Today's menu:

Euler and Kolyvagin system machinery.

Theme

the size/str. of arithmetically interesting groups $\longleftrightarrow$ values of L-ftns

$(Cl(F),\ Sel(\mathbb{Q}, E[p^\infty]),$
$Ш(E/\mathbb{Q})[p^\infty])$

The role of Euler systems

$\exists$ a non-zero Euler system $z_{\mathbb{Q}}$.

$\Downarrow$ ← "Euler system machinery"

an upper bound of Selmer gp. (Rubin, Kato, Perrin-Riou)

today's focus
(+ Kolyvagin systems)

Euler systems

$$z_{\mathbb{Q}(\zeta_{mp^n})} \in H^1(\mathbb{Q}(\zeta_{mp^n}), V).$$

↑ a certain p-adic rep'n.

(1) norm compatibility.

(2) the connection with L-values

$$z_{\mathbb{Q}} \longleftrightarrow L(V^*(1), 0).$$

General setup.

$p > 2$. $F/\mathbb{Q}_p$ fin. ext'n.

$F \supset \mathcal{O} \supset m_F = (\varpi)$.

T, a free $\mathcal{O}$-module of finite rank n. with conti. $\mathcal{O}$-linear action of $G_{\mathbb{Q}}$.

T is geometric ("reasonably good")

if . T is unram. outside a finite set of primes Σ.

. "de Rham at p" in the sense of Fontaine.

$$\rho : G_{\mathbb{Q}} \twoheadrightarrow G_{\mathbb{Q},\Sigma} = \mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$$
$$\longrightarrow \mathrm{Aut}_{\mathcal{O}}(T) \simeq GL_n(\mathcal{O})$$

(e.g. $T = T_p E$)

$V = T \otimes F$, $W = V/T$ (e.g. $E[p^\infty]$).

For $m \geq 0$, $W_m := \varpi^{-m}T/T = W[\varpi^m]$.

$$\overset{\text{\rotatebox{90}{$\simeq$}}}{T/\varpi^m T}.$$

$T^*(1) = \mathrm{Hom}_{\mathcal{O}}(T, \mathcal{O}(1))$
$V^*(1) = \mathrm{Hom}_F(V, F(1))$
$W^*(1) = V^*(1)/T^*(1)$.

L-fns and Selmer gps

For $\ell \notin \Sigma$,

$$P_\ell(T, X) := \det(I_n - X \cdot \rho(\underset{\substack{\downarrow \\ \text{arithmetic.}}}{\mathrm{Frob}_\ell})|_T)$$

Set

$$L^\Sigma(T, s) := \prod_{\ell \notin \Sigma} P(T, \ell^{-s})^{-1}.$$

(converges for $\mathrm{Re}(s) \gg 0$)

Local condins.

$$H^1_f(\mathbb{Q}_\ell, V) = \mathrm{Ker}(H^1(\mathbb{Q}_\ell, V) \xrightarrow{\text{res}} H^1(\underset{\substack{\uparrow \\ \text{inertia at } \ell.}}{I_\ell}, V))$$

for $\forall \ell \neq p, \infty$.

$\rightsquigarrow$ $H^1_f(\mathbb{Q}_\ell, T)$ and $H^1_f(\mathbb{Q}_\ell, W)$

are defined as preimg and img of $H^1_f(\mathbb{Q}_\ell, V)$

w.r.t. $T \to V \to W$, respectively.

$\underline{\ell = \infty}$. $H^1_f(\mathbb{R}, -) = 0$ (if $p > 2$).

$\underline{\ell = p}$. $H^1_f(\mathbb{Q}_p, V) = 0$ for p-strict Selmer group

$\mathrm{Sel}_{\{p\}}$ or Sel_0.

$$H^1_f(\mathbb{Q}_p, V) = \mathrm{Ker}(H^1(\mathbb{Q}_p, V) \to H^1(\mathbb{Q}_p, V \otimes \mathbb{B}_{\mathrm{cris}}))$$

: Bloch-Kato local condition.

Write.

$$H^1_{/f} = \frac{H^1}{H^1_f} \quad (= H^1_{\mathrm{sing}}).$$

<u>Def.</u> Σ', another finite set of primes
$(\Sigma \cap \Sigma' = \phi)$

$$\mathrm{Sel}^{\Sigma'}(\mathbb{Q}, W) = \mathrm{Ker}\Big(H^1(\mathbb{Q}, W) \to \prod_{\ell \notin \Sigma'} H^1_{/f}(\mathbb{Q}_\ell, W) \Big)$$

$\mathrm{Gal}(\mathbb{Q}_{\Sigma \cup \Sigma'}/\mathbb{Q})$

: Σ'-relaxed Selmer.

