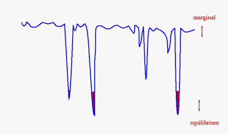
The replica-dynamic correspondence

in finite dimensions.

ICTS 2021

1. Parisi scheme

Equilibrium: $\overline{Z_J^n}$ $\qquad n = 1, 2, ...$
Parisi ansatz and continue to $n \to 0$ **gives** $\overline{\ln Z_J}$
Replica trick + Parisi ansatz



Equilibrium states

$\overline{Z_J^n}$ $\qquad NQ_{ab} = \sum_i \langle s_i^a s_i^b \rangle$

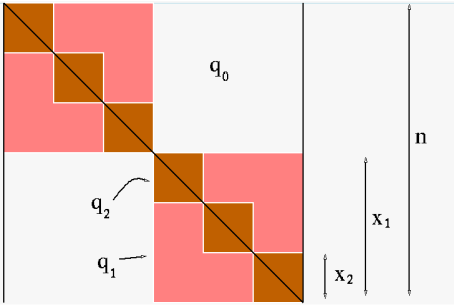
$a, b = 1, ..., n$

and continue to $n \to 0$ to get the 'good object' $\overline{\ln \ Z_J}$

Replica trick + Parisi ansatz

$0 \leq x_1 \leq x_2 \leq 1$ **nb!**

$q_0 \leq q_1 \leq q_2 \leq 1$

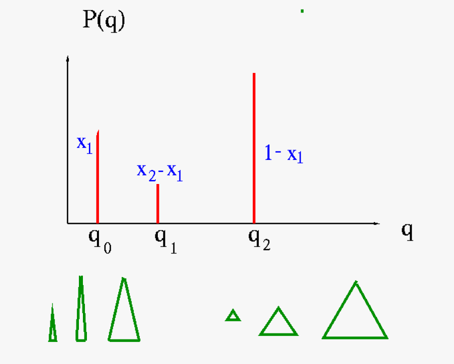


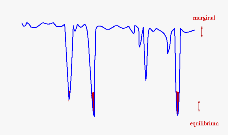
Probability of finding two states at overlap q

$$P(q) \equiv \frac{dX}{dq}$$

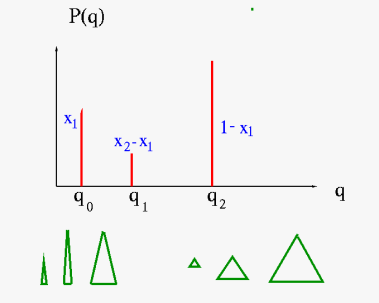
on average over disorder

$$q_{31} = \min[q_{32}, q_{21}]$$



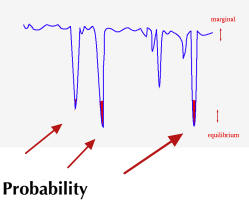


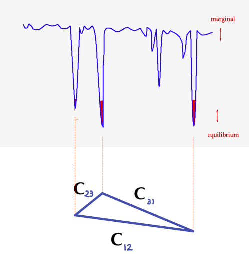
Possible triangles



Ultrametricity $\equiv$ isosceles with equal larger sides

This is what theory gives us:





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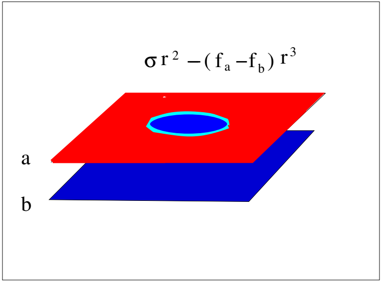
nb: these states are never visited by a physical dynamics.

