Multifractal approach to fully developed turbulence

Angelo VULPIANI

Dept. of Physics, University of Rome "Sapienza", Italy

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Summary of the talk

Difficulties with an approach to turbulence using only first principles

Richardson and the self similarity

Kolmogorov and scaling laws

Anomalous scaling and multifractals

Prediction of statistical features from the multifractal approach

From Richardson to Anomalous Scaling in Multifractals

On the shoulders of giants

L.F. RICHARDSON & A.N. KOLMOGOROV

And the collaboration of friends and coworkers

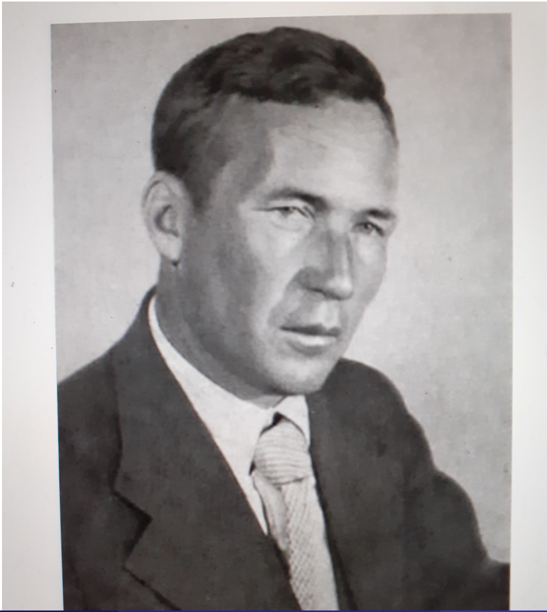
Giovanni Paladin, Giorgio Parisi, Roberto Benzi
+ (in alphabetic order):
E. Aurell, L. Biferale, G. Boffetta, T. Bohr,
A. Celani, M. Cencini, A. Crisanti, M.H. Jensen,
A. Mazzino, P. Muratore-Ginanneschi,
S. Musacchio, M. Vergassola, D. Vergni

Leonardo da Vinci (1452 - 1519)

The first description of turbulence

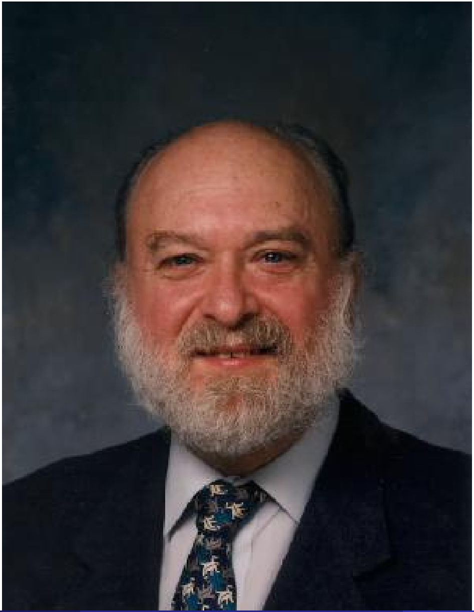
Lewis Fry Richardson (1881 - 1953)

Andrei Nikolaevich Kolmogorov (1903 - 1987)



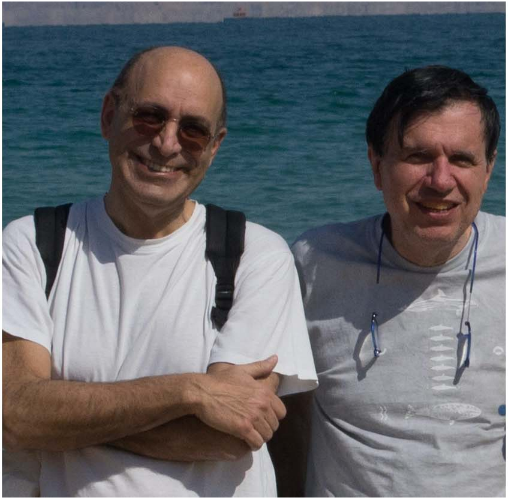
Benoit Mandelbrot (1924 - 2010)

Leo Kadanoff (1937 - 2015)



Uriel Frisch

Roberto Benzi & Giorgio Parisi



AV & Giovanni Paladin (1958 - 1996)

Why it is difficult to understand fully developed turbulence

Fully developed turbulence indicates the phenomena
at very high Reynolds number

One has the following unpleasant properties:

- Many degrees of freedom
- Strong nonlinearities
- Non Hamiltonian
- Non equilibrium
- Non Gaussian
- Difficulties in the direct numerical simulations

The troubles in the building a theory from the first principle

The Navier-Stokes equations

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \Delta \mathbf{u} + \mathbf{F} \;\; , \;\; \nabla \cdot \mathbf{u} = 0$$

the limit $\nu \to 0$, has no relation at all with the Euler equation

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p \;\; , \;\; \nabla \cdot \mathbf{u} = 0$$

The limit $\nu \to 0\,,\, \mathbf{F} \to 0$ in NS equation is different from the Euler equations, we are in presence of a singular limit and the statistical features are completely different.
For inviscid fluid, once an ultraviolet cutoff is introduced in the Fourier series of the velocity field, it is possibile to build in a simple way a statistical theory: it is enough to use the Liouville theorem and the energy conservation and follow the usual approach used for the standard statistical mechanics of Hamiltonian system.

