Today's menu:

Sketch of the construction of Beilinson–Kato zeta elements (+ε?)

: the most painful part of this lecture series

⚠ only sketch, no proof, no norm compatibility is checked.

$$\underbrace{{}_{c,d}z_m^{(p)}(f, r, r', \xi, S)}_{\substack{\downarrow \\ \mathbb{Q}(\zeta_m)}} \ \cdots \ ?$$

mod. forms

$\rightsquigarrow z_{\mathbb{Q}(\zeta_m)}$

$\exists$ explicit formula.

Need geometry of modular curves and specific ftns on them.

Modular forms and modular curves

$N \geq 3$.

$Y(N)$ = modular curve$/\mathbb{Q}$ of level $\Gamma(N)$ w/o cusps.

$$\mathbb{Q}\text{-sch} \longrightarrow \text{Sets}.$$

$$S \mapsto \left\{ (E, e_1, e_2) : \begin{array}{l} E : \text{ell. curve}/S \\ (e_1, e_2) \text{ a pair of sections of } E \\ \text{forms a } \mathbb{Z}/N\mathbb{Z}\text{-basis of } E[N] \end{array} \right\} \Big/ \simeq.$$

: smooth irred. affine curve.

(cf. the constant field of $Y(N) = \mathbb{Q}(\zeta_N)$)

$X(N) \supseteq Y(N)$ smooth compactification.

$$\begin{array}{ccc} Y(N\cdot n) & (E, e_1, e_2) & \\ \text{finite étale} \downarrow & \downarrow & \to \mathcal{O}(Y(N)) \subseteq \mathcal{O}(Y(Nn)) \\ & & \uparrow \\ Y(N) & (E, ne_1, ne_2) & \text{via pull-back.} \end{array}$$

<u>The very starting pt of the construction</u>:

<u>Prop.</u> E, ell. curve$/S$. $(c, 6) = 1$.

(1) $\exists!\ {}_c\theta_E \in \mathcal{O}(E \setminus E[c])^\times$ s.t.

(i) $\mathrm{Div}({}_c\theta_E) = c^2 \cdot (0) - E[c]$ on E.

(ii) $N_a({}_c\theta_E) = {}_c\theta_E$ for $\forall a$ with $(a, c) = 1$.

where $E \setminus E[ac] \xrightarrow{\times a} E \setminus E[c]$

$$\mathcal{O}(E \setminus E[c]) \to \mathcal{O}(E \setminus E[ac])$$

norm.

$$\mathcal{O}(E \setminus E[ac])^\times \xrightarrow{N_a} \mathcal{O}(E \setminus E[c])^\times$$

(2) $(d, 6) = 1$. Then

$$({}_d\theta_E)^{c^2} \cdot (c^*({}_d\theta_E))^{-1}$$

$$= ({}_c\theta_E)^{d^2} \cdot (d^*({}_c\theta_E))^{-1}.$$

in $\mathcal{O}(E \setminus E[cd])^\times$.

where $E \xrightarrow{\times c} E$, $E \xrightarrow{\times d} E$.

$$\rightsquigarrow \quad c^* : \mathcal{O}(E \setminus E[d])^\times \to \mathcal{O}(E \setminus E[cd])^\times.$$
$$d^* : \mathcal{O}(E \setminus E[c])^\times \to \mathcal{O}(E \setminus E[cd])^\times.$$

complex analytic.

(3) $\tau \in \mathfrak{h}$, $z \in \mathbb{C} \setminus c^{-1}(\mathbb{Z}\tau + \mathbb{Z})$.

${}_c\theta(\tau, z)$ = the value of ${}_c\theta_{\mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z})}$ (over $\mathbb{C}$) at z

$$\overset{!}{=} q^{\frac{1}{12}\cdot(c^2-1)} \cdot (-t)^{\frac{1}{2}\cdot(c-c^2)} \cdot \gamma_q(t)^{c^2} \cdot \gamma_q(t^c)^{-1}.$$

where $q = e^{2\pi i \tau}$, $t = e^{2\pi i z}$, and.

$$\gamma_q(t) = \prod_{n \geq 0} (1 - q^n \cdot t) \cdot \prod_{n \geq 1} (1 - q^n \cdot t^{-1}).$$

(4) If $h : E \to E'$ an isogeny of ell. curves$/S$ of degree prime to c,

then the corresponding norm map
$$h_* : {}_c\theta_E \mapsto {}_c\theta_{E'}.$$

* Everything will be obtained from this ${}_c\theta_E$.