A general consideration, however:



If the Hamiltonian H is short range and $H + \mu A$ is also short range

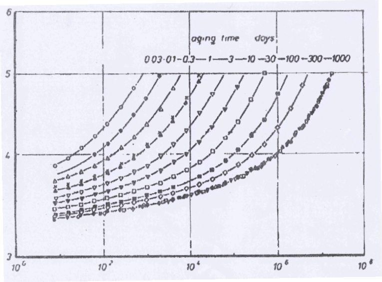
then for large $O(1)$ times $\chi_A = \frac{1}{N}\frac{\delta\langle A\rangle_t}{\delta\mu} = \frac{1}{N}\frac{\delta\langle A\rangle_{equil}}{\delta\mu}$

and in particular $\frac{\langle E\rangle_t}{N} = \frac{\langle E\rangle_{equil}}{N}$

2. Three versions of glass dynamics

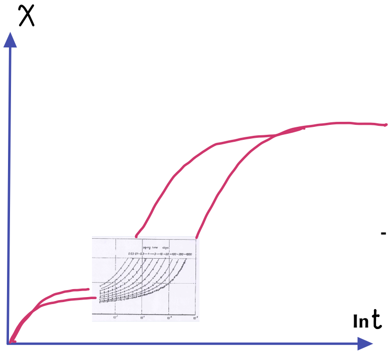
- aging
- weak shear
- slowly changing interactions

Aging



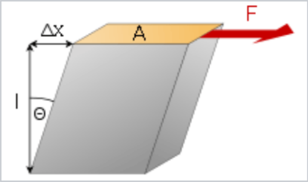
the stretching of a plastic bar, from an hour to four years old (Struik)

α-scale grows with waiting time



Weak shear

(the secret of eternal old age)



Shear strain

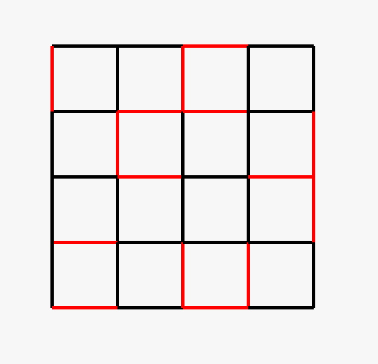
bend the ruler back and forth a bit every day

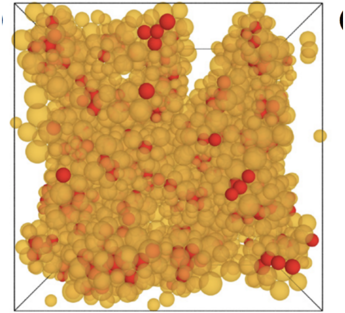
α-scale constant, longer for small shear

Slowly evolving interactions

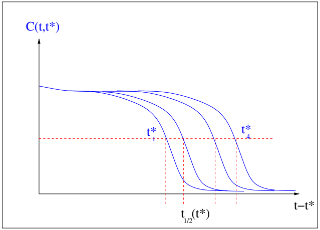
change couplings, the radii of spheres

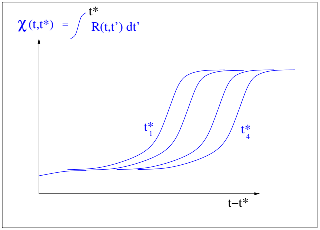
(the secret of eternal old age)

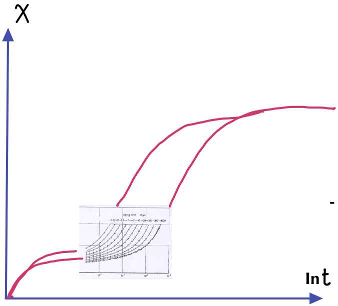


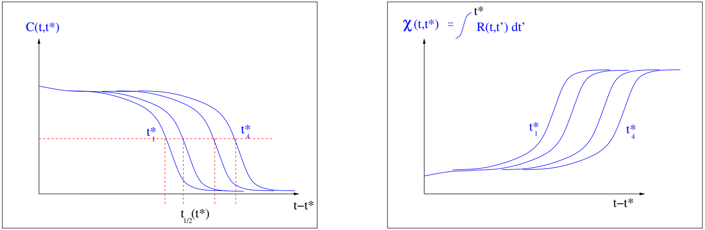


α-scale constant, longer for small evolution rate


C(t,t*)
t*1
t*4
t1/2(t*)
t−t*


χ(t,t*) = ∫^{t*} R(t,t') dt'
t*1
t*4
t−t*


χ
ln t



Slowness parameters $t^*_{1/2}$

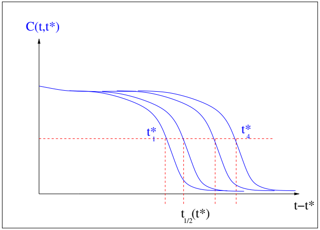
- waiting time t_w
- shear rate $\frac{1}{\dot\gamma}$
- characteristic time of parameter change τ_1

