Introduction to Beilinson–Kato elements and their applications.

<u>The goal.</u>

① <u>Illustrate</u> how ~~#~~ Kato's Euler systems work are constructed. (not very rigorous)

② Help you a little bit to read Kato's Astérisque 2004 paper.

<u>Our focus</u> : arithmetic of elliptic curves (<u>not</u> general modular forms)

How to understand rat'l pts on ell. curves?

<u>E.g.</u> $E : \underline{y^2 = x^3 - x.}$ LMFDB. 32.a3

$\rightsquigarrow E(\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$

$E : \underline{y^2 + y = x^3 - x.}$ LMFDB 37.a1.

$\rightsquigarrow E(\mathbb{Q}) \simeq \mathbb{Z}.$

We do <u>not</u> know whether $\# E(\mathbb{Q}) = \infty$, but L-ftns know.

" Human beings < L-ftns ".

Thm. (Wiles, Taylor-Wiles, .., BCDT).

$$\left\{ \begin{array}{l} \mathbb{Q}\text{-isogeny classes} \\ \text{of rat'l ell. curves} \\ \text{of cond. } N \end{array} \right\} \longleftrightarrow \begin{array}{l} \text{wt 2 cuspidal newforms} \\ \text{with rat'l Fourier coeff.} \\ \text{of level } \Gamma_0(N) \end{array}.$$

$$E \longmapsto f_E.$$

satisfying $L(E,s) = \sum_{n \geq 1} \frac{a_n}{n^s}$ $\quad (\mathrm{Re}(s) > \frac{3}{2})$.

where $f_E = \sum_{n \geq 1} a_n \cdot q^n$.

Conj (BSD). $E/\mathbb{Q}$.

(1). $\mathrm{rk}_{\mathbb{Z}} E(\mathbb{Q}) = \mathrm{ord}_{s=1} L(E,s)$ $(=r)$.

(2) If (1) holds and $Ш(E/\mathbb{Q})$ is finite, then.

$$\lim_{s \to 1} \frac{L(E,s)}{(s-1)^r \cdot \Omega_E^+ \cdot \mathrm{Reg}_E} = \frac{\# Ш(E/\mathbb{Q}) \cdot \mathrm{Tam}(E)}{\# (E(\mathbb{Q})_{\mathrm{tors}})^2}.$$

where $\mathrm{Tam}(E) = \prod_{\ell | N} [E(\mathbb{Q}_\ell) : \underbrace{E_0(\mathbb{Q}_\ell)}_{\text{preimg. of nonsing pts.}}]$

Beilinson-Kato's zeta elements.

p, a prime (including 2)

E, an ell. curve w/o CM.

$T = T_pE = \varprojlim_n E(\bar{\mathbb{Q}})[p^n]$.

$\rho \circlearrowright V = V_p E) = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, 2-dim'l $\mathbb{Q}_p$-v.sp.

$G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$

$\mathbb{Q}_\Sigma \leftarrow$ max'l ext'n unram. outside a finite set $\Sigma \ni p, \infty$, bad primes

$\mathbb{Q}$

$\rightarrow \mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$

$H^1(\mathbb{Q}, V)$ $(= H^1(\mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q}), V))$.

contains the info. of $E(\mathbb{Q})$ via Kummer map?

$$\underset{\text{geometry}}{E(\mathbb{Q}) \otimes \mathbb{Q}_p} \longrightarrow \underset{\text{cohomology.}}{H^1(\mathbb{Q}, V)}$$

Local nature at p.

