<u>Today's</u> <u>menu.</u>: Cyclotomic Iwasawa theory for elliptic curves.

$$\begin{tikzcd}
\text{p-adic L-ftn.} \arrow[rr, leftrightarrow, dashed, "\text{main conj.}"] & & \text{Selmer gps} \\
& & \text{Kato's K.S.} \arrow[u, "\text{bound}"'] \\
\boxed{\text{modular symbols. ("Betti")}} \arrow[uu] & \text{Kato's E.S. ("étale")} \arrow[l, "\exp^*"] \arrow[uul, "\text{Col.}"'] \arrow[ur, squiggly] & \\
& E. \arrow[ul] \arrow[u] &
\end{tikzcd}$$

modular symbols? "analytic and computable incarnation of zeta elts".

("de Rham"? : Rankin-Selberg method (omitted))

p-adic L-ftns and modular symbols (Betti construction)

<u>L-value?</u>

$$E/\mathbb{Q} \underset{\text{modularity}}{\rightsquigarrow} f_E(z) = \sum_{n\geq 1} a_n \cdot q^n$$

level N. wt 2.

$a_n \in \mathbb{Z}$.

$$L(E,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Express L-values in terms of integrals:
(Will ignore all the convergence issues!)

$$\Gamma(s) = \int_0^\infty e^{-x} \cdot x^{s-1} dx.$$

$$= (2\pi i)^s \cdot \int_0^\infty e^{-2\pi n x} \cdot x^{s-1} \cdot dx.$$

$$L(E,s) \cdot \Gamma(s) = \sum_{n\geq 1} a_n \cdot \frac{\Gamma(s)}{n^s}.$$

$$= (2\pi)^s \cdot \int_0^\infty \underbrace{\left(\sum_{n\geq 1} a_n \cdot e^{2\pi n x}\right)}_{= f(ix)} \cdot x^{s-1} dx$$

$$\underline{L(E,1)} = 2\pi \cdot \int_0^\infty f(iy) dy.$$

$$= 2\pi i \cdot \underline{\int_{i\infty}^0 f(z) dz}.$$

<u>Twists</u>

$\chi : (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{C}^\times$ Dirichlet char of cond d with $(d, N) = 1$.

$\to L(E, \chi, 1)$?

$$\tau(n, \chi) = \sum_{a \in (\mathbb{Z}/d\mathbb{Z})^\times} \chi(a) \cdot e^{2\pi i n \cdot (a/d)}.$$

$\tau(\chi) := \tau(1, \chi)$ Gauss sum.

$$\tau(n, \chi) = \overline{\chi(n)} \cdot \tau(\chi)$$

$$f_{\bar{\chi}}(z) := \sum_{n\geq 1} \bar{\chi}(n)\cdot a_n \cdot e^{2\pi i n z}.$$

$$= \frac{1}{\tau(\chi)} \cdot \sum_{n\geq 1} \tau(n,\chi)\cdot a_n \cdot e^{2\pi i n z}.$$

$$= \frac{1}{\tau(\chi)} \cdot \sum_{n\geq 1} \sum_{a\in(\mathbb{Z}/d\mathbb{Z})^\times} \chi(a)\cdot a_n \cdot e^{2\pi i n(z+\frac{a}{d})}.$$

Birch's lemma. (Mazur-Tate-Teitelbaum)

$$L(E,\bar{\chi},1) = \frac{2\pi i}{\tau(\chi)} \cdot \sum_{a\in(\mathbb{Z}/d\mathbb{Z})^\times} \chi(a)\cdot \underline{\int_{i\infty}^{a/d} f(z)\,dz}.$$

$(f(z+1) = f(z).)$

$$= \frac{\tau(\bar{\chi})}{d}\cdot 2\pi i \cdot \sum_{a} \chi(a)\cdot \int_{i\infty}^{-a/d} f(z)\,dz.$$

These computations lead to ...

$$X_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash(\mathfrak{h}\cup\mathbb{P}^1(\mathbb{Q})).$$

For a pair α, β in $\mathbb{P}^1(\mathbb{Q})$,

$$\{\alpha,\beta\}_f = 2\pi i \cdot \int_\alpha^\beta f(z)\,dz.$$

(<u>unnormalized) modular symbols</u>.

Modular elts. of Mazur-Tate.