$\cup|$

$\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$

$$\mathrm{Sel}(\mathbb{Q}, W) = \mathrm{Ker}\Big(H^1(\mathbb{Q}, W) \to \prod_{\ell \in \Sigma} H^1_{/f}(\mathbb{Q}_\ell, W) \Big)$$

: Selmer

$\cup|$

$$\mathrm{Sel}_{\Sigma'}(\mathbb{Q}, W) = \mathrm{Ker}\Big(\mathrm{Sel}^{\Sigma'}(\mathbb{Q}, W) \to \prod_{\ell \in \Sigma'} H^1(\mathbb{Q}, W) \Big)$$

: Σ'-strict Selmer.

A weak form of Bloch-Kato conj.

$$\underset{\substack{s=k \\ \text{"c.c.pt"}}}{\mathrm{ord}} \big(L(V, s) \big) \overset{?}{=} \dim_F \mathrm{Sel}(\mathbb{Q}, V^*(1)) - \dim_F H^0(\mathbb{Q}, V^*(1)).$$

Def. (Euler systems)

ρ, T as before.

$$\begin{tikzcd}[row sep=small] \mathbb{Q}^{ab.} \arrow[d, dash] \\ \mathcal{K} \text{ : "big"} \arrow[d, dash] \\ \mathbb{Q}. \end{tikzcd}$$

$\mathcal{K}$ contains $\mathbb{Q}_\infty$.
$\mathbb{Q}(q)$ for $\forall q \equiv 1\ (p)$

where.

$$\begin{tikzcd}[row sep=small] \mathbb{Q}(\zeta_q) \arrow[d, dash] \\ \mathbb{Q}(q) \arrow[d, dash] \\ \mathbb{Q}. \end{tikzcd}$$

$\mathbb{Q}(q)/\mathbb{Q}$: max'l p-subext'n.

An Euler system for $(T, \mathcal{K})$

= a collection of cohomology classes

$$\underline{z} = \{ z_K \in H^1(K, T) : \mathbb{Q} \underset{\text{fin.}}{\subseteq} K \subseteq \mathcal{K} \}.$$

satisfying

$$\operatorname{cores}_{\mathbb{Q}(\zeta_{n\ell})/\mathbb{Q}(\zeta_n)} (z_{\mathbb{Q}(\zeta_{n\ell})}).$$

$$= \begin{cases} z_{\mathbb{Q}(\zeta_n)} & \text{if } \ell \mid n \text{ or } \ell \in \Sigma. \\ P_\ell(V^*(1), \mathrm{Fr}_\ell^{-1}) \cdot z_{\mathbb{Q}(\zeta_n)} & \text{otherwise} \end{cases}$$

A rough picture

$$z_{\mathbb{Q}(\xi_n)} \longleftrightarrow L^{\Sigma_n}(V, \text{"c.c. pt"})$$

$$\Sigma_n = \Sigma \cup \{\ell \mid n\}.$$

Thm (Rubin)

Let $\underline{z}$ be an E.S. for $(T, \mathcal{K})$

Assume $\underline{Hyp(\mathbb{Q}, V)}$:

(1) $\exists\, \tau \in Gal(\overline{\mathbb{Q}}/\mathbb{Q}(\xi_{p^\infty}))$ s.t.

$V/(\rho(\tau)-1)V$ is 1-dim'l over F

(2) V is irred.

If $z_{\mathbb{Q}} \in H^1(\mathbb{Q}, T)$ is not a torsion, then $Sel_{\{p\}}(\mathbb{Q}, W^*(1))$ is finite.

idea of proof.

Bound $Sel_{\{p\}}(\mathbb{Q}, W_m^*(1))$ independent of m.

Tools : local and global dualities.

Local duality.

$$\langle\, ,\, \rangle_\ell : H^1(\mathbb{Q}_\ell, W_m) \times H^1(\mathbb{Q}_\ell, W_m^*(1)) \to H^2(\mathbb{Q}_\ell, \mathcal{O}/\varpi^m\mathcal{O}(1)) \simeq \mathcal{O}/\varpi^m\mathcal{O}.$$

$$\Rightarrow H^1_f(\mathbb{Q}_\ell, W_m)^\perp = H^1_f(\mathbb{Q}_\ell, W_m^*(1)).$$

Σ'.