Exer. $H^1(\mathbb{Q}_p, V)$ is 2-dim'l $\mathbb{Q}_p$-v.sp

(use local Euler characteristic formula)

$$\underset{\text{1-dim'l}}{E(\mathbb{Q}_p) \otimes \mathbb{Q}_p} \hookrightarrow \underset{\text{2-dim'l.}}{H^1(\mathbb{Q}_p, V)}$$

Weil pairing. Tate twist.

$$V \times V \longrightarrow \mathbb{Q}_p(1)$$

induces a nondeg. cup. product pairing.

$$\langle\ ,\ \rangle_p : H^1(\mathbb{Q}_p, V) \times H^1(\mathbb{Q}_p, V) \xrightarrow{\cup} H^2(\mathbb{Q}_p, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p.$$

Under this pairing,

$$E(\mathbb{Q}_p) \otimes \mathbb{Q}_p \perp E(\mathbb{Q}_p) \otimes \mathbb{Q}_p \quad \text{(orthogonal)}$$

($\Leftarrow$ local Tate duality.).

The commutative diagram.

$$\begin{tikzcd}
 & H^1(\mathbb{Q}_p, V) \arrow[d, phantom, "\cup"] & \times & H^1(\mathbb{Q}_p, V) \arrow[dd, "\exp^*"] & \longrightarrow & \mathbb{Q}_p \arrow[dd, equal] \\
\mathrm{Exp}_{\hat{E}}(a) \in & \underline{E(\mathbb{Q}_p) \otimes \mathbb{Q}_p} & & & & \\
a \in \underset{\underset{\hat{\mathbb{G}}_a(p\mathbb{Z}_p)}{\shortparallel}}{\underline{p\mathbb{Z}_p}} \subseteq \arrow[u, mapsto] & \underset{\text{tangent sp. of } E/\mathbb{Q}_p}{\mathbb{Q}_p} \arrow[u, "\exp", "\cong"'] & \times & \mathbb{Q}_p & \longrightarrow & \mathbb{Q}_p
\end{tikzcd}$$

$$\mathbb{Q}_p \cong H^0(E/\mathbb{Q}_p, \Omega^1) = \underline{\mathbb{Q}_p \cdot \omega_E}$$

$(a, b) \mapsto a \cdot b$ multiplication.

$$\mathrm{Ker}(\exp^*) = E(\mathbb{Q}_p)\otimes\mathbb{Q}_p \subseteq H^1(\mathbb{Q}_p, V)$$

announced early 90's, but published in 2004.

<u>Thm</u> (Kato)

$\exists\, z_{\mathbb{Q}} \in H^1(\mathbb{Q}, V)$ s.t.

$$H^1(\mathbb{Q}, V) \xrightarrow{\mathrm{loc}_p} H^1(\mathbb{Q}_p, V) \xrightarrow{\exp^*} \mathbb{Q}_p\cdot\omega_E.$$

$$z_{\mathbb{Q}} \longmapsto \exp^*(\mathrm{loc}_p(z_{\mathbb{Q}}))$$

$$= \frac{L(E,1)}{\Omega_E^+}\cdot\omega_E.$$

<u>Cor</u> (Kato, cf. Kolyvagin)

$$\mathrm{rk}_{\mathbb{Z}} E(\mathbb{Q}) > 0 \Rightarrow L(E,1) = 0$$

(pf) Let $P \in E(\mathbb{Q})$ be a pt of ∞ order.

Under $E(\mathbb{Q})\otimes\mathbb{Q}_p \to E(\mathbb{Q}_p)\otimes\mathbb{Q}_p$,

the img of P generates $E(\mathbb{Q}_p)\otimes\mathbb{Q}_p$.

Since $z_{\mathbb{Q}} \in H^1(\mathbb{Q}, V)$ and $P \in E(\mathbb{Q})$ are global,

the Artin reciprocity law says

$$\sum_{\ell\le\infty} \langle \mathrm{loc}_\ell z_{\mathbb{Q}}, \mathrm{loc}_\ell P\rangle_\ell = 0.$$

We have

$$H^1(\mathbb{Q}_\ell, V) = 0 \quad \forall \ell \neq p.$$

$$\Rightarrow \langle \mathrm{loc}_p z_{\mathbb{Q}}, \underbrace{\mathrm{loc}_p P}_{\substack{\cap \\ E(\mathbb{Q}_p)\otimes\mathbb{Q}_p}} \rangle_p = 0.$$

$$\mathrm{loc}_p z_{\mathbb{Q}} \in E(\mathbb{Q}_p) \otimes \mathbb{Q}_p$$

$$\Rightarrow \exp^* \mathrm{loc}_p(z_{\mathbb{Q}}) = 0$$

$$\Rightarrow L(E, 1) = 0 \qquad \square.$$

This is the very starting pt. of Kato's E.S.

