Houches lectures (2005) for this and other applications of matrix models to geometry and duality.
- Unitary matrix models became useful in the study of black holes in AdS (Aharony, Marsano, Minwalla, Papadodimas, Raamsdonk; Alvarez- Gaume, Basu, Gomez, Liu, Marino, SRW; Maldacena, Milekhin;…)

Sachdev-Ye-Kitaev (SYK) model and 2-dim gravity

- N real fermions at a space point interacting by a random all to all interaction. There is no spin glass phase.
- Low temperature behavior governed by elements of Diff S^1 i.e., f(t) with f'(t) > 0. The effective action is the Schwarzian $\int \{f(t),t\}dt$.
- It turns out that this model is holographically dual to a models of 2-dim gravity called JT gravity at finite temperature and the effective action is the same Schwarzian.
- Here too RMT comes in and I will refer you to the literature: e.g., see Shenker et al ArXiv:[1611.04650]; Stanford, Witten ArXiv:[1907.03363]
- I will leave you with a question: One could add interactions to the SYK model so that one can possibly have a transition to a spin glass phase.
- Q: What would be the description of the gravity dual of this phase?

Congratulations Giorgio!

Your scientific journey is an inspiration

Thank you.

Thanks to Abhishek Dhar, Chandan Dasgupta, Samriddhi Sankar Ray Smarajit Karmakar and the ICTS for organizing this celebration of the work of Giorgio Parisi.