Siegel units

$${}_c g_{\alpha,\beta} \in \bigcup_N \mathcal{O}(Y(N))^\times \quad \rightsquigarrow c, d\ \mathbb{Z} \cdots \text{ integral.}$$

$$g_{\alpha,\beta} \in \bigcup_N \mathcal{O}(Y(N))^\times \underline{\otimes \mathbb{Q}} \quad \rightsquigarrow \mathbb{Z} \cdots$$

denominators.

where $(\alpha,\beta) \in (\mathbb{Q}/\mathbb{Z})^{\oplus 2} \setminus \{(0,0)\}$.

$$(c, 6 \cdot \mathrm{ord}(\alpha) \cdot \mathrm{ord}(\beta)) = 1.$$

properties :

$$\left. \begin{array}{l} {}_c g_{\alpha,\beta} \in \mathcal{O}(Y(N))^\times. \\ g_{\alpha,\beta} \in \mathcal{O}(Y(N))^\times \otimes \mathbb{Q}. \end{array} \right) \text{ if } N\alpha = N\beta = 0.$$

Also, ~~${}_c g_{\alpha,\beta} \in \mathcal{O}$~~

$$ {}_c g_{\alpha,\beta} = (g_{\alpha,\beta})^{c^2} \cdot (g_{c\alpha, c\beta})^{-1}.$$

$$\in \mathcal{O}(Y(N))^\times \otimes \mathbb{Q}.$$

How to produce these elements?

E univ. ell. curve.

$\downarrow \uparrow e_1, e_2$: sections $\rightsquigarrow {}_c g_{\alpha,\beta}$

$Y(N) \quad N \geq 3.$

where N is chosen to satisfy $N\alpha = N\beta = 0$.

Write

$$(\alpha, \beta) = \left(\tfrac{a}{N}, \tfrac{b}{N}\right) \in \left(\tfrac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^{\oplus 2} \setminus \{(0,0)\}.$$

with $a, b \in \mathbb{Z}$.

$$\rightsquigarrow \quad \iota_{\alpha,\beta} = ae_1 + be_2 : Y(N) \to E \setminus E[c].$$

$$\left(\begin{array}{l} \mathrm{Im}(\iota_{\alpha,\beta}) \cap E[c] = \phi. \\ \text{since} \quad (c, \mathrm{ord}(\alpha)\cdot\mathrm{ord}(\beta)) = 1 \\ \qquad\quad \text{and } (\alpha,\beta) \neq (0,0) \end{array} \right)$$

$$\underline{\underline{{}_c g_{\alpha,\beta}}} := \iota_{\alpha,\beta}^{*}({}_c\Theta_E) \in \underline{\mathcal{O}(Y(N))^{\times}}.$$

$$g_{\alpha,\beta} = \underbrace{{}_c g_{\alpha,\beta} \otimes (c^2-1)^{-1}}_{\text{indep. of } c.} \in \mathcal{O}(Y(N))^{\times} \otimes \mathbb{Q}.$$

$\Leftarrow$ we take $(c, 6) = 1$
$c \equiv 1 \pmod N$
$c \neq \pm 1.$

$${}_c g_{\alpha,\beta} = (g_{\alpha,\beta})^{c^2} \cdot (g_{c\alpha, c\beta})^{-1}.$$

$\forall_c$ with $(c, 6N) = 1.$

A little bit of K-thy.

X, a regular separate Noeth. sch.

$\to K_i(X)$: Quillen K-gp. $(i \geq 0)$

with cup product

$$\cup : K_i(X) \times K_j(X) \to K_{i+j}(X)$$

① $\exists$ canonical hom.

$$\mathcal{O}(X)^\times \longrightarrow K_1(X)$$

② Steinberg symbol.

$$\mathcal{O}(X)^\times \times \mathcal{O}(X)^\times \longrightarrow K_2(X)$$

$$(u, v) \longmapsto \{u, v\}.$$

<u>Group action of $GL_2(\mathbb{Z}/N\mathbb{Z})$ on $Y(N)$</u>

$GL_2(\mathbb{Z}/N\mathbb{Z}) \circlearrowright Y(N)$ from the left.

$$\sigma = \begin{pmatrix} a & c \\ b & d \end{pmatrix} : (E, e_1, e_2) \mapsto (E, e_1', e_2')$$

where $\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$

and

$\sigma(\zeta_N) = \zeta_N^{\det(\sigma)}$ on the constant field.