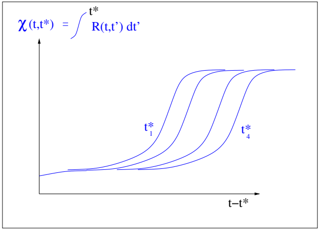
3. Reparametrization softness

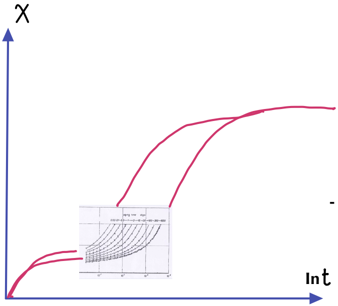
$$\underbrace{C_{slow}(t,t') \to C_{slow}(h(t),h(t'))}_{\textit{a correlation}}$$

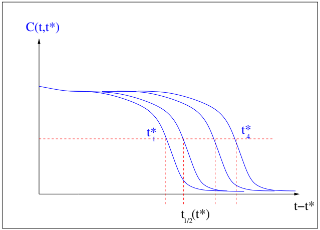
$$\underbrace{\chi_{slow}(t,t') \to \chi_{slow}(h(t),h(t'))}_{\textit{response field acting } t' \to t}$$

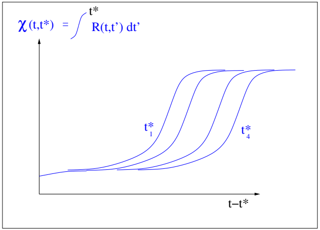
It seems reasonable to concentrate first in reparametrization-invariant quantities


C(t,t*)
t*1
t*4
t1/2(t*)
t−t*

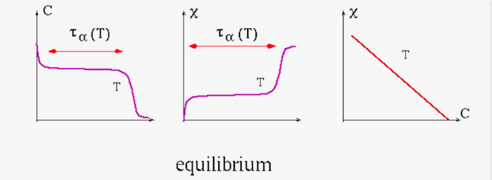

χ(t,t*) = ∫ t* R(t,t') dt'
t*1
t*4
t−t*


χ
ln t





In equilibrium, we would have:



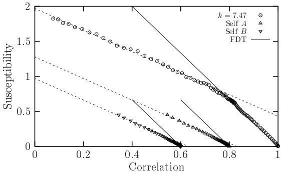
Fluctuation-Dissipation: a symmetry between the fields C, R

Reparametrization-invariant quantities

$$C_a(t,t') = g[C_b(t,t')] \rightarrow C_a = g[C_b]$$

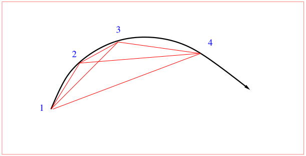
$\chi(t,t')$ vs. $C(t,t') \rightarrow \chi(C)$

