

# Integrable dissipative spin chains

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Acknowledgment:

Naoyuki Shibata (UTokyo)

N. Shibata and H.K.,

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- *Phys. Rev. B* **99**, 224432 (2019)
- *PTEP* **2020**, 12A108 (2020)



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# Outline

## Introduction

- Open quantum systems, GKSL equation
- NESS, Liouvillian gap
- Dissipative quantum spin chains

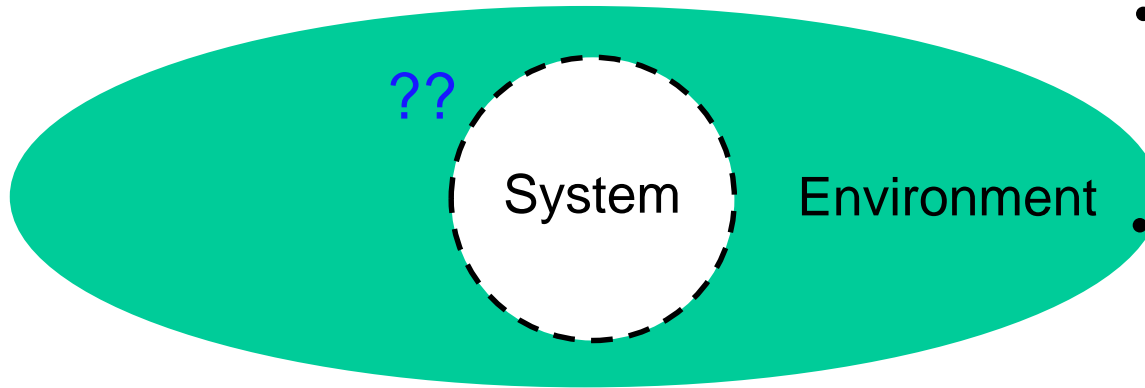
Model I: Quantum compass chain + dissipation

Model II: Quantum Ising chain + dissipation

Summary

# Open quantum systems

## ■ System & environment



- Total Hamiltonian

$$H = H_S + H_E + H_{SE}$$

- System density matrix

$$\rho(t) = \text{Tr}_E[\rho_{\text{tot}}(t)]$$

## ■ GKSL (or Lindblad) equation

Gorini, Kossakowski, Sudarshan, *JMP* **17**, Lindblad, *CMP* **48** (1976)

- Markovian, CPTP (completely positive trace-preserving)

$$\frac{d\rho}{dt} = \mathcal{L}[\rho] := -i[H, \rho] + \sum_i \left( L_i \rho L_i^\dagger - \frac{1}{2} \{L_i^\dagger L_i, \rho\} \right)$$

Liouvillian

$\rho$  : density matrix

$L_i$  : Lindblad operator

(Hermitian, positive semi-definite,  $\text{Tr } \rho = 1$ )

# State engineering by dissipation!?

4/25

nature  
physics

LETTERS

PUBLISHED ONLINE: 20 JULY 2009 | DOI: 10.1038/NPHYS1342

## Quantum computation and quantum-state engineering driven by dissipation

Frank Verstraete<sup>1</sup>★, Michael M. Wolf<sup>2</sup> and J. Ignacio Cirac<sup>3</sup>★

- Steady states = MPS, toric code g.s., ...

nature  
physics

ARTICLES

PUBLISHED ONLINE: 2 OCTOBER 2011 | DOI: 10.1038/NPHYS2106

## Topology by dissipation in atomic quantum wires

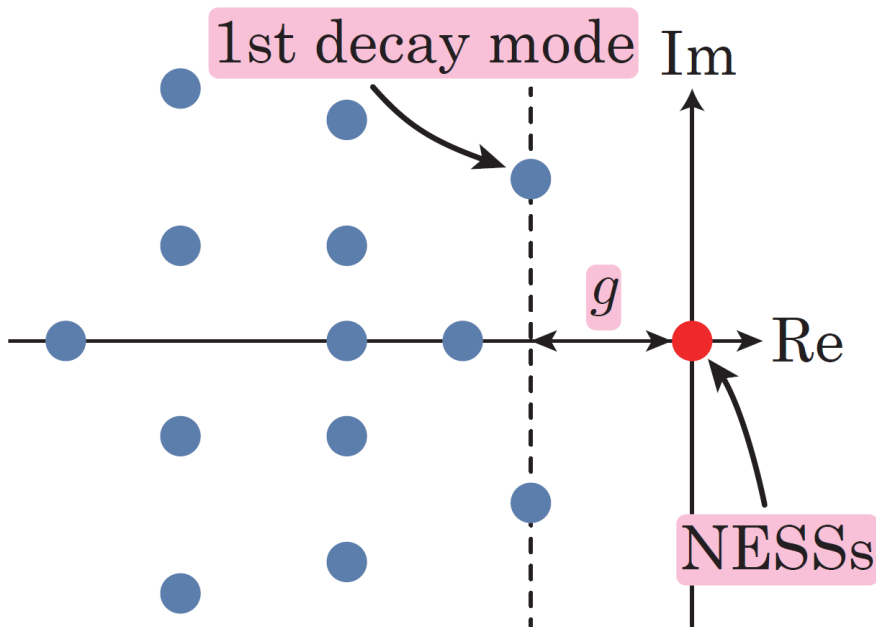
Sebastian Diehl<sup>1,2</sup>★, Enrique Rico<sup>1,2</sup>, Mikhail A. Baranov<sup>1,2,3</sup> and Peter Zoller<sup>1,2</sup>

# NESS and Liouvillian gap

## ■ Eigenvalue problem

$$\mathcal{L}[\rho_i] = \lambda_i \rho_i$$

1.  $\text{Re}(\lambda_i) \leq 0 \quad \forall i$ , 2.  $\mathcal{L}[\rho_i] = \lambda_i \rho_i \Rightarrow \mathcal{L}[\rho_i^\dagger] = \lambda_i^* \rho_i^\dagger$



## • Non-Equilibrium Steady State

$$\mathcal{L}[\rho_0] = 0 \quad (\lambda_0 = 0)$$

## • Liouvillian gap

$$g := - \max_{\substack{i \\ \lambda_i \neq 0}} \text{Re}(\lambda_i)$$

determined by the **1st decay mode**

Relaxation time:  $\tau = 1/g$

## Two-level examples

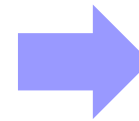
- Dephasing  $H = 0, L = \sqrt{\gamma}\sigma^z$  ( $\gamma > 0$ )

$$\mathcal{L}[\rho] = \frac{d\rho}{dt} = \gamma(\sigma^z \rho \sigma^z - \rho)$$

- Eigen-operators

$$\mathcal{L}[1_2] = \mathcal{L}[\sigma^z] = 0,$$

$$\mathcal{L}[\sigma^x] = -2\gamma\sigma^x, \quad \mathcal{L}[\sigma^y] = -2\gamma\sigma^y$$



Two NESS

Liouvillian gap =  $2\gamma$

- Spontaneous emission  $H = \sigma^z, L = \sqrt{\gamma}\sigma^-$   $\sigma^\pm = \frac{\sigma^x \pm i\sigma^y}{2}$

$$\mathcal{L}[\rho] = \frac{d\rho}{dt} = -i[\sigma^z, \rho] + \gamma \left( \sigma^- \rho \sigma^+ - \frac{1}{2} \{ \sigma^+ \sigma^-, \rho \} \right)$$

- Eigen-operators

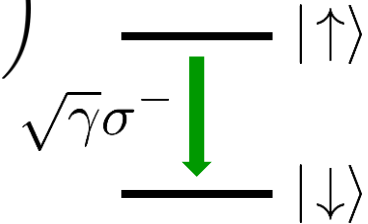
$$\mathcal{L} \left[ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] = 0, \quad \mathcal{L} \left[ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = -\gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathcal{L}[\sigma^\pm] = \left( -\frac{\gamma}{2} \pm 2i \right) \sigma^\pm$$



Single NESS

Liouvillian gap =  $\gamma/2$



# Super-operator formalism

## ■ Vectorization of density matrix

$$\rho = \sum_{m,n} \langle m|\rho|n\rangle |m\rangle\langle n| \quad \{|m\rangle\} : \text{ orthonormal basis}$$

$$|\sigma\rangle\langle\tau| \Leftrightarrow |\sigma\rangle \otimes |\tau\rangle \quad \rho \mapsto |\rho\rangle = \sum_{m,n} \langle m|\rho|n\rangle |m\rangle|n\rangle$$

$$P\rho Q \mapsto \sum_{m,n} \langle m|\rho|n\rangle (P|m\rangle)(Q^T|n\rangle)$$

## ■ Example: dephasing

$$\mathcal{L}[\rho] = \frac{d\rho}{dt} = \gamma(\sigma^z \rho \sigma^z - \rho)$$

$$\text{vec} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} = \begin{pmatrix} \rho_{11} \\ \rho_{12} \\ \rho_{21} \\ \rho_{22} \end{pmatrix} \quad \frac{d}{dt} \begin{pmatrix} \rho_{11} \\ \rho_{12} \\ \rho_{21} \\ \rho_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\gamma & 0 & 0 \\ 0 & 0 & -2\gamma & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_{11} \\ \rho_{12} \\ \rho_{21} \\ \rho_{22} \end{pmatrix}$$

Clearly, the eigenvalues of  $\mathcal{L}$  are 0 and  $2\gamma$ .

# Dissipative quantum spin chains

## ■ Exact results

- Boundary dissipation: Prosen, *PRL* **107**, 137201 (2011)
- Reducible to free fermions/bosons: Prosen, *NJP* **10**, 043026 (2008)
- Reducible to imaginary- $U$  Hubbard:  
Medvedyeva, Essler, Prosen, *PRL* **117**, 137202 (2016)
- Yang-Baxter integrable: Ziolkowska, Essler, *SciPost* **8**, 044 (2020)

## ■ Our results

Two spin-chain models with bulk dissipation.

Integrable! 'Phase transition' in the 1st decay mode.

	Hamiltonian	Dissipation	Liouvillian (after vec.)
I	$\sigma_{2j-1}^x \sigma_{2j}^x, \sigma_{2j}^y \sigma_{2j+1}^y$	$\sigma_j^z$ (dephasing)	Non-Hermitian Kitaev Ladder
II	$\sigma_j^z, \sigma_j^x \sigma_{j+1}^x$	$\sigma_j^z, \sigma_j^x \sigma_{j+1}^x$	Non-Hermitian Ashkin-Teller ( $\rightarrow$ XXZ)



# Outline

Introduction I

Introduction II

**Model I: Quantum compass chain + dissipation**

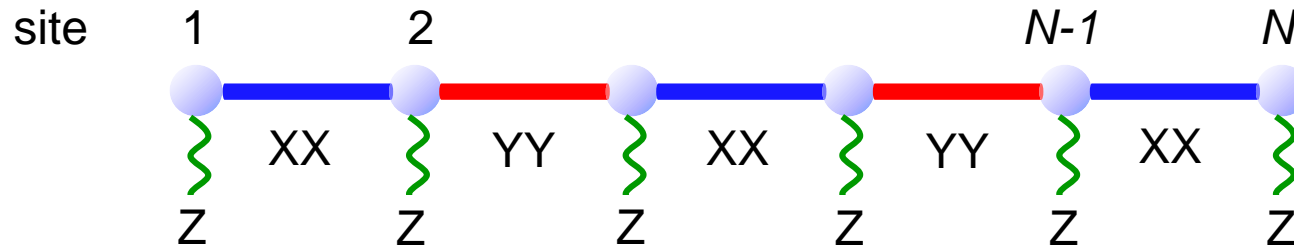
- Hamiltonian, dissipation
- Liouvillian  $\rightarrow$  Kitaev-ladder  $\rightarrow$  non-Hermitian SSH
- Liouvillian gap, autocorrelator of the edge spin

Model II: Quantum Ising chain + dissipation

Summary

# Model I

## ■ Setup: spin-1/2 chain with OBC



## ■ Hamiltonian

- Quantum compass chain

Brzezicki, Dziarmaga, and Oles, *PRB* **75**, 134415 (2007)

$$H = H_{XX} + H_{YY} = -J_x \sum_{j=1}^{N/2} \sigma_{2j-1}^x \sigma_{2j}^x - J_y \sum_{j=1}^{N/2-1} \sigma_{2j}^y \sigma_{2j+1}^y$$

Half of the XY chain. **Reducible to free fermions.**

## ■ Dissipation: dephasing $L_j = \sqrt{\gamma} \sigma_j^z$ ( $j = 1, \dots, N$ )

N.B.  $L_j^\dagger = L_j$ ,  $(L_j)^2 = \gamma$

- XY and XXZ with dephasing: Znidaric, *PRE* **92**, 042143 (2015)

# Steady states

- Z2 parity  $P = \prod_{j=1}^N \sigma_j^z$   $[P, H] = 0, [P, L_j] = 0 \forall j$

- NESS are prop. to the projections to the parity sectors

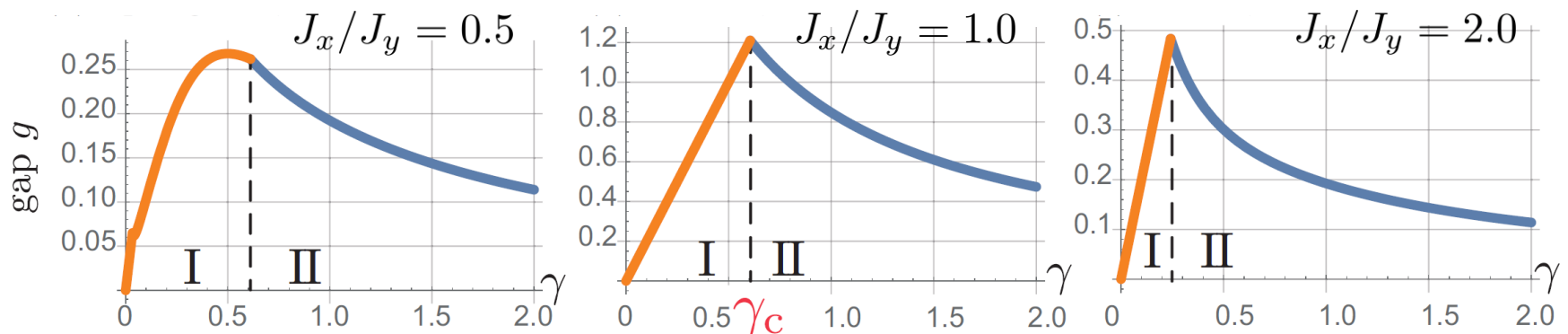
$$\rho_{\pm} = \frac{1 \pm P}{2^N}$$

Ex.)  $N=2$   $P_+ = \frac{1}{2}(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|)$

- Proof  $\mathcal{L}[\rho_{\pm}] = -i[H, \rho_{\pm}] + \sum_j [L_j, \rho_{\pm}]L_j = 0$

- Unique NESS: Numerically checked uniqueness for small  $N$

## Liouvillian gap



# Mapping to Kitaev ladder

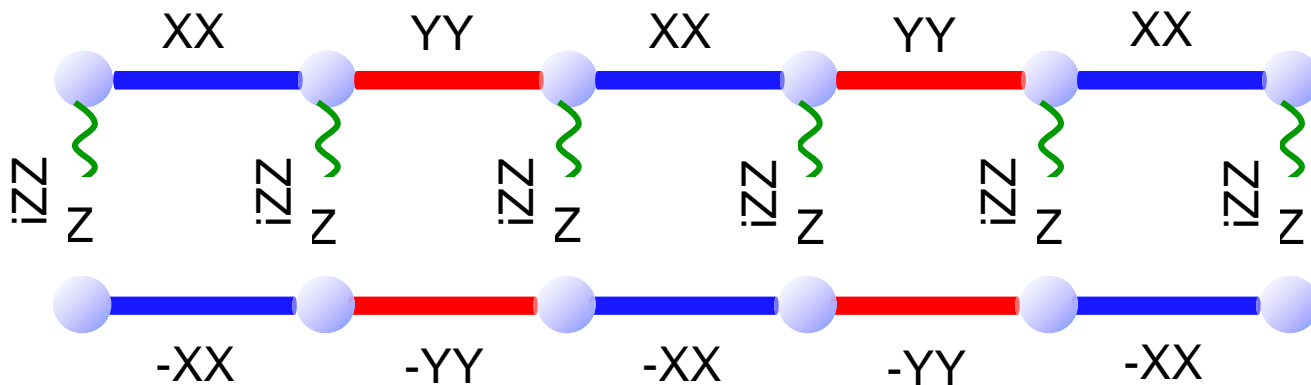
## ■ Vectorization

$$|\sigma_1, \dots, \sigma_N\rangle \langle \tau_1, \dots, \tau_N| \Leftrightarrow |\sigma_1, \dots, \sigma_N\rangle \otimes |\tau_1, \dots, \tau_N\rangle$$

With this identification,  $\rho$  can be thought of as a state on **2-leg ladder**.

## ■ Non-Hermitian 'Hamiltonian'

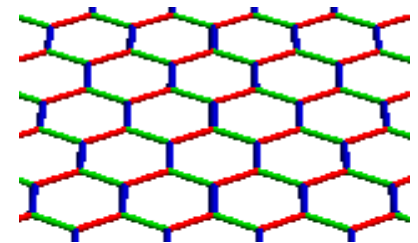
$$\mathcal{H} = i\mathcal{L} + \text{const.}$$



**Kitaev model on a ladder!**

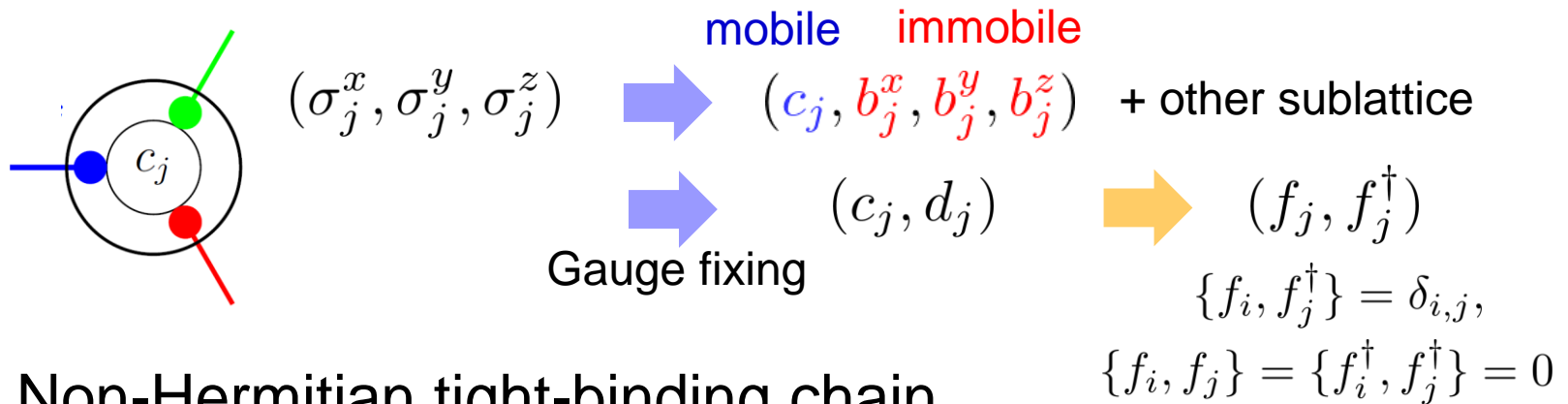
A. Kitaev, *Ann. Phys.* **321**, 2 (2006)

Solvable in the same manner as Kitaev honeycomb.



# From Kitaev ladder to SSH

- Spin  $\rightarrow$  Majorana  $\rightarrow$  complex fermions

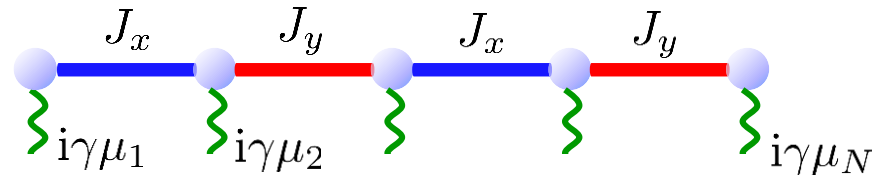


- Non-Hermitian tight-binding chain

$$\mathcal{H} = 2J_x \sum_{i=1}^{N/2} (f_{2i-1}^\dagger f_{2i} + \text{h.c.}) + 2J_y \sum_{i=1}^{N/2-1} (f_{2i}^\dagger f_{2i+1} + \text{h.c.})$$

$$+ 2i\gamma \sum_{j=1}^N \mu_j \left( f_j^\dagger f_j - \frac{1}{2} \right)$$

$\mu_j = \pm 1$



Su-Schrieffer-Heeger chain (*PRL*1979) with imaginary  $\mu$ !  
 Solvable sector-by-sector (Sector~configuration of  $\mu$ s)

# Liouvillian gap

## ■ Fermionic picture

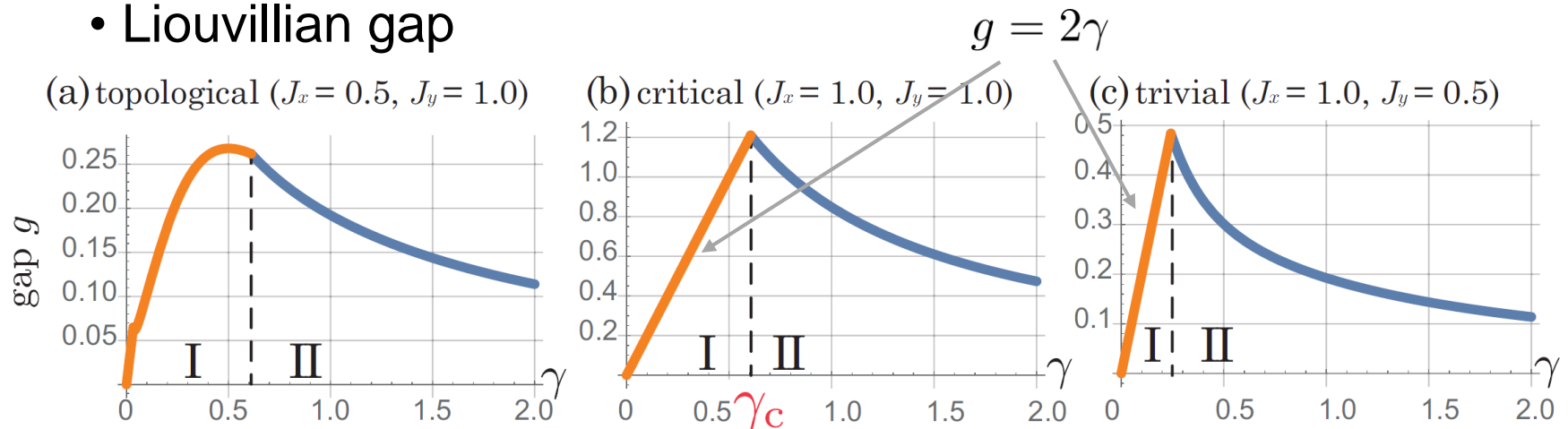
- Steady state: fully filled state in sector  $\mu = (+, +, \dots, +)$
- 1st decay modes: must be in the other sectors

## ■ Results

- Three regimes

1. Topological ( $J_x < J_y$ ), 2. critical ( $J_x = J_y$ ), 3. Trivial ( $J_x > J_y$ )

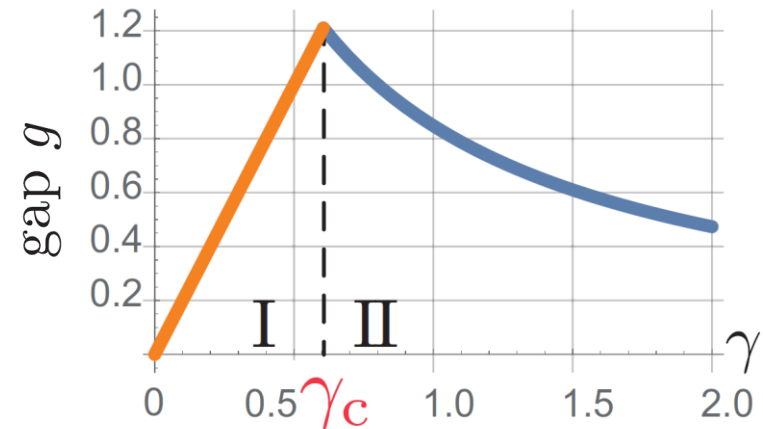
- Liouvillian gap



Config. of  $\mu$ s for the 1st decay mode in I is different from that in II.  
 → Nonmonotonic behavior in the Liouvillian gap  $g$ !

# Liouvillian gap in the 'critical' case ( $J_x=J_y$ )

- Numerical result for  $N=10$
- $J_x = J_y = 1.0$



## ■ Analytical results

- 1st decay mode lives in the sector  $\mu = (+, \dots, +, -, +, \dots, +)$  for small  $\gamma$ , while  $\mu = (-, -, +, \dots, +)$  for large  $\gamma$ .
- Exact result for  $g$  ( $J_x=J_y=1.0$ )

$$g = \begin{cases} 2\gamma & (0 \leq \gamma \leq \gamma_c) \\ \frac{6^{1/3} \left( 9\gamma^2 + \sqrt{48\gamma^6 + 81\gamma^4} \right)^{2/3} - 2 \cdot 6^{2/3} \gamma^2}{3\gamma \left( 9\gamma^2 + \sqrt{48\gamma^6 + 81\gamma^4} \right)^{1/3}} & (\gamma_c \leq \gamma) \end{cases}$$

- Critical dissipation strength  $\gamma_c = \sqrt{\frac{\sqrt{3}-1}{2}} = 0.605\dots$

# Autocorrelator of the edge spin

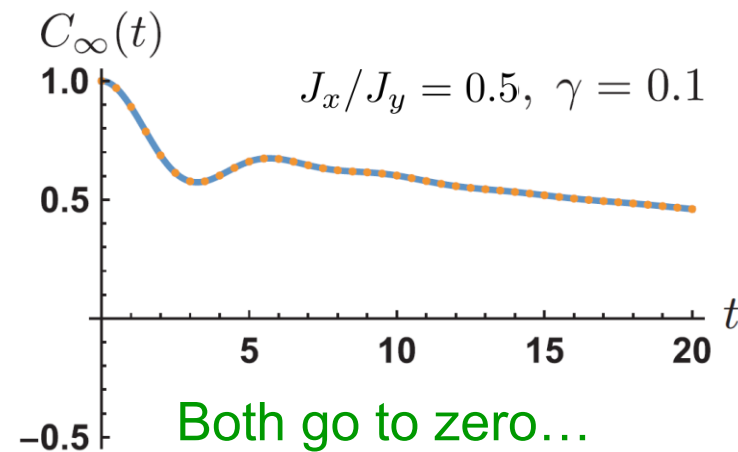
## ■ Signature of long-time coherence

$$C_\infty(t) = \langle \sigma_1^z(t) \sigma_1^z(0) \rangle_{T=\infty} = \frac{1}{2^N} \text{tr} \left( e^{t\mathcal{L}^*} [\sigma_1^z] \sigma_1^z \right)$$

- XYZ chain: Kemp, Yao, Laumann, Fendley, *JSM* (2017) 063105
- Dissipative quantum Ising: Vasiloiu *et al.*, *PRB* **98**, 094308 (2018)

## ■ Exact results

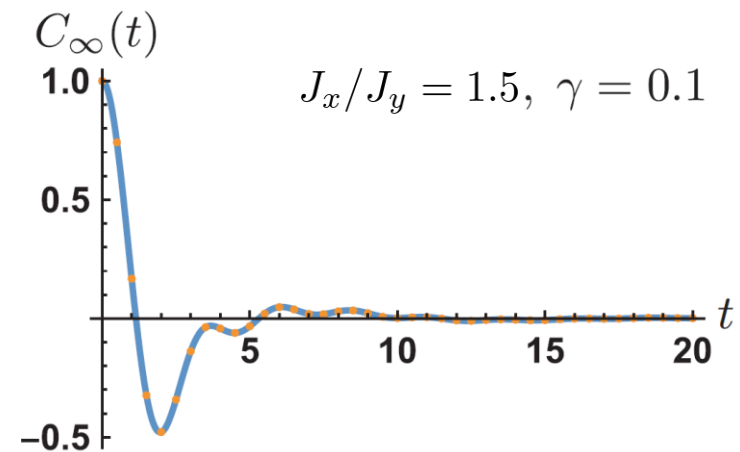
- Tour de force of 19C math.
- Topological regime



Both go to zero...