Consider the diagram..

$$0 \to Sel^{\{p\}}(\mathbb{Q}, W_m) \to Sel^{\Sigma\cup\{p\}}(\mathbb{Q}, W_m) \xrightarrow{loc^s_{\Sigma'}} \sum_{\ell\in\Sigma'} H^1_{/f}(\mathbb{Q}_\ell, W_m)$$

$$\times$$

$$0 \to Sel_{\Sigma\cup\{p\}}(\mathbb{Q}, W_m^*(1)) \to \underline{\underline{Sel_{\{p\}}(\mathbb{Q}, W_m^*(1))}} \xrightarrow{loc^f_{\Sigma'}} \sum_{\ell\in\Sigma'} H^1_f(\mathbb{Q}_\ell, W_m^*(1))$$

$$\Big\downarrow \sum_{\ell\in\Sigma'} \langle\, ,\, \rangle_\ell$$

$$\mathcal{O}/\varpi^m\mathcal{O}$$

Global duality

$$\mathrm{Im}(loc^s_{\Sigma'})^\perp = \mathrm{Im}(loc^f_{\Sigma})$$

get larger $\rightsquigarrow$ get smaller?

Construct Σ' and elements $\kappa_{\Sigma'}$

$$\kappa_{\Sigma'} \in \mathrm{Sel}^{\{p\} \cup \Sigma'}(\mathbb{Q}, W_m)$$

(Kolyvagin derivative classes)

from z.

s.t. $\# \operatorname{coker}(\mathrm{loc}^s_{\Sigma'})$ and

$\# \mathrm{Sel}_{\Sigma' \cup \{p\}}(\mathbb{Q}, W_m^*(1))$

are bounded <u>independently of m</u>.

$\Sigma' \rightsquigarrow$ "Kolyvagin primes".

ℓ is a <u>Kolyvagin prime</u>

if $(\ell, pN) = 1$, $\ell \equiv 1 \pmod{p^m}$

$a_\ell \equiv \ell + 1 \pmod{p^m}$

<u>Rem</u> There are infinitely many Kolyvagin primes due to Chebotarev density thm.

$\rightarrow$ finiteness of $\mathrm{Sel}_{\{p\}}(\mathbb{Q}, W^*(1))$.

Upgrade to the finiteness of $\mathrm{Sel}(\mathbb{Q}, W^*(1))$?

$\rightarrow$ ~~[illegible]~~ Restrict ourselves to the case of elliptic curves.

The difference between

$$\begin{array}{ccc} \mathrm{Sel}_{\{p\}}(\mathbb{Q}, W^*(1)) & \text{and} & \mathrm{Sel}(\mathbb{Q}, W^*(1)) \\ \parallel & & \cong \\ \mathrm{Sel}_0(\mathbb{Q}, E[p^\infty]) & & \mathrm{Sel}(\mathbb{Q}, E[p^\infty]) \end{array}$$

The same diagram works with the BK local condition at p.

$$0 \to \mathrm{Sel}(\mathbb{Q}, T) \to \mathrm{Sel}^{\{p\}}(\mathbb{Q}, T) \xrightarrow{\mathrm{loc}_p^s} H^1_{/f}(\mathbb{Q}_p, T)$$

$$\times$$

$$0 \to \mathrm{Sel}_0(\mathbb{Q}, E[p^\infty]) \to \underline{\mathrm{Sel}(\mathbb{Q}, E[p^\infty])} \xrightarrow{\mathrm{loc}_p^f} H^1_f(\mathbb{Q}_p, E[p^\infty])$$

$$\begin{array}{c} \downarrow \\ \mathbb{Q}_p/\mathbb{Z}_p. \end{array}$$

with $\mathrm{Im}(\mathrm{loc}_p^s)^\perp = \mathrm{Im}(\mathrm{loc}_p^f)$

In order to show $\mathrm{Sel}(\mathbb{Q}, E[p^\infty])$ is finite, enough to show:

$$\begin{array}{ccc} \underline{\mathrm{loc}_p^s} : \mathrm{Sel}^{\{p\}}(\mathbb{Q}, \underline{V}) & \xrightarrow{\mathrm{loc}_p} & H^1(\mathbb{Q}_p, V) \\ \rotatebox{90}{$\in$} & \searrow^{\mathrm{loc}_p^s} & \downarrow \\ z_{\mathbb{Q}}. & & H^1_{/f}(\mathbb{Q}_p, V) \\ & & \cong \downarrow \exp^* \\ \downarrow & & \mathrm{Fil}^0(\mathbb{D}_{\mathrm{cris}}(V)) \\ \dfrac{L^{(p)}(E,1)}{\Omega_E^+}\omega_E \neq 0. & & = \mathbb{Q}_p \cdot \omega_E. \end{array}$$

surjective.