Fix an isom. $(\mathbb{Z}/d\mathbb{Z})^\times \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})$

$$a \mapsto (\sigma_a : \zeta_d \mapsto \zeta_d^a).$$

$$\chi : (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{C}^\times$$

$$\updownarrow$$

$$\chi : \mathbb{C}[\mathrm{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})] \to \mathbb{C}.$$

$$\sigma \mapsto \chi(\sigma).$$

$$\underbrace{\sum_{a \in (\mathbb{Z}/d\mathbb{Z})^\times} \{i\infty, a/d\}_f \cdot \sigma_a}_{\text{(unnormalized) Mazur-Tate elts.}} \mapsto \tau(\chi) \cdot L(E, \bar{\chi}, 1).$$

$\rightsquigarrow$ know twisted L-values.

Some hint on p-adic L-ftns

$$\chi : (\mathbb{Z}/p^n\mathbb{Z})^\times \to \mathbb{C}^\times.$$

$$\to L(E, \bar{\chi}, 1) = \frac{2\pi i}{\tau(\chi)} \cdot \underbrace{\sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(a) \cdot \int_{i\infty}^{a/p^n} f(z)\,dz}.$$

looks like a Riemann sum

in the following sense.

$$\mathbb{Z}_p^\times = \coprod_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} (a + p^n\mathbb{Z}_p)$$

if χ is a loc. const. ftn. on $\mathbb{Z}_p^\times$ $\rightsquigarrow$ p-adic integration

Normalization.

Write $\{r\}_f = \{i\infty, r\}_f$ for $r \in \mathbb{P}^1(\mathbb{Q})$.

Symmetrize integrals

$$\{r\}_f^{\pm} = 2\pi i \cdot \left(\int_{i\infty}^{r} f(z)\,dz \pm \int_{i\infty}^{-r} f(z)\,dz \right).$$

Thm (Shimura)

$\exists\, \Omega_f^{\pm} \in \mathbb{C}^\times$ s.t. $\{r\}_f^{\pm} \in \mathbb{Q}(a_n(f):n) \cdot \Omega_f^{\pm}$.

Write $[r]_f^{\pm} = \dfrac{\{r\}_f^{\pm}}{\Omega_f^{\pm}} \in \mathbb{Q}(a_n(f):n) \subseteq \overline{\mathbb{Q}}$.

| finite / $\mathbb{Q}$.

$$[r]_f = [r]_f^+ + [r]_f^-.$$

Hecke action and distribution relations.

$\{\alpha, \beta\} \in H_1(X_0(N), \mathbb{Q})$, $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$.

For a prime ℓ not dividing N,

$$T_\ell \{\alpha, \beta\} = \{\ell\alpha, \ell\beta\} + \sum_{r=0}^{\ell-1} \left\{ \frac{\alpha+r}{\ell}, \frac{\beta+r}{\ell} \right\}.$$

$\ell = p$? $\rightsquigarrow$ want to remove $\{p\alpha, p\beta\}$ to obtain the Riemann sum. $\rightsquigarrow$ p-stabilization.

$$\theta_{\mathbb{Q}(\zeta_M)} = \sum_{a \in (\mathbb{Z}/M\mathbb{Z})^\times} \left[\frac{a}{m}\right]_f \cdot \sigma_a$$

$$\in \mathbb{Q}[\mathrm{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})].$$

: modular elements
(Stickelberger elts. for elliptic curves).

$$\pi : \mathbb{Q}[\mathrm{Gal}(\mathbb{Q}(\zeta_{M\ell})/\mathbb{Q})] \to \mathbb{Q}[\mathrm{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})]$$

natural projection.

$$\nu : \mathbb{Q}[\mathrm{Gal}(\mathbb{Q}(\zeta_M)/\mathbb{Q})] \to \mathbb{Q}[\mathrm{Gal}(\mathbb{Q}(\zeta_{M\ell})/\mathbb{Q})]$$

norm. $\quad \sigma \quad \longmapsto \quad \sum_{\pi(\tau)=\sigma} \tau.$

"sum of liftings".

<u>Prop.</u>

$$\underline{\pi(\theta_{\mathbb{Q}(\zeta_{M\ell})})} = \begin{cases} a_\ell \cdot \theta_{\mathbb{Q}(\zeta_M)} - \nu(\theta_{\mathbb{Q}(\zeta_{M/\ell})}) & \text{if } \ell \mid M. \\ (a_\ell - \sigma_\ell - \sigma_\ell^{-1}) \cdot \theta_{\mathbb{Q}(\zeta_M)} & \text{if } \ell \nmid M. \end{cases}$$

(pf) use Hecke relation $\square$.