Fluid in a box L^3 with periodic boundary conditions and a cutoff

$$\mathbf{u}(\mathbf{x},t) = \frac{1}{L^{3/2}} \sum_{|\mathbf{k}|<K_M} \hat{\mathbf{u}}(\mathbf{k},t) e^{i\mathbf{k}\cdot\mathbf{x}}$$

using the independent variables $\{X_n\}$ one has

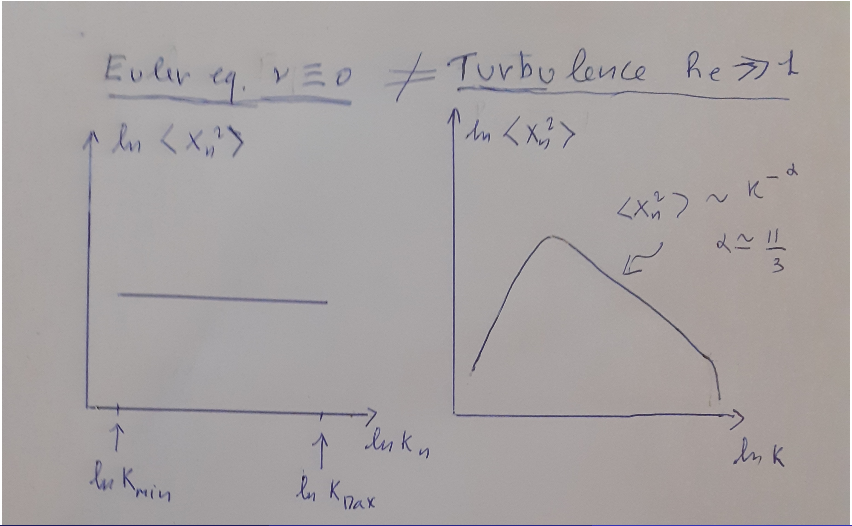
$$\frac{dX_n}{dt} = -\nu k_n^2 X_n + \sum_{j,l} M_{njl} X_j X_l + f_n \quad , \quad n = 1, 2, ..., N \sim K_M^3$$

• Euler equation $\nu = f_n = 0$ one has $\sum_n \frac{\partial}{\partial X_n} \frac{dX_n}{dt} = 0$ and $\frac{1}{2} \sum_n X_n^2 = E = constant$.
Following the same reasoning used for the statistical mechanics of Hamiltonian system one has $P(\{X_n\}) = C\,\delta\Big(\frac{1}{2}\sum_n X_n^2 - E\Big)$ and equipartition $\langle X_n^2 \rangle = 2E/N$, in the limit of large N one has a gaussian distirubution

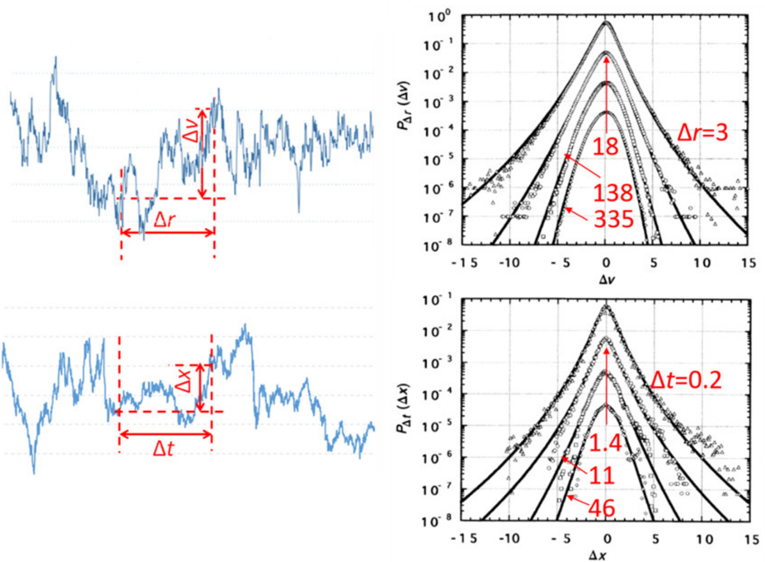
$$P(\{X_n\}) \sim exp - \frac{\beta}{2} \sum_n X_n^2$$

But the Euler equation is not the limit $R_e \to \infty$...