$\square$

Move to Kolyvagin systems.

First, why? 2 advantages.

① sharp bound via the primitivity.

$\Uparrow$ a mod p criterion.

② str. thm. (not only size)

(e.g. Kolyvagin's ~~work on~~ conjecture on the non-vanishing of Heegner pt K.S.

$\Downarrow$

can determine the str. of $Sel(K, E[p^\infty])$ in terms of Heegner pt K.S.)

$\{z_K\}_K$, $z_{\mathbb{Q}} \neq 0 \Rightarrow Sel_{\{p\}}$ is finite.

$\{$ $\Updownarrow$ $\{$ upgrade.

$\{\kappa_n\}_n$, $\kappa_1 \neq 0 \Rightarrow$ str. of $Sel_{\{p\}}$ in terms of $\{\kappa_n\}_n$.

What's the cost?

Σ' = a finite set of Kolyvagin primes we've chosen.

$n = \prod_{\ell \in \Sigma'} \ell$.

$\rightarrow \kappa_n \in Sel^{\{p\} \cup \Sigma'}(\mathbb{Q}, W_m)$

Mazur–Rubin organized κ_n with more controlled local conditions at Σ'.
("rigidity" in their book)

<u>If</u> $\ell \equiv 1 \pmod{p^m}$,

$$H^1_{tr}(\mathbb{Q}_\ell, W_m) = \mathrm{Ker}\left(H^1(\mathbb{Q}_\ell, W_m) \to H^1(\mathbb{Q}_\ell(\xi_\ell), W_m) \right)$$

: transverse local condition at ℓ.
($\approx$ the complement of the finite local condition.)

and $\exists$ the finite-singular comparison map.

$$H^1_f(\mathbb{Q}_\ell, W_m) \xrightarrow{\varphi_\ell^{fs}} H^1_{/f}(\mathbb{Q}_\ell, W_m)$$

$$H^1_{/f}(\mathbb{Q}_\ell, W_m) \simeq H^1_{tr}(\mathbb{Q}_\ell, W_m)$$

obtained from. the Euler factor at ℓ.

<u>Hope</u> $\kappa_n \in \mathrm{Sel}^{\{p\}}_{n\text{-}tr}(\mathbb{Q}, W_m)$

i.e. if $\ell \mid n$,

$$\mathrm{loc}_\ell(\kappa_n) \in H^1_{tr}(\mathbb{Q}_\ell, W_m).$$

The axiom of Kolyvagin system.

$$\mathrm{loc}_\ell^{s.}(\kappa_{n\ell}) = \varphi_\ell^{fs}(\mathrm{loc}_\ell(\kappa_n))$$

Prop. · H^1_f are orthogonal complement to each other via local Tate pairing.

· H^1_{tr} "

Rem.

① We only consider the core rk 1 case.

($\exists$ higher core rk K.S., Stark systems ...)

② "The img of Galois rep'n is large" required.

③ $p > 3$ if self-dual. in Mazur-Rubin

$p = 3$ by Sakamoto recently.

$\underline{z} = (z_K)_K$ Kato's E.S. for E.

$z_K \in H^1(K, T)$ is characterized by

$$\sum_{\sigma \in Gal(K/\mathbb{Q})} \sigma(\exp^* \mathrm{loc}_p^s z_K) \cdot \chi(\sigma)$$

$$= \frac{L^{(Sp)}(E, \chi, 1)}{\Omega_E^+} \cdot \omega_E$$

where χ is an even char of $Gal(K/\mathbb{Q})$
(odd $\to 0.$).

S = the product of ramified primes of $K/\mathbb{Q}$.

For $m \geq 0$, $\mathcal{P}_m$ = the set of primes ℓ.

- $(\ell, Np) = 1$.
- $a_\ell \equiv \ell + 1 \ (p^m)$
- $\ell \equiv 1 \ (p^m)$

For each $\ell \in \mathcal{P}_m$, fix a prim. root η_ℓ.
and.