Fact. If $E[p]$ is irred., E has good red. at p, then. $\Theta_{\mathbb{Q}(\xi_{p^n})} \in \underline{\underline{\mathbb{Z}_p}}[\mathrm{Gal}(\mathbb{Q}(\xi_{p^n})/\mathbb{Q})]$ $\forall n \geq 1$.

<u>p-adic</u> <u>L-ftns</u>. p-stabilization.

3 term relation $\xrightarrow{\downarrow}$ norm-compatible sequence.

Take α a root of $X^2 - a_p \cdot X + p = 0$, and put.

$$\Theta^{\alpha}_{\mathbb{Q}(\xi_{p^n})} := \frac{1}{\alpha^n} \cdot \left(\Theta_{\mathbb{Q}(\xi_{p^n})} - \frac{1}{\alpha} \cdot \nu\, \Theta_{\mathbb{Q}(\xi_{p^{n-1}})} \right)$$

$\rightsquigarrow$ forms a projective system.

$$\pi\left(\Theta^{\alpha}_{\mathbb{Q}(\xi_{p^{n+1}})}\right) = \Theta^{\alpha}_{\mathbb{Q}(\xi_{p^n})}.$$

Suppose α is a p-adic unit.
(E has good ord. red. at p)

$$\Theta^{\alpha}_{\mathbb{Q}(\xi_{p^\infty})} = \varprojlim_n \Theta^{\alpha}_{\mathbb{Q}(\xi_{p^n})} \in \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}(\xi_{p^\infty})/\mathbb{Q})]].$$

$$\begin{array}{ccc} \Theta^{\alpha}_{\mathbb{Q}(\xi_{p^\infty})} & \in & \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}(\xi_{p^\infty})/\mathbb{Q})]] \\ \big\downarrow & & \big\downarrow \text{trivial Teichmuller cpnt.} \\ L_p(E) & \in & \Lambda = \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]] \end{array}$$

Interpolation formula.

$$\mathbb{1}\left(\theta^{\alpha}_{\mathbb{Q}(\zeta_{p^\infty})}\right) = \left(1-\frac{1}{\alpha}\right)^2 \cdot \theta_{\mathbb{Q}}$$

$$= \left(1-\frac{1}{\alpha}\right)^2 \cdot \frac{L(E,1)}{\Omega_E^+}.$$

$\chi\ (\neq \mathbb{1})$ Dirichlet char of cond p^n.

$$\rightarrow \chi\left(\theta^{\alpha}_{\mathbb{Q}(\zeta_{p^\infty})}\right) = \frac{1}{\alpha^n} \cdot \tau(\chi) \cdot \frac{L(E,\bar{\chi},1)}{\Omega_E^{\chi(-1)}}.$$

Selmer gps over the cyclotomic $\mathbb{Z}_p$-ext'n.

$$\begin{array}{c} \mathbb{Q}_\infty \\ | \\ \mathbb{Q} \end{array} \Big) \Gamma \qquad \Lambda = \mathbb{Z}_p[\![\Gamma]\!].$$

M. f.g. torsion Λ-module.

$$M \sim \Lambda/f_1^{e_1} \oplus \cdots \oplus \Lambda/f_r^{e_r}.$$

upto finite kernel and cokernel.

$$\mathrm{char}_\Lambda(M) = \left(\prod_{i=1}^r f_i^{e_i}\right) \subseteq \Lambda.$$

$\rightsquigarrow$ (linear alg $/\Lambda$ version of char. poly.)

<u>Thm</u> (Kato)

$E/\mathbb{Q}$. good red. at p.

The img. of Galois rep'n $\supseteq SL_2(\mathbb{Z}_p)$.

$Sel(\mathbb{Q}_\infty, E[p^\infty])^\vee \left(= Hom_{\mathbb{Z}_p}(Sel, \mathbb{Q}_p/\mathbb{Z}_p) \right)$

is fin. gen. <u>torsion</u> Λ-module..

<u>IMC with p-adic L-fn. (Mazur)</u>

E has good ord. at p

$(L_p(E)) = char_\Lambda(Sel(\mathbb{Q}_\infty, E[p^\infty])^\vee)$ in Λ.

<u>Thm</u> (Kato)

- When the img is large ($Im(\rho) \supseteq SL_2(\mathbb{Z}_p)$),

  $\subseteq$ holds in Λ.

- Even without the img. assumption,

  $\subseteq$ holds in $\Lambda \otimes \mathbb{Q}_p$.

<u>Thm</u> (Skinner-Urban)

If $\exists\, \ell \| N$ and $E[p]$ is ramified at ℓ,

then $\supseteq$ holds in Λ.