<u>Euler systems in K_2.</u> 2 Siegel units $\rightsquigarrow$ 1 elts

$M, N \geq 1 \rightsquigarrow Y(M, N)$

Choose (a big) $L \geq 3$. s.t. $M \mid L$.

$N \mid L$.

$\rightsquigarrow Y(M,N) = {}_G \backslash Y(L)$ (indep. of L)

where $G = \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}/L\mathbb{Z}) :$

$: \begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix}$ (mod M)

$\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ (mod N) $\Big\}$

$X(M,N) \supseteq Y(M,N)$ smooth cptification.

$M+N \geq 5$, $Y(M,N)$ represents the functor.

$\mathbb{Q}$-sch $\longrightarrow$ Sets

$S \longmapsto \left\{ \begin{array}{l} (E, e_1, e_2) : E/S. \\ \quad e_1, e_2 \text{ sections of } E/S \\ \quad \text{s.t.} \cdot Me_1 = Ne_2 = 0. \\ \qquad \cdot\ \mathbb{Z}/M\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z} \to E \\ \qquad\quad (a,b) \mapsto ae_1 + be_2 \\ \qquad\quad \text{is injective} \end{array} \right\} / \simeq.$

$M \mid L$, $N \mid L$.

$Y(L) \longrightarrow Y(M,N)$

$(E, e_1, e_2) \longmapsto (E, (\frac{L}{M})\cdot e_1, (\frac{L}{N})\cdot e_2).$

Zeta elements

$M, N \geq 2, \ M+N \geq 5.$

Choose $c, d \in \mathbb{Z}_{>0}$ s.t. $(c, 6M) = 1$
$(d, 6N) = 1$

$${}_{c,d}z_{M,N} := \{ {}_c g_{1/M, 0}, {}_d g_{0, 1/N} \}$$

$$\in K_2(Y(M,N)).$$

$$z_{M,N} := \{ g_{1/M, 0}, g_{0, 1/N} \}$$

$$\in K_2(Y(M,N)) \otimes \mathbb{Q}.$$

$\rightsquigarrow$ satisfy all the expected norm compatibilities.

The corresponding zeta ftn

$$Z_{M,N}(s) = \sum_{(n,M)=1} \underbrace{T'(n)}_{\text{dual Hecke operator}} \cdot \begin{pmatrix} 1/n & 0 \\ 0 & 1 \end{pmatrix}^* \cdot n^{-s}$$

$\circlearrowleft$

$$\delta_{M,N} \in H^1(Y(M,N)(\mathbb{C}), \mathbb{C})$$

$\uparrow$ $\qquad \cong$

$$(0, i\infty) \in H_1(X(M,N)(\mathbb{C}), \{\text{cusps}\}, \mathbb{Z}) \otimes \mathbb{C}.$$

Thm (Beilinson)

$$\exists\ \mathrm{reg}_{M,N} : K_2(Y(M,N)) \to H^1(Y(M,N)(\mathbb{C}), \mathbb{R}\cdot i)$$

$$Z_{M,N} \longmapsto \lim_{s\to 0} \frac{1}{s} \cdot Z_{M,N}(s) \cdot 2\pi i \cdot \underset{\mathbb{Z}}{\overline{\delta_{M,N}}}$$

Upto now: elts in $\underline{Y(M,N)_{/\mathbb{Q}}}$.

We want elts in $\underline{Y_1(N)_{/\mathbb{Q}(\xi_m)}}$

(to have Euler systems)

<u>Euler systems on $X_1(N) \otimes \mathbb{Q}(\xi_m)$.</u>

elements in $K_2(Y(M,N))$

$\downarrow \quad \leftarrow (\xi, S)$

element in $K_2(Y_1(N) \otimes \mathbb{Q}(\xi_m))$

2 types of zeta elements in terms of (ξ, S)

①. $\xi = a(A)$ (symbol)

$a, A \in \mathbb{Z}, A \in \mathbb{Z}_{\geq 1}$.

$S = \{\ell : \ell \mid m\cdot A\}$.

② $\xi \in SL_2(\mathbb{Z})$

$S = \{\ell : \ell \mid m\cdot N\}$.