$$Gal(\mathbb{Q}(\zeta_\ell)/\mathbb{Q}) \simeq (\mathbb{Z}/\ell\mathbb{Z})^\times$$

$$\sigma_{\eta_\ell} \leftarrow\!\mapsto \eta_\ell$$

$$D_\ell = \sum_{i=1}^{\ell-2} i \cdot \sigma_{\eta_\ell}^i$$

$$D_n = \prod_{\ell | n} D_\ell$$

$$z_{\mathbb{Q}(\xi_n)} \in H^1(\mathbb{Q}(\xi_n), T)$$

$$\downarrow \qquad\qquad \downarrow D_n$$

$$D_n z_{\mathbb{Q}(\xi_n)} \in H^1(\mathbb{Q}(\xi_n), T)$$

$$\downarrow \qquad\qquad \searrow \text{mod } p^m$$

$$D_n z_{\mathbb{Q}(\xi_n)} \pmod{p^m} \in \left(\frac{H^1(\mathbb{Q}(\xi_n), T)}{p^m}\right)^{G_n} \subseteq H^1(\mathbb{Q}(\xi_n), T)/p^m$$

$$G_n = \mathrm{Gal}(\mathbb{Q}(\xi_n)/\mathbb{Q})$$

$$\downarrow$$

$$H^1(\mathbb{Q}(\xi_n), T/p^mT)^{G_n}$$

$$\downarrow \qquad\qquad \downarrow \text{res}^{-1}$$

$$\kappa_n \in H^1(\mathbb{Q}, T/p^mT)$$

Rem. In general, $\exists$ a complicated formula for κ_n from Kolyvagin derivative classes, but we do not need the formula.

The idea of K.S. argument
(str. thm.)

<u>Notation</u> $\underline{\kappa} = (\kappa_n)_n$, $n \in \mathcal{N}_1$.

$\overline{\mathcal{N}_1}$: $\square$-free products of primes in $\mathcal{P}_1$.

For $n \in \mathcal{N}_m$,

$$\lambda(n, E[p^m]) = \mathrm{length}_{\mathbb{Z}/p^m\mathbb{Z}}(\mathrm{Sel}_{0,n\text{-tr}}(\mathbb{Q}, E[p^m]))$$

$$\partial^{(r)}(\underline{\kappa}^{(m)}) = \min\{ m - \mathrm{length}(\mathbb{Z}/p^m\mathbb{Z} \cdot \kappa_n^{(m)}) : n \in \mathcal{N}_m, \nu(n) = r \}.$$

where ~~$\underline{\kappa}$~~ $\kappa_n^{(m)} = \kappa_n \pmod{p^m}$.

$$e_i(\underline{\kappa}^{(m)}) = \partial^{(i)}(\underline{\kappa}^{(m)}) - \partial^{(i+1)}(\underline{\kappa}^{(m)}),$$

$i \geq 0$

<u>Thm</u> <u>$\kappa_n^{(m)} \neq 0$ for some $n \in \mathcal{N}_m$.</u>

$$\Rightarrow \mathrm{Sel}_{\{p\}}(\mathbb{Q}, E[p^m]) \simeq \bigoplus_{i \geq 1} \mathbb{Z}/p^{d_i}\mathbb{Z}.$$

with $d_1 \geq d_2 \geq \cdots$

and $\partial^{(r)}(\underline{\kappa}^{(m)}) = \min\{m, j + \sum_{i>r} d_i\}$.

where $j \geq 0$ satisfies $\mathbb{Z}/p^m\mathbb{Z} \cdot \kappa_n^{(m)} = p^{j+\lambda(n, E[p^m])} \cdot \mathrm{Sel}^{\{p\}}_{n\text{-tr}}(\mathbb{Q}, E[p^m])$

indep. of n.

(pf)

$(\supseteq)$ easy when $r=0$ (we get $=$)

$$\mathrm{Sel}_{\{p\}}(\mathbb{Q}, E[p^m]) \to \bigoplus_{\ell \mid n} E(\mathbb{Q}_\ell) \otimes \mathbb{Z}/p^m\mathbb{Z}$$
$$\simeq \left(\mathbb{Z}/p^m\mathbb{Z}\right)^{\oplus \nu(n)}.$$

$\rightsquigarrow$ its kernel $\subseteq \mathrm{Sel}_{\{p\},\, n\text{-tr}}(\mathbb{Q}, E[p^m])$.

$(\supseteq)$ Using Chebotarev density thm,

We can choose a prime $\ell \in \mathcal{P}_m$

s.t. $\mathrm{Sel}_{\{p\},\, \ell\text{-str.}}(\mathbb{Q}, E[p^m]) \simeq \underbrace{\bigoplus_{i \geq 2}}_{\text{kill a generator.}} \mathbb{Z}/p^{d_i}\mathbb{Z}.$

and

$\mathrm{Sel}_{\{p\},\, \underline{\underline{\ell\text{-str}}}} \simeq \mathrm{Sel}_{\{p\},\, \underline{\underline{\ell\text{-tr}}}}.$

use induction on $\nu(n)$

$\square$