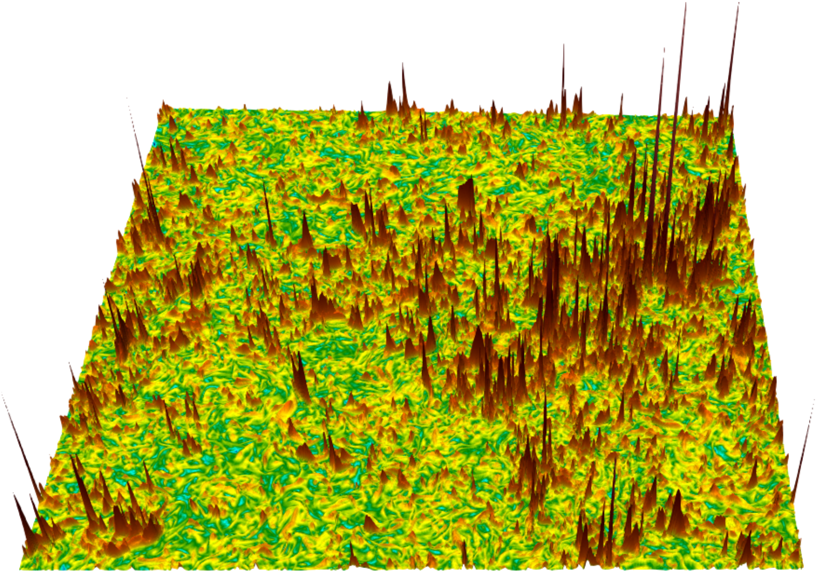
For $R_e \to \infty$ **one has** $\epsilon = \frac{\nu}{2} \sum_{i,j} \langle (\partial_j u_i + \partial_i u_j)^2 \rangle = O(1)$



Turbulence (top) and financial market (bottom)



Intermittent behaviour

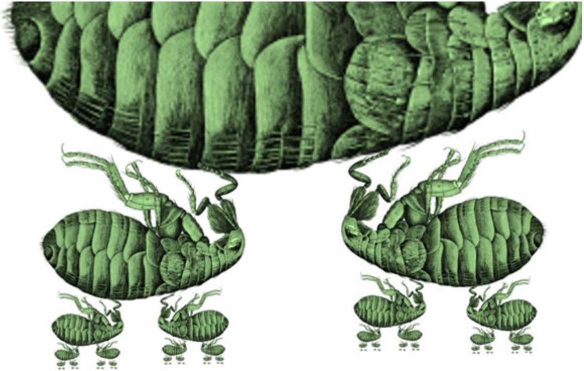


It's now rather clear that an analytical approach is very difficult (impossible?)

Because of the nonlinear character of the system in order to compute $\langle X_n^2 \rangle$ one has to deal with $\langle X_n X_j X_k \rangle$, then for these correlation one need $\langle X_n X_j X_k X_m \rangle$,and so on, this is the well know problem of the hierarchy.
It is necessary to **close** the infinite set of the equations.

The situation is similar to the BGGKY hierarchy in kinetic theory for dilute gases, but here a simple approach, e.g. assuming that
$\langle X_n X_j X_k X_m \rangle = \langle X_n X_j \rangle \langle X_k X_l \rangle + \langle X_n X_k \rangle \langle X_j X_l \rangle + \langle X_n X_l \rangle \langle X_k X_m \rangle$
gives inconsistent results.
Therefore for the closure path one is forced to use some phenomenological ideas (e,g. Heisenberg 1948, Eddy Damped Quasi Normal Markovian approximation, etc)

Fleas and self-similarity



So, nat'ralists observe, a flea
Hath smaller fleas that on him prey;
And these have smaller yet to bite 'em,
And so proceed ad infinitum.

Jonathan Swift

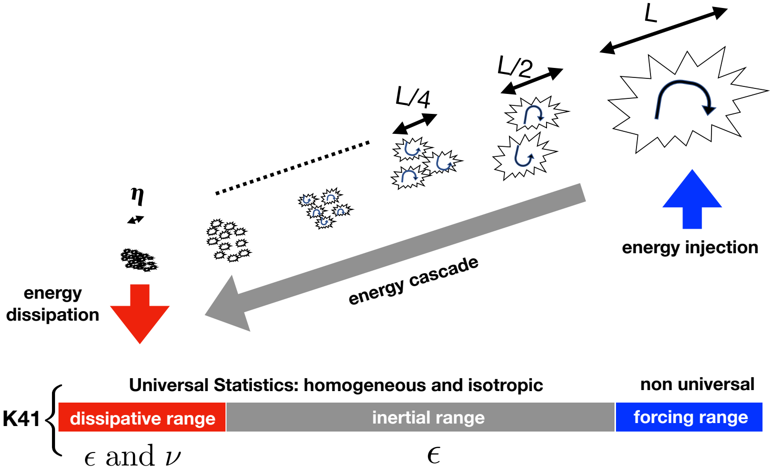
Richardson and self-similarity

Richardson seriously asked himself the (apparently crazy) question
Does the wind have a speed?
Starting from just a few empirical data, he guessed the self-similar structure of turbulence; here is how he summarised his insight in a verse (inspired by a satirical one by Swift):

Big whirls have little whirls
that feed on their velocity,
and little whirls have lesser whirls
and so on to viscosity
in the molecular sense.