Zeta elements from ① $\rightsquigarrow$ cover bad Euler factors

② $\rightsquigarrow$ integrality of zeta elements.

Define $z_{1,N,m}(\xi, S) \in K_2(Y_1(N) \otimes \mathbb{Q}(\zeta_m)) \otimes \mathbb{Q}$.

① Choose $M \geq 1$, $L \geq 4$.

s.t. $mA \mid M$, $N \mid L$, $M \mid L$.

$S = \{\ell : \ell \mid M\}$

$L = \{\ell : \ell \in S \text{ or } \ell \mid N\}$.

$$\left.\begin{aligned} \mathfrak{h} &\to \mathfrak{h} \\ \tau &\mapsto A^{-1}(\tau + a) \end{aligned}\right) \star$$

$\exists !$ $Y(M, L) \to Y_1(N) \otimes \mathbb{Q}(\zeta_m)$

which is compatible with $\star$.

$\leadsto$ norm map

$t_{m,a(A)} : K_2(Y(M,L))_{\otimes \mathbb{Q}} \to K_2(Y_1(N) \otimes \mathbb{Q}(\zeta_m)) \otimes \mathbb{Q}$.

$z_{M,L} \mapsto z_{1,N,m}(a(A), S)$

indep. of M, L.

② Choose $L \geq 3$ s.t.

$m | L$, $N | L$, $S = \{ \ell : \ell | L \}$.

$\exists$ canonical projection.

$$Y(L) \to Y_1(N) \otimes \mathbb{Q}(\zeta_m)$$

$$\rightsquigarrow$$

norm map.

$$K_2(Y(L)) \otimes \mathbb{Q} \to K_2(Y_1(N) \otimes \mathbb{Q}(\zeta_m)) \otimes \mathbb{Q}.$$

$$\xi^*(Z_{L,L}) \mapsto \underbrace{Z_{1,N,m}(\xi, \delta)}_{\text{indep. of } L.}$$

<u>Rem</u> $\exists$ "with (c,d)" version
$\rightsquigarrow$ remove $\otimes \mathbb{Q}$ $\rightsquigarrow$ integral elements

Summary.

${}_c\Theta_E \rightsquigarrow$ Siegel units ${}_c g_{\alpha,\beta}$ $\xrightarrow{\cup} K_2(Y(M,N))$ $\underbrace{(k, r, r')}_{\Sigma}$ $\xrightarrow{}$ H^2.

$\downarrow$

Eisenstein series.

$\downarrow$

zeta mod. forms $=$

$\downarrow$ Rankin – Selberg

zeta value formula.

Chern char

Soulé twist

HSSS.

$H^1(\mathbb{Q}, H^1_{\acute{e}t}(Y(M,N), \mathbb{Z}_p)(k-r))$

$\downarrow \exp^*$.

zeta mod. forms.

can rewrite everything here with $Y_1(N) \otimes \mathbb{Q}(\zeta_m)$

$\downarrow$

generating Euler systems

$\rightsquigarrow$

$p_f \subseteq T$.

$\otimes\, T/p_f$

$\rightarrow$ zeta elts for f.

More refined applications of Kato's Kolyvagin systems

$\ell \in \mathcal{P}_{k_c} \iff \ell \nmid Np.$

$\ell \equiv 1 \pmod{p^{k_c}}$

$a_\ell \equiv \ell + 1 \pmod{p^{k_c}}$

$\underline{n \in \mathcal{N}_{k_c}}$ $\square$-free products

$I_\ell \mathbb{Z}_p = (\ell - 1, 1 - a_\ell + \ell) \cdot \mathbb{Z}_p \subseteq p^{k_c}\mathbb{Z}_p.$

$I_n \mathbb{Z}_p = \sum_{\ell \mid n} I_\ell \mathbb{Z}_p .$

Define.

$$\tilde{\delta}_n = \sum_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} \overline{\left[\frac{a}{n}\right]^+} \cdot \left(\overline{\prod_{\ell \mid n} \log_{\eta_\ell}(a)} \right)$$

$$\in \mathbb{Z}_p / I_n \mathbb{Z}_p .$$

Kurihara #'s

(the case of class gps: higher annihilators.)

<u>Why $\tilde{\delta}_n$?</u>

- makes Kato's K.S. κ_n "computable"
- also appears in refined Iwasawa theory à la Kurihara.