Prop. (Mazur's control thm)
E, good ord at p.

$$\mathrm{Sel}(\mathbb{Q}, E[p^\infty]) \longrightarrow \mathrm{Sel}(\mathbb{Q}_\infty, E[p^\infty])^\Gamma.$$

has finite kernel and cokernel.

Application of IMC.

$$L(E,1)=0 \iff \mathbb{1}(L_p(E)) = 0.$$

$$\underset{\text{IMC}}{\iff} \left(\mathrm{Sel}(\mathbb{Q}_\infty, E[p^\infty])^\vee\right)_\Gamma$$

$$\| $$

$$\left(\mathrm{Sel}(\mathbb{Q}_\infty, E[p^\infty])^\Gamma\right)^\vee$$

is infinite.

$$\underset{\substack{\text{control}\\\text{thm.}}}{\iff} \mathrm{Sel}(\mathbb{Q}, E[p^\infty])$$

is infinite.

Conj. (p-adic BSD).
E, good. ord. red'n at p.

$$\mathrm{rk}_{\mathbb{Z}} E(\mathbb{Q}) = \mathrm{ord}_{T=0} L_p(E)$$

$$(\chi = \mathbb{1})$$

Cor. of Kato's thm.

$$\mathrm{rk}_{\mathbb{Z}} E(\mathbb{Q}) \le \mathrm{ord}_{T=0} L_p(E)$$

What are needed to have $\textcircled{=}$?

- IMC.
- $\# Ш < \infty$.
- non-degeneracy of p-adic height pairing.

From Kato's zeta elements
to modular elements of Mazur-Tate
(following the exposition of Kurihara)

$$\begin{tikzcd}
\text{Kato's zeta elts.} \arrow[d, "\exp^*"'] \arrow[dd, phantom, ""] \arrow[rrd, bend left, "\text{Col.}"] & & \\
\text{modular symbols} \arrow[r, squiggly] & \text{modular elements} \arrow[r] & p\text{-adic } L\text{-ftns}
\end{tikzcd}$$

in the middle.

p-stabilization + limit.

Fix $(\xi_{p^n})_n \in \mathbb{Z}_p(1)$. a compatible system of primitive p^n-th roots of unity

(A very little of p-adic Hodge theory) good red'n

$\varphi \subset D = \mathbb{D}_{dR}(V) \leftarrow$ filtered φ-module.
dR cohom.

$\cup|$

crys cohom.

$D^0 = \mathbb{D}^0_{dR}(V) = \mathbb{Q}_p \cdot \omega_E. \leftarrow$ invariant differential.

$\rightsquigarrow$ D^0 = the cotangent sp. of E.

D/D^0 = the tangent sp. of E.

$$\begin{tikzcd} D \otimes \mathbb{Q}_p(\zeta_{p^{n+1}}) \arrow[r] \arrow[rr, bend right, "\widetilde{\mathrm{Exp}}_n(E)"] & D/D^0 \otimes \mathbb{Q}_p(\zeta_{p^{n+1}}) \arrow[r] & H^1(\mathbb{Q}_p(\zeta_{p^{n+1}}), V) \end{tikzcd}$$

exp.

extending the formal exponential map.

Consider an element

$$\gamma_n = \sum_{i=0}^{n} \varphi^{i+1}(\omega_E) \otimes \zeta_{p^{n+1-i}} + (1-\varphi)^{-1} \cdot (\varphi^{n+2}(\omega_E))$$

$$\in D \otimes \mathbb{Q}_p(\zeta_{p^{n+1}})$$

and a homomorphism.

$$P_n : H^1(\mathbb{Q}_p(\zeta_{p^{n+1}}), T_p E) \to \mathbb{Q}_p[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^{n+1}})/\mathbb{Q})].$$

$$a \longmapsto \frac{1}{[\varphi(\omega_E), \omega_E]_{dR}} \cdot \sum_{\sigma} \langle \exp(\gamma_n^{\sigma}), a \rangle_p \cdot \sigma.$$

where

$$\begin{array}{ccccc}
[\ ,\ ]_{dR} : D\otimes \mathbb{Q}_p(\zeta_{p^{n+1}}) & \times & D\otimes \mathbb{Q}(\zeta_{p^{n+1}}) & \to & \mathbb{Q}_p(\zeta_{p^{n+1}}) \\
\gamma_n^{\psi} \quad \downarrow \exp & & \uparrow \exp^* & & \downarrow \mathrm{Tr}. \\
\langle\ ,\ \rangle_p : H^1(\mathbb{Q}_p(\zeta_{p^{n+1}}), V) & \times & H^1(\mathbb{Q}_p(\zeta_{p^{n+1}}), V) & \to & \mathbb{Q}_p. \\
\cup & & & & \\
\exp(\gamma_n) & & & &
\end{array}$$

<u>Prop.</u> $\mathrm{Im}(\mathcal{L}_n) \subseteq \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}(\zeta_{p^{n+1}})/\mathbb{Q})]$.