$z_{\mathbb{Q}}$ is a part of a bigger thing.

Recall Selmer gp. of $E[p^\infty]$ $(= \bigcup_{n \geq 0} E[p^n])$.

$$\mathrm{Sel}(\mathbb{Q}, E[p^n]) := \mathrm{Ker}\left(H^1(\mathbb{Q}_\Sigma/\mathbb{Q}, E[p^n]) \to \prod_{\ell \in \Sigma} \frac{H^1(\mathbb{Q}_\ell, E[p^n])}{E(\mathbb{Q}_\ell)\otimes \mathbb{Z}/p^n\mathbb{Z}} \right).$$

$$\varinjlim_n \rightsquigarrow \mathrm{Sel}(\mathbb{Q}, E[p^\infty])$$

$$0 \to E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}(\mathbb{Q}, E[p^\infty]) \to Ш(E/\mathbb{Q})[p^\infty] \to 0.$$

<u>Thm</u> (Kato, cf. Kolyvagin)

$E/\mathbb{Q}$, no CM. ; p, <u>any</u> prime.

(i). If $L(E,1) \neq 0$, then $\mathrm{Sel}(\mathbb{Q}, E[p^\infty])$ is finite, so $E(\mathbb{Q})$ and $Ш(E/\mathbb{Q})[p^\infty]$ are finite.

(ii). Let $F/\mathbb{Q}$ be a finite abel. ext'n.
$\chi : \mathrm{Gal}(F/\mathbb{Q}) \to \mathbb{C}^\times$ a character.
If $L(E, \chi, 1) \neq 0$,
then $E(F)^\chi = E(F) \otimes_{\mathbb{Z}[\mathrm{Gal}(F/\mathbb{Q})], \chi} \mathbb{Z}[\chi]$.
$Ш(E/F)[p^\infty]^\chi$
are finite.

This thm. needs more than <u>one</u> elt $z_{\mathbb{Q}}$:

The theory of <u>Euler systems</u> comes in.

"cohomological" shape of L-fns".

- families of Galois cohomology classes $z_{\mathbb{Q}(\zeta_{mp^n})}$ over abel. ext'ns of $\mathbb{Q}$. with <u>norm compatibility</u>.
  ↳ remembers all Euler factors of L-fns.

) def'n.

· recover L-values (explizit reciprocity law)
deep thm.

Need a little bit of Iwasawa theory.

<u>Motto.</u> A counter-intuitive principle of Iwasawa thy
" ∞ ext'ns are easier than finite ext'ns ".

Cyclotomic deformation of BSD?
(= Iwasawa main conj.)

$E/\mathbb{Q}$ good ordinary reduction at $p > 2$.
$c_p \nmid N \cdot a_p(E)$)

$$\mathbb{Z}_p \simeq \left(\begin{array}{c} \mathbb{Q}_\infty \\ | \\ \mathbb{Q} \end{array} \!\!\begin{array}{c} \diagdown \\ \mathbb{Q}_n \\ \diagup \end{array} \right) \simeq \cancel{\mathbb{Z}/p\mathbb{Z}}\ \mathbb{Z}/p^n\mathbb{Z}.$$

$$\begin{aligned} \Lambda &= \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]]. \\ &= \varprojlim \mathbb{Z}_p[\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})] \\ &\simeq \mathbb{Z}_p[[T]]. \\ \gamma &\mapsto 1+T. \end{aligned}$$

M, a fin. gen. torsion Λ-module.

$$M \sim \bigoplus \Lambda/(f_i) \quad , f_i \in \Lambda.$$

kernel, cokernel are finite.

$$\mathrm{char}_\Lambda(M) = (\prod f_i) \subseteq \Lambda.$$

$L_p(E) \in \Lambda$ p-adic L-ftn.