Now we know that such a kind of behaviours is rather common and it appears in many natural phenomena; for instance, in turbulence, cosmology, geophysics.

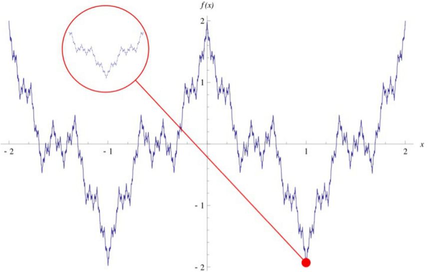
A cartoon of the cascade



A simple self-similar function (by Weierstrass)

$$\mathbf{f(x) = \sum_{n=1}^{\infty} A^{-n} \cos(2\pi B^{n-1} x)} \ , \ \ \mathbf{0 \leq x \leq 1}$$

B is integer and $A < B$, $f(x)$ is not differentiable in any point.



One has $f(x + \Delta x) - f(x) \sim \Delta x^h$ where $h = 2 - D_F = \ln A / \ln B < 1$, this is the simplest case on self-similarity with **a single exponent**.

Kolmogorov realized that it is not necessary (impossible?) to insist for a theory from the first principles.

When Kolmogorov was close to eighty I asked him about the history of his discoveries of the scaling laws. He gave me a very astonishing answer by saying that for half a year he studied the results of concrete measurements. ... Kolmogorov was never seriously interested in the problem of existence and uniqueness of solutions of the Navier-Stokes system. He also considered his theory of turbulence as purely phenomenological and never believed that it would eventually have a mathematical framework.

(Ya.G.Sinai in **The Kolmogorov Legacy in Physics** page V, R.Livi and AV Ed.s 2003)

• Kolmogorov devised (in part, following Richardsons's ideas about the turbulent cascade) the first modern theory of turbulence (K41). From some physical argument as well as exact result from the Navier-Stokes equation, and the (experimental) remark that the energy dissipation $\epsilon \sim \nu |\nabla \mathbf{u}|^2$ is $O(1)$ in the limit $R_e \to \infty$

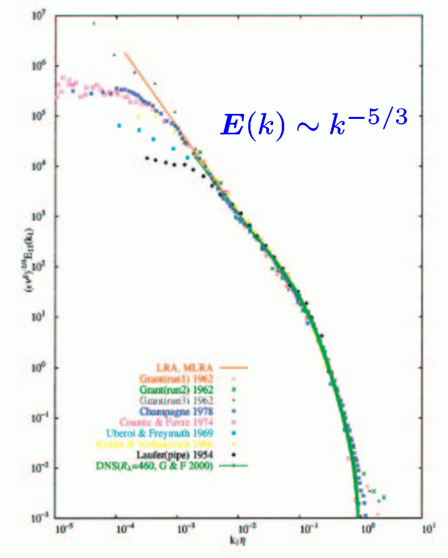
$$\delta v(\ell) \sim \ell^h \ , \ \ h = \frac{1}{3}$$

in the inertial range $\eta \ll \ell \ll L$, where L and U are the typical length and velocity respectively, $\eta = L R_e^{-\frac{3}{4}}$ is the Kolmogorov length, $R_e = \frac{UL}{\nu}$ is the Reynolds number and $\delta v(\ell)$ is the (longitudinal) velocity difference between two points at distance ℓ.

• Landau's remark: since the K41 is a "sort of mean field" it cannot be exact, it is necessary to take into account the fluctuations.

Experimental results: the 5/3 spectrum

The K41 $h = 1/3$ corresponds to $E(k) \sim k^{-5/3}$



• **Experimental data about intermittency** support Landau's criticism: one exponent is not enough

$$\langle |\delta v(\ell)|^p \rangle \sim \ell^{\zeta_p} \ , \ \zeta_p \neq \frac{p}{3}$$

• **Again Kolmogorov K62**, a lognormal approach

$$\zeta_p = \frac{p}{3} + \frac{\mu}{18} p(3-p)$$

where μ is a measure of the fluctuations; K62 is surely better than K41 (two parameters $1/3$ and μ) but there are still some troubles.

• **Multifractal model** 1983-1984 (Benzi, Frisch, Paladin, Parisi, AV) Something less than a theory, but it allows for the possibility to do precise previsions in terms of a unique "ingredient" which can be obtained from experimental data, e.g. the existence of an intermediate dissipation range (Frisch and Vergassola), the PdF for the gradient of the velocity (Benzi et al) and of the acceleration of particle advected by a turbulent field (Biferale et al).

Our first paper on multifractals

J. Phys. A: Math. Gen. 17 (1984) 3521-3531. Printed in Great Britain

On the multifractal nature of fully developed turbulence and chaotic systems

Roberto Benzi†, Giovanni Paladin‡, Giorgio Parisi§ and Angelo Vulpiani‡

† Centro scientifico IBM di Roma, via Giorgione 129, 00100, Roma, Italy
‡ Università di Roma 'La Sapienza', Dipartimento di Fisica, and GNSM-CNR unità di Roma, P.le Aldo Moro 2, 00100, Roma, Italy
§ Università di Roma II, 'Tor Vergata', Dipartimento di Fisica, 00173, Roma and Laboratori Nazionali INFN, Frascati, Italy

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Abstract. It is generally argued that the energy dissipation of three-dimensional turbulent flow is concentrated on a set with non-integer Hausdorff dimension. Recently, in order to explain experimental data, it has been proposed that this set does not possess a global dilatation invariance: it can be considered to be a multifractal set. In this paper we review the concept of multifractal sets in both turbulent flows and dynamical systems using a generalisation of the β-model.

The multifractal model in a nutshell

The Navier-Stokes equations

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \Delta \mathbf{u} + \mathbf{F} \ , \ \ \nabla \cdot \mathbf{u} = 0$$

are invariant under the scaling transformation

$$\mathbf{x} \to \lambda \mathbf{x} \ , \ \mathbf{u} \to \lambda^h \mathbf{u} \ , t \to \lambda^{1-h} t \ , \ \nu \to \lambda^{1+h} \nu$$

The exponent h cannot be determined with only symmetry considerations, following a dimensional argument of the K41 and an exact result (the so-called "4/5 law") the natural candidate is $h = 1/3$.

The K41 theory corresponds to a global invariance with $h = 1/3$ and $\zeta_p = p/3$, in disagreement with several experimental investigations. This phenomenon, which goes under the name of **intermittency** is a consequence of the breakdown of self- similarity and implies that the scaling exponents cannot be determined on a simple dimensional basis.

The multifractal model of turbulence assumes that the velocity has a local scale-invariance, i.e. there is continuous spectrum of exponents h , each of which belonging to a given fractal set.

• The assumption is that in the inertial range one has $\delta v(\ell, \mathbf{x}) \sim \ell^h$ if $\mathbf{x} \in S_h$ where S_h is a fractal set with dimension $D(h)$ and $h \in (h_m, h_M)$

• Noting that probability to have a given scaling exponent h at the scale ℓ is

$$P(\ell, h) \sim \ell^{3-D(h)}$$

with simple steepest descent estimation one has

$$\zeta_p = \inf_h \Big(hp + 3 - D(h) \Big)$$

• For each value p one has a dominant singularity $\tilde{h}$ determined by the equation

$$p = \frac{dD}{dh}\Big|_{\tilde{h}} \;\; \rightarrow \;\; \zeta_p = \Big(p\tilde{h} + 3 - D(\tilde{h}) \Big)$$

• From a technical point of view the idea of the multifractal is basically contained in the large deviation theory, however, the introduction of the multifractal description in 1980s had an important role in statistical physics, chaos and disordered systems. In particular, to clarify in a neat way that the usual idea, coming from critical phenomena, that just few scaling exponents are relevant, is wrong, and an infinite set of exponents is necessary for a complete characterization of the scaling features.

• Of course the computation of $D(h)$ from the NSE is not at present an attainable goal. A first step is a phenomenological approach using multiplicative processes which generalize the K62 lognormal model, corresponding to a parabolic $D(h)$.

• It is not so astonishing to find a model to fit the experimental data, e.g. our multiplicative random β-model gives:

$$D(h) = 3 + (3h-1)\left[1 + \ln_2\left(\frac{1-3h}{1-x}\right)\right] + 3h\ln_2\left(\frac{x}{3h}\right)\ ,\ x = \frac{7}{8}\,.$$

Rome band (JPA 1984) $\longleftrightarrow$ Chicago band (PRE 1985)

- The $f(\alpha)$ vs α formalism (Halsey et al PRE 1985)

Given a singular measure $\mu(\mathbf{x}) \to$ partition with cells of size ℓ

$$P_i(\ell) = \int_{\Lambda_i(\ell)} d\mu(\mathbf{x}) \;\to\; \sum_i P_i(\ell)^q \sim \ell^{(q-1)d_q}$$

Denoting $f(\alpha)$ ithe fractal dimension of the regions such that $P_i(\ell) \sim \ell^{\alpha}$, one has the Renyi dimensions

$$d_q = \frac{1}{q-1} \inf_{\alpha} (\alpha q - f(\alpha))$$

- For each value q one has a dominant singularity $\tilde{\alpha}$ determined by the equation

$$q = \left.\frac{df}{d\alpha}\right|_{\tilde{\alpha}} \;\to\; d_q = \frac{1}{q-1}\Big(\tilde{\alpha} q - f(\tilde{\alpha})\Big)$$

There is a rather close relation between the two approaches, i.e. $f(\alpha)$ vs α and $D(h)$ vs h, it is enough to remind the K62 theory.

Formulating the K62 approach in term of multifractals one has $\delta v(\ell, \mathbf{x}) \sim (\epsilon_\ell(\mathbf{x})\ell)^{1/3}$ where $\epsilon_\ell(\mathbf{x})$ is the energy dissipation on the cell of size ℓ and center in $\mathbf{x}$.
Since the energy dissipation is non negative one can introduce a measure

$$\mu(\mathbf{x}) = Const.\, \epsilon(\mathbf{x})$$

Simply manipulations show $\zeta_q = \frac{q}{3} + \left(\frac{q}{3} - 1\right)(d_{\frac{q}{3}} - 3)$

$$h \longleftrightarrow \frac{\alpha - 2}{3} \; , \; D(h) \longleftrightarrow f(\alpha) \; , \; f(\alpha) \le \alpha \longleftrightarrow D(h) \le 3h + 2$$

$$\exists \alpha^* : \alpha^* = f(\alpha^*) = d_1 \longleftrightarrow \exists h^* : 3h^* + 3 - D(h^*) = \zeta_3 = 1$$

A multiplicative process: random β model

Energy is injected at scale L; at the n-th step of the cascade a mother eddy of size $\ell_n = L2^{-n}$ splits into daughter eddies of size ℓ_{n+1} and the daughter eddies cover a fraction $\beta_j \in (0,1)$ of the mother volume.
Since the energy transfer is constant throughout the cascade one has for the velocity difference v_n on the scale ℓ_n is $v_n = v_0 \ell_n^{1/3} \prod_{j=1}^{n} \beta_j$ where β_j are independent, identically distributed random variables.

$$\zeta_q = \frac{q}{3} - \ln_2 \langle \beta^{1-q/3} \rangle$$

Phenomenological arguments suggest $\beta_j = 1$ with probability x and $\beta_j = 1/2$ with probability $1 - x$. The scaling exponents are

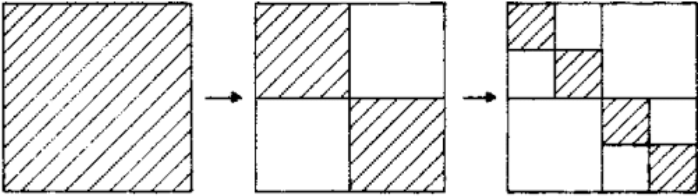
$$\zeta_q = \frac{q}{3} - \ln_2 \left(x + (1-x) 2^{\frac{q}{3}-1} \right)$$

The two limit cases are $x = 1$ (the K41), and $x = 0$ which is the fractal β -model. Using $x = 7/8$ one has a good fit for the ζ_q of the experimental data at high Reynolds numbers.

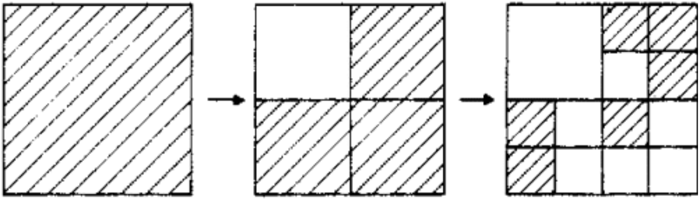
(a) Schematic view of the β -model

(b) Schematic view of the random β-model.

The shaded areas are the zones active during the fragmentation process.

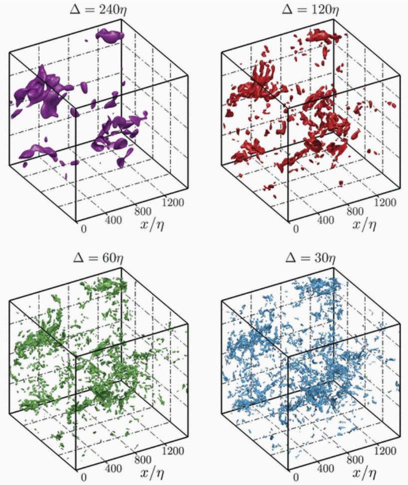


(a)

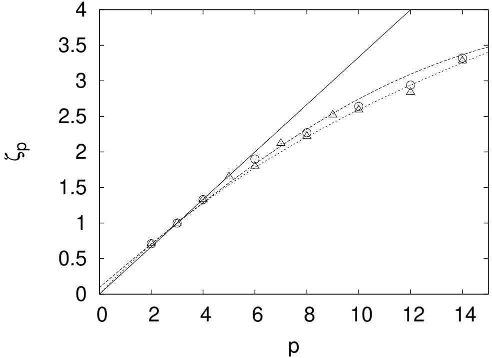


(b)

A more artistic sketch



Scaling exponents ζ_p vs p, of the structure functions



$\langle|\delta v(\ell)|^p\rangle \sim \ell^{\zeta_p}$ Circles and triangles are the experimental data (Anselmet et al.) The solid line corresponds to K41 scaling $p/3$; the dashed line is the random β-model prediction of Benzi et al, the dotted line is the She-Leveque model.

A non unique Kolmogorov length...