<u>Thm.</u> $\mathcal{L}_n(z_{\mathbb{Q}(\zeta_{p^{n+1}})}) = \theta_{\mathbb{Q}(\zeta_{p^{n+1}})}$.

$$\theta_{\mathbb{Q}(\zeta_{p^{n+1}})} \downarrow \theta_{\mathbb{Q}_n}.$$

: "étale" construction of Mazur-Tate elements.

<u>Thm</u> $\exists$ Coleman map.

$\mathrm{Col} : H^1_{Iw}(\mathbb{Q}_p, T) \longrightarrow \Lambda$ <u>good ord.</u>

$\left(\varprojlim_n H^1(\mathbb{Q}_{n,p}, T)\right)$

s.t. $\mathrm{Col}(\log_p(z_{\mathbb{Q}_\infty})) = L_p(E)$.

Constructions?

- elliptic curves or wt 2 mod. forms
  → explicit local pts. (Rubin, Kurihara, Kobayashi!)
- higher wt mod. forms / p-adic Galois rep'ns
  → p-adic Hodge thy. (Perrin-Riou).

Kato's main conj. w/o p-adic L-fns.

global duality

$$
\begin{tikzcd}
H^1(\mathbb{Q},T) \arrow[r] & \dfrac{H^1(\mathbb{Q}_p,T)}{H^1_f(\mathbb{Q}_p,T)} \arrow[r] \arrow[d, "\exp^*"] & \mathrm{Sel}(\mathbb{Q},E[p^\infty])^\vee \arrow[d] \\
 & H^0(E/\mathbb{Q}_p,\Omega^1) & \mathrm{Sel}_0(\mathbb{Q},E[p^\infty])^\vee \arrow[d] \\
 & & 0
\end{tikzcd}
$$

$(= E(\mathbb{Q}_p)\otimes\mathbb{Z}_p)$

$z_{\mathbb{Q}} \mapsto (L\text{-value})\cdot\omega_E$

$H^0(E/\mathbb{Q}_p,\Omega^1) = \mathbb{Q}_p\cdot\omega_E \ni (L\text{-value})\cdot\omega_E.$

Sel_0: p-strict

Interpolate this over the cyclotomic tower $\mathbb{Q}_\infty$.

$$z_{\mathbb{Q}_\infty} = \varprojlim_n z_{\mathbb{Q}_n} \in H^1_{Iw}(\mathbb{Q}, T)$$

good ord.

$$\rightsquigarrow \quad 0 \to T^0 \to T \to T^1 \to 0 \quad \text{as } \underbrace{G_{\mathbb{Q}_p}\text{-module}}_{\text{reducible.}}$$

$$\begin{tikzcd}
0 \arrow[r] & \dfrac{H^1_{Iw}(\mathbb{Q},T)}{\Lambda\cdot z_{\mathbb{Q}_\infty}} \arrow[r, "\mathrm{loc}_p"] & \dfrac{H^1_{Iw}(\mathbb{Q}_p,T)}{H^1_{Iw}(\mathbb{Q}_p,T^0)+\Lambda\cdot \mathrm{loc}_p z_{\mathbb{Q}_\infty}} \arrow[d, "\cong / \mathrm{Col}"'] \arrow[dr] & \\
& & \Lambda / L_p(E). & \mathrm{Sel}(\mathbb{Q}_\infty, E[p^\infty])^\vee \arrow[d] \\
& & & \mathrm{Sel}_0(\mathbb{Q}_\infty, E[p^\infty])^\vee \arrow[d] \\
& & & 0.
\end{tikzcd}$$

<u>Thm</u> (Kato) When img is large.

- $\mathrm{Sel}_0(\mathbb{Q}_\infty, E[p^\infty])^\vee$ is Λ-torsion.
- $\mathrm{char}_\Lambda \dfrac{H^1_{Iw}(\mathbb{Q},T)}{z_{\mathbb{Q}_\infty}} \subseteq \mathrm{char}_\Lambda(\mathrm{Sel}_0^\vee)$.

$=?$ ← Kato's IMC.

This works for any reduction type.