Rem "computable"?
$=$ divisibility of κ_n.

$\kappa_n \in p^{?} \cdot \mathrm{Sel}_{rel, n\text{-}tr}(\mathbb{Q}, T/I_n T)$?

compute this via $\hat{\delta}_n$.

Picture

Kato's K.S.

$\downarrow$

str. of p-strict
Selmer gp
(Mazur–Rubin)
in terms of κ_n

$\dashrightarrow$
in progress
(K).

Kolyvagin systems
of Gauss sum type.

$\downarrow$

str. of Selmer gps
in ~~th~~ terms of $\tilde{\delta}_n$.
(Kurihara).

Arithmetic applications of $\tilde{\delta}_n$.

E.g. (Kurihara) $p=5$.

$$E : y^2 + y = x^3 - 17034726259173x - 27061436852750306309.$$

LMFDB 423801.ci1.

$$\rightsquigarrow \tilde{\delta}_1 = \frac{L(E,1)}{\Omega_E^+} = 10000 = \underbrace{2^4}_{\text{Tam.}} \cdot \underbrace{5^4}_{\text{Ш}}.$$

Ques $\text{Ш}(E/\mathbb{Q})[5^\infty] \simeq (\mathbb{Z}/25\mathbb{Z})^{\oplus 2}$ or $\cancel{(\mathbb{Z}/5\mathbb{Z})^{\oplus 4}}$?

$\tilde{\delta}_{11\cdot 41} \not\equiv 0 \pmod 5$ $\quad \nu(11\cdot 41)$

$\Rightarrow \mathrm{Sel}(\mathbb{Q}, E[5]) \simeq (\mathbb{Z}/5\mathbb{Z})^{\oplus 2}$.

$\Rightarrow \text{Ш}(E/\mathbb{Q})[5^\infty] \simeq (\mathbb{Z}/25\mathbb{Z})^{\oplus 2}$.

: more info. than the BSD conj.

Furthermore, we have an isom.

$$\mathrm{Sel}(\mathbb{Q}, E[5]) \simeq E(\mathbb{Q}_{11}) \otimes \mathbb{Z}/5\mathbb{Z}$$
$$\oplus\, E(\mathbb{Q}_{41}) \otimes \mathbb{Z}/5\mathbb{Z}.$$

: explicit semi-local description of mod p Selmer gp.

Thm (Kurihara)

$E/\mathbb{Q}$.

- p, non-anomalous good ordinary for E. or $a_p(E) = 0$.
- $p \nmid \mathrm{Tam}(E)$
- E has large img.
- $\mu = 0$.
- IMC holds.
- the p-adic height pairing is nondeg.

Then

① $\tilde{\delta}_n \not\equiv 0 \pmod p$ for some $n \in \mathcal{N}_1$.

② $\mathrm{Sel}(\mathbb{Q}, E[p^\infty])$ is completely determined by $\tilde{\delta}_n$.

(pf) Kolyvagin systems of Gauss sum type.

<u>Thm</u> (Sakamoto)

- p, non-anomalous good ord. for E.
- E has large img.
- $\mu = 0$
- $p \nmid \mathrm{Tam}(E)$

Then IMC $\Rightarrow$ $\tilde{\delta}_n \not\equiv 0 \ (p)$ for some n.

pf) "Kolyvagin system of rank zero".

<u>Converse</u> (K-Kim-Sun, K-Nakamura)

- non-anomalous good reduction.
  or additive reduction.
- large img.
- $p \nmid \mathrm{Tam}(E)$

If $\tilde{\delta}_n \not\equiv 0 \pmod{p}$ for some n,

then IMC holds.

More precisely, $\tilde{\delta}_n \not\equiv 0 \ (p)$

$\Rightarrow$ $\kappa_n \not\equiv 0 \ (p)$ "primitivity".

$\Rightarrow$ IMC.

Rem. If $p \mid \mathrm{Tam}(E)$,

then $\kappa_n \equiv 0 \pmod p$ $\forall n \in \mathbb{N}$.

(Mazur–Rubin, Büyükboduk)

Cor.

- E, good ord at p (non-anomalous)
- E has large img.
- $\mu = 0$
- $p \nmid \mathrm{Tam}(E)$
- IMC holds

$\Rightarrow$ $\kappa_n \not\equiv 0 \pmod p$.

: "cyclotomic version of Wei Zhang's result".