or $\quad \frac{d\chi}{dC} = \underbrace{X(C)}_{\text{remember}}$



$C_{31} = \mathcal{F}[C_{32}, C_{21}]$

triangle relations

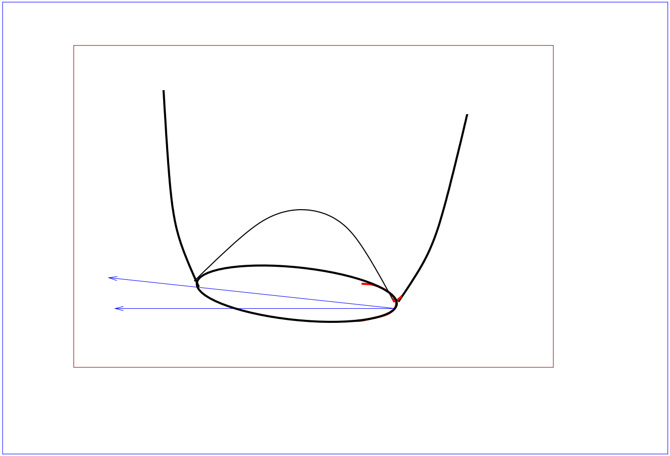


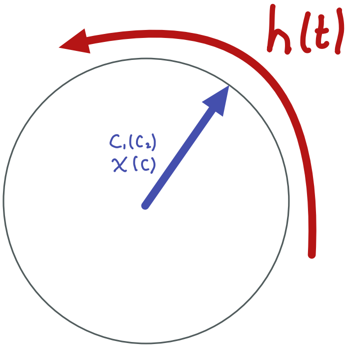
$f(a,b)$ is an associative function, and all possibilities may be classified

$$\text{If } \mathcal{F}[C(t_3,t_2),C(t_2,t_1)] = \min[C(t_3,t_2),C(t_2,t_1)]$$

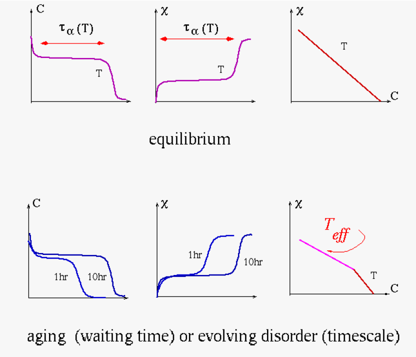
then (t_3, t_2) and (t_2, t_1) are in different timescales

because one of the relaxations takes
negligible time respect to the other



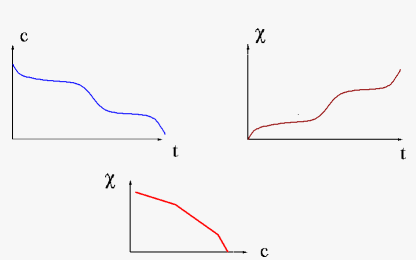

h(t)
$C_1(C_2)$
$\chi(C)$

4. Multithermalization



The system has developed a new temperature!

some systems have <u>three</u> nested timescales



and develop <u>two</u> new temperatures! If:

$$T(t_3, t_2) \neq T(t_2, t_1) \quad \Rightarrow \quad \mathcal{F}[C_{32}, C_{21}] = \min[C_{32}, C_{21}]$$

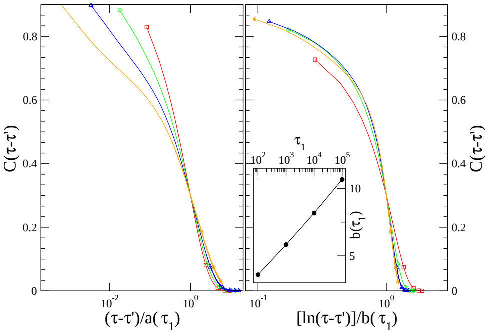
All in all: Multithermalization

- two or more scales
- a temperature for each timescale
- the same for all observables at the same timescale

An important case is infinitely many timescales!

$$C_{31} = f(C_{21}, C_{32}) = \min[C_{21}, C_{32}] \quad \forall C$$

$$C(t,t') = \mathcal{C}\left(\frac{\ln(t-t'+1)}{\ln \tau_1}\right) \qquad \to e^{\ln(\tau_1)\,\mathcal{C}_{31}} = e^{\ln(\tau_1)\,\mathcal{C}_{32}} + e^{\ln(\tau_1)\,\mathcal{C}_{21}}$$



one slow scale

infinitely many scales

Sherrington-Kirkpatrick, couplings evolving with timescale τ_1

5. Relation between statics and dynamics

another way to introduce $P(q) \equiv \frac{dX}{dq}$

add interactions

$$H \to H + \epsilon^1 \sum h_i s_i + \epsilon^2 \sum h_{ij} s_i s_j + \epsilon^3 \sum h_{ijk} s_i s_j s_k + \dots$$

the h's are Gaussian and fully-connected

$$I^p = \frac{\delta}{\delta \epsilon^p} \left\langle \sum h_{i_1 \dots i_p} s_{i_1 \dots i_p} \right\rangle = \int dq \, \frac{dx^{Parisi}}{dq} \, q^p \qquad \forall p$$

$$I^p(t \to \infty) = \frac{\delta}{\delta \epsilon^p} \left\langle \sum h_{i_1 \dots i_p} s_{i_1 \dots i_p} \right\rangle \to \int dq \, \frac{dx^{dynamic}}{dq} \, q^p \qquad \forall p$$

Dynamic and Parisi $x(q)$ are generators of the same susceptibilities

Franz, Mézard, Parisi and Peliti:

In finite dimensions: $X^{dynamic}(q) = x^{Parisi}(q)$

and 'replica' quantities become measurable

Outline of the argument

1. Nucleation argument: observe that in finite dimensions, with short range interactions, all susceptibilities tend to the equilibrium value for long times ($t \to \infty$ after $N \to \infty$, as in experiments).

2. Original perturbations $h_{i_i,\dots,i_p}$ are traded for special 'diluted' ones, and it is argued that this changes nothing. Those interactions may be considered as short-range.

Recently, under the same assumptions:

$$\frac{\gamma}{N}\sum_{ijk}\left(h^{(1)}_{ij}h^{(2)}_{kj}+h^{(1)}_{ij}h^{(3)}_{kj}+h^{(2)}_{ij}h^{(3)}_{kj}\right)s_k s_i$$

$$\frac{\gamma}{N^{(3p-1)/2}}\sum_{IJK}\left(h^{(1)}_{IJ}h^{(2)}_{KJ}+h^{(1)}_{IJ}h^{(3)}_{KJ}+h^{(2)}_{KJ}h^{(3)}_{IJ}\right)s_{i_1}\dots s_{i_p}\; s_{k_1}\dots s_{k_p}$$

with $I=i_1,\dots,i_p$, $J=j_1,\dots,j_r$, $K=k_1,\dots,k_s$

Proof à la Franz, Mézard, Parisi and Peliti:

Parisi ultrametricity if and only if dynamic ultrametricity.

6. Importance of reparametrization softness

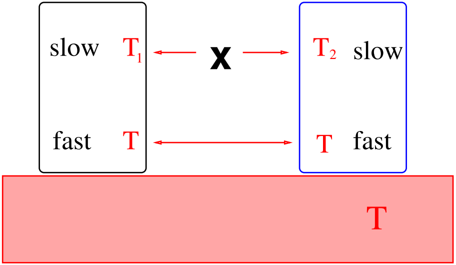
Multithermalization

- two or more scales
- a temperature for each timescale
- the same for all observables at the same timescale

Implies Parisi solution and reciprocally

A problem:

How can this even work?



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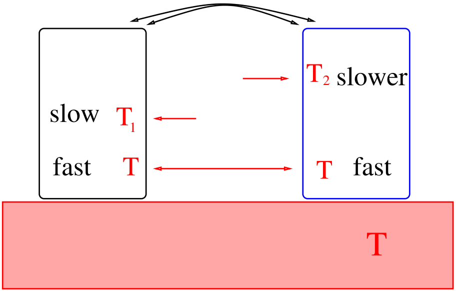
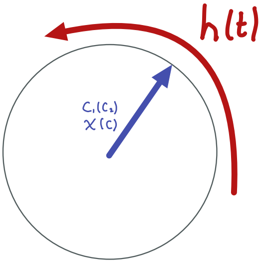


Reparametrizations arrange timescales

and save the scenario.

7. The sigma model



write a theory for $h(t, x)$ and fix the rest!

- In dynamics Castillo, Chamon, Cugliandolo, Kennett 2003-2005
  an effective theory for shear response, aging and fluctuations

- In replicas Brézin and deDominicis 2002-2004
  the first step (with problems)

- In imaginary-time quantum dynamics Kitaev, Maldacena-Shenker-Stanford 2016
  complete theory $\rightarrow$ QFT,
  σ-model of reparametrizations $\rightarrow$ gravity

In replicas Brézin and deDominicis 2002-2004
the first step (with problems)

- **Start from K-step RSB but X's fixed by marginality may perhaps eliminate contradictions**

- **what would this theory give us?**
  Perhaps an understanding of the $3d$ case

Worth an effort...