<u>remembers</u> all twisted L-values
$L(E, \chi, 1)$

by the interpolation formula.

$$\chi(L_p(E)) \doteq \tau(\bar{\chi}) \cdot \frac{L(E,\chi,1)}{\Omega_E^+} \cdots$$

where. $\chi : \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \to \mathbb{C}^\times$ finite order.

<u>Thm</u> (Rohrlich) $L_p(E) \neq 0$.

<u>Conj</u> (IMC)

$$(L_p(E)) = \mathrm{char}_\Lambda(\underbrace{\mathrm{Sel}(\mathbb{Q}_\infty, E[p^\infty])^\vee}_{\text{f.g. torsion } \Lambda\text{-module!}})$$

<u>Thm</u> (Kato)

If the img of Galois rep'n contains a ~~conj~~ a conjugate of $SL_2(\mathbb{Z}_p)$, then.

"$\subseteq$" holds.

($\rightsquigarrow$ an upper bound of Selmer gps).

Thm $E/\mathbb{Q}$ non-CM., $L(E,1) \neq 0$

For almost all primes p, we have

$$\# Ш(E/\mathbb{Q})[p^\infty] \;\Big|\; \frac{L(E,1)}{\Omega_E^+}.$$

Sketch of pf.

$$\text{non-CM} \overset{\text{Serre}}{\Longrightarrow} \bar{\rho} \text{ is surj. } \forall p \gg 0.$$

$$\left(\Rightarrow \cancel{\rho \text{ contain}}\ \mathrm{Im}(\rho) \supseteq SL_2(\mathbb{Z}_p). \right).$$

Remove finitely many primes:

① all bad reduction primes

② all primes dividing Euler factors at $s=1$ at bad reduction primes

③ $p=2$.

④ primes dividing a constant C_E (indep. of p)

s.t. "$\exp^* \log z_\ell = C_E \cdot \dfrac{L(E,1)}{\Omega_E^+}$".

⑤ primes dividing $\prod_{q \mid N} \#(E(\mathbb{Q}_q)_{\mathrm{tors}})$.

One issue: <u>anomalous</u> good ordinary primes

Kato's thm on IMC.

\+ no nonzero finite Λ-submodule (Greenberg)

\+ control thm. (Mazur)

$\square$

<u>Cor.</u> $E/\mathbb{Q}$ no CM.

$L(E,1) \neq 0 \Rightarrow Ш(E/\mathbb{Q})$ is finite.

<u>Kato's thm (ii)</u>

$\Rightarrow$ <u>Cor.</u> $E/\mathbb{Q}$ no CM.

$E(\mathbb{Q}_\infty)$ is (<u>still!</u>) fin. gen.

$$\begin{tikzcd}
\mathbb{Q}_\infty \arrow[d, dash, dotted] & & E(\mathbb{Q}_\infty) \arrow[d, dash, dotted] \\
\mathbb{Q}_n \arrow[rr, "``E''"] \arrow[d, dash, dotted] & & E(\mathbb{Q}_n) \arrow[d, dash, dotted] \\
\mathbb{Q} & & E(\mathbb{Q})
\end{tikzcd}$$

does not grow too much.

shows the philosophy of Iwasawa thy very well.

Back to IMC.

Eisenstein congruence.

Thm (Skinner-Urban)

If $\exists \ell \| N$ and $E[p]$ is ramified at ℓ,

then " $\supseteq$ " holds

($\rightsquigarrow$ lower bound of Selmer grps)

Where does Kato's Euler systems appear in Kato's thm on IMC.