In the K41, since one has a unique scaling exponent $\delta v(\ell) \sim \ell^{1/3}$, then there is just a unique Kolmogorov length η:

$$\frac{\delta v(\eta)\eta}{\nu} \sim 1 \;\; \rightarrow \;\; \eta \sim \left(\frac{\nu}{U}\right)^{\frac{3}{4}}$$

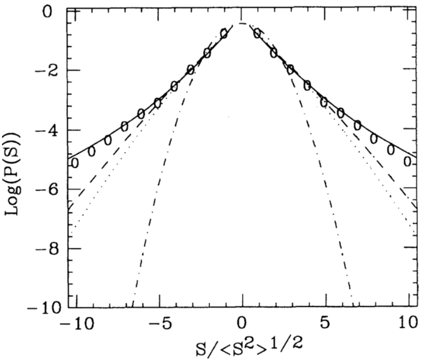
In the multifractal model one has $\delta v(\ell) \sim \ell^{h}$, therefore for each valure of h there is Kolmogorov length $\eta(h)$:

$$\frac{\delta v(\eta)\eta}{\nu} \sim 1 \;\; \rightarrow \;\; \eta \sim \left(\frac{\nu}{U}\right)^{\frac{1}{1+h}}$$

As consequence of many Kolmogorov lengths one can
• determine the Pdf of the velocity gradient, of the acceleration etc
• show the existence of a new kind of scaling for the intermediate dissipative range (Frisch and Vergassola 1991, Jensen et al 1991)

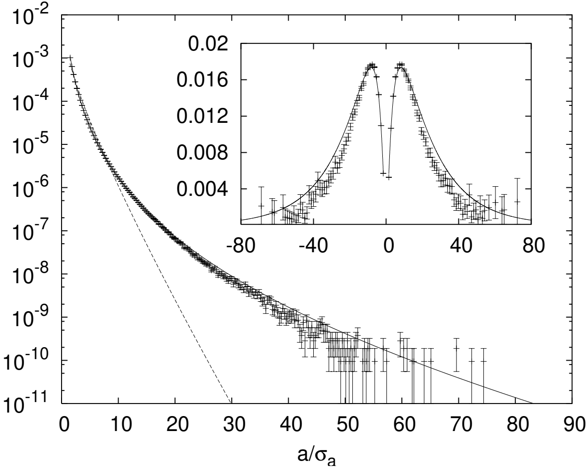
$D(h) \to$ the PdF of the velocity gradient s

$$p(s) \sim \int dh \Big(\frac{\nu}{|s|}\Big)^{2-\frac{h+D(h)}{2}} exp\Big(-\frac{\nu^{1-h}|s|^{1+h}}{2U^2}\Big)$$



Points represent experimental data, solid line is the multifractal prediction with the random β- model, dotted and dashed lines represent the K41 and β-model results, respectively.

The PdF of the acceleration



Points are the DNS data, the solid line is the multifractal prediction. the dashed line is the K41 prediction. **NOTE THE VALUES OF** a/σ_a **!!**

Relative diffusion in turbulence: beyond Gaussian processes

The problem is the behavior of the distance R between two particles advected by a turbulent field.

Richardson (1926) in order to take into account the turbulent velocity field, proposed a diffusion equation in which the diffusion coefficient depends on the distance.

$$\frac{\partial}{\partial t} P(R,t) = \frac{1}{R^{d-1}} \frac{\partial}{\partial R}\left(R^{d-1} D(R) \frac{\partial}{\partial R} P(R,t)\right)$$

where d is the spatial dimension, and the diffusion coefficient $D(R)$ depends on the distance between the two particles. From (few) data LFR had been able to guess a scaling law: for $d = 3$, $D(R) \sim R^{4/3}$, obtaining the non standard diffusion

$$< R^2(t) > \sim t^3 \, .$$

Now we know that this approach basically follow from K41; what about the intermittency?
One has an anomalous scaling

$$< R^p(t) > \sim t^{\alpha(p)}$$

with $\alpha(p) \neq \frac{3}{2}p$.

From the multifractal model one has a prediction for $\alpha(p)$ in terms of $D(h)$ (Boffetta et al):

$$\alpha(p) = \inf_h \left[\frac{p + 3 - D(h)}{1 - h} \right] .$$

It is remarkable that, even in presence of intermittency, the Richardson scaling $\alpha(2) = 3$ is exact; the multifractal prediction has been checked in synthetic turbulence, where the velocity field is a random process with the proper statio-temporal statistics (Boffetta et al) as well as in direct numerical simulations of the NS equations (Boffetta and Sokolov).

Intemediate dissipative range

• Standard scaling:

$$E(k) = F(k/k_D) \ , \ k_D = \frac{1}{\eta} \sim R_e^{3/4}$$

$F(z) \sim z^{-5/3}$ for $z \ll 1$, $F(z) \sim e^{-cz}$ for $z \gg 1$

• Generalized scaling (consequence of the multifractality)

$$\frac{\ln E(k)}{\ln R_e} = G\Big(\frac{\ln k}{\ln R_e}\Big)$$

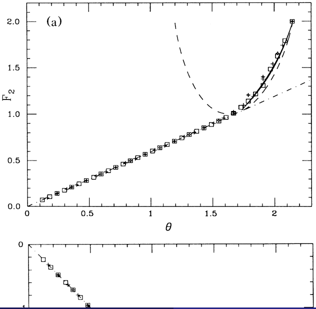
the shape of $G(z)$ depends on the $D(h)$ for $z < z_*$ one has the usual scaling shape with the minor change that $5/3$ is replaced by $1 + \zeta_2$, while for larger value of z the shape of $D(h)$ plays a role.

• The Generalized scaling, in particular in the intermediate dissipative range had been observed in the experimental data (Gagne and Castaing 1991), and for lagrangian turbulence (Arnéodo et al 2008).

An example of generalized scaling in dynamical systems

The invariant set at the accumulation point of period- doubling bifurcations of $x_{t+1} = \lambda(1 - 2x_t^2)$, $\lambda = 0.837005134....$, with a cutoff
MH Jensen et *Multiscaling in Multifractals* PRL 1991

VOLUME 67, NUMBER 2 PHYSICAL REVIEW LETTERS 8 JULY 1991



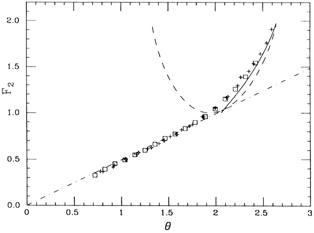


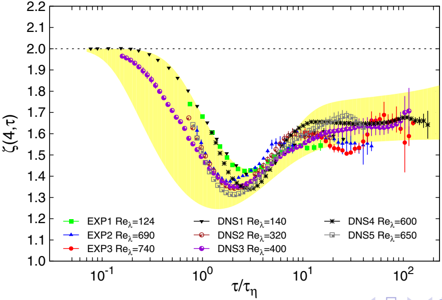
FIG. 3. $F_2(\theta) = \ln C_2/\ln\epsilon$ vs $\theta = \ln(l/l_0)/\ln\epsilon$ for the period-doubling repeller. The data for $C_2(l,\epsilon)$ are obtained at $\epsilon = 0.0015$ (squares), $\epsilon = 0.003$ (crosses), and $\epsilon = 0.006$ (crossed circles). The lines are the same as in Fig. 2, with $\tau(2) = 0.495$.

A very accurate test of the intermediate dissipative range

For the (rescaled) Lagrangian structure function

$$\frac{\langle (v_i(t+\tau) - v_i(t))^4 \rangle}{\langle (v_i(t+\tau) - v_i(t))^2 \rangle} \sim \tau^{\zeta(4,\tau)}$$

PRL **100,** 254504 (2008) PHYSICAL REVIEW LETTERS week ending 27 JUNE 2008

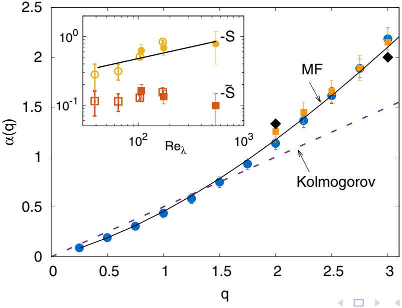


Again on Lagrangian properties: for the scaling of $p = \mathbf{v} \cdot \mathbf{a}$

$$\langle p^q \rangle \sim \epsilon^q Re_\lambda^{\alpha(q)}$$

(M. Cencini et al PHYS. REV. FLUIDS **2**, 104604 (2017))

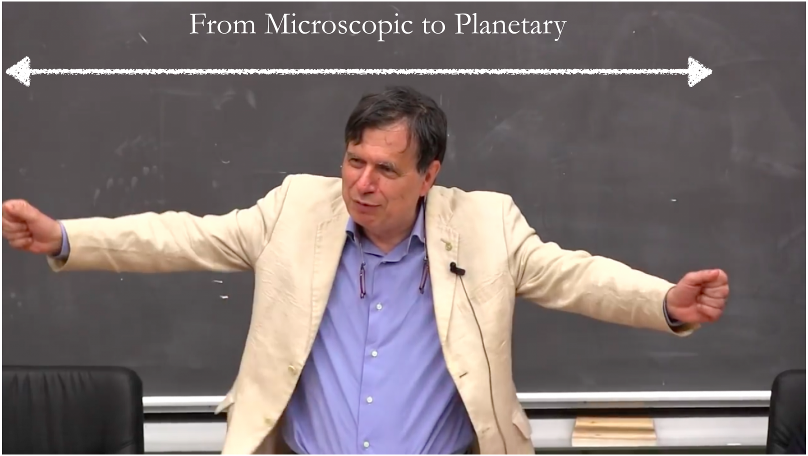
TIME IRREVERSIBILITY AND MULTIFRACTALITY OF ...



No conclusion (for now)

After ~ 40 **years of nice and amusing research, we'll try to go on...**

A Master of Disorder and Stochasticity



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