: involves Kato's IMC w/o p-adic L-ftns.

$\boxed{\exists z_{\mathbb{Q}(\zeta_{mp^n})}}$

$$z_{\mathbb{Q}_\infty} = \varprojlim_{n} z_{\mathbb{Q}_n} \in H^1_{Iw}(\mathbb{Q}, T)$$

$$\underset{\text{cores.}}{} \qquad = \varprojlim_n H^1(\mathbb{Q}_n, T).$$

a part of E.S. norm relation.

$z_{\mathbb{Q}_\infty}$ plays the role of $L_p(E)$

More precisely, $\exists$ Coleman map.

$$\mathrm{Col} : H^1_{Iw}(\mathbb{Q}_p, T) \to \Lambda.$$
$$\mathrm{loc}_p(z_{\mathbb{Q}_\infty}) \mapsto L_p(E)$$

and Kato's IMC. w/o $L_p(E)$ $\leftarrow$ no reduction type assumption.

$$\Updownarrow \text{ Col}$$

IMC with $L_p(E)$

Remarks on the Euler system argument

In the Euler system argument, need the derivative process.

$\rightsquigarrow$ can organize these cohomology classes (more?)

theory of Kolyvagin systems (Mazur-Rubin?)

(1) a criterion to have equality.

(2) the structure thm.
(cf Kolyvagin 1991 Annalen)

Mazur's diagram. (big picture)

$$\begin{tikzcd}
\text{L-values} \arrow[rr, leftrightarrow, "\text{explicit reciprocity law}", "\text{Gross-Zagier formula}"'] \arrow[d, leftrightarrow, dashed, "\text{CNF, BSD, Bloch-Kato}"'] & & \text{algebraic treasures (Euler systems)} \arrow[d, "\text{derivatives}"] \\
\text{arithmetic (Selmer gps, class gps, ...)} & & \text{Kolyvagin systems} \arrow[ll, "\text{control by global duality}"]
\end{tikzcd}$$

The question reduces to ... :

Where do Euler systems come from?
(the most difficult questions)

- cyclotomic units
- elliptic units
- Beilinson-Kato. ← Siegel units.
- Heegner pts / cycles.
- Beilinson-Flach

⋮

- Loeffler-Zerbes' program.

Very^2 rough picture. (as a guideline to read Kato's paper)

$c,d\,Z_m(f, r, r', \xi, S)$ ......?

2 Siegel units

$c\,g_{\alpha,\beta}$, $d\,g_{\gamma,\delta}$, .. $\in \mathcal{O}$(open modular curve with certain level str.)$^\times$.

$\longrightarrow K_2(Y_1(N) \otimes \mathbb{Q}(\zeta_m))$ Steinberg symbol. + certain twist. (ξ, S)

$\downarrow$ Chern char.

$H^2(Y_1(N)_{\mathbb{Q}(\zeta_m)}, \mathbb{Z}_p(2))$

$\downarrow$ HSSS

$H^1(\mathbb{Q}(\zeta_m), H^1_{et}(X_1(N)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(2)))$

$= H^1_{et}(X_1(N)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1))(1)$

$= \mathrm{Jac}(X_1(N))$

$\downarrow$ " $\otimes(\zeta_{p^n})_n^{\otimes -1}$ " $\leftarrow$ $q^{\mathbb{Z}/p^n\mathbb{Z}}$ p^n level. $(\varprojlim_n)$

$H^1(\mathbb{Q}(\zeta_m), H^1_{et}(X_1(N)_{\bar{\mathbb{Q}}}, \mathbb{Z}_p(1))$

$\downarrow$ $\exp^*$.

space of modular forms

$\cup$

zeta modular forms.

$\sim$ wt 1 Eisen series $\times$ wt 1 Eisen. series

$\downarrow$ Rankin-Selberg method.

L-value formula.

$c\theta_E$, $d\theta_E$
certain ftns. on univ. ell. curve.

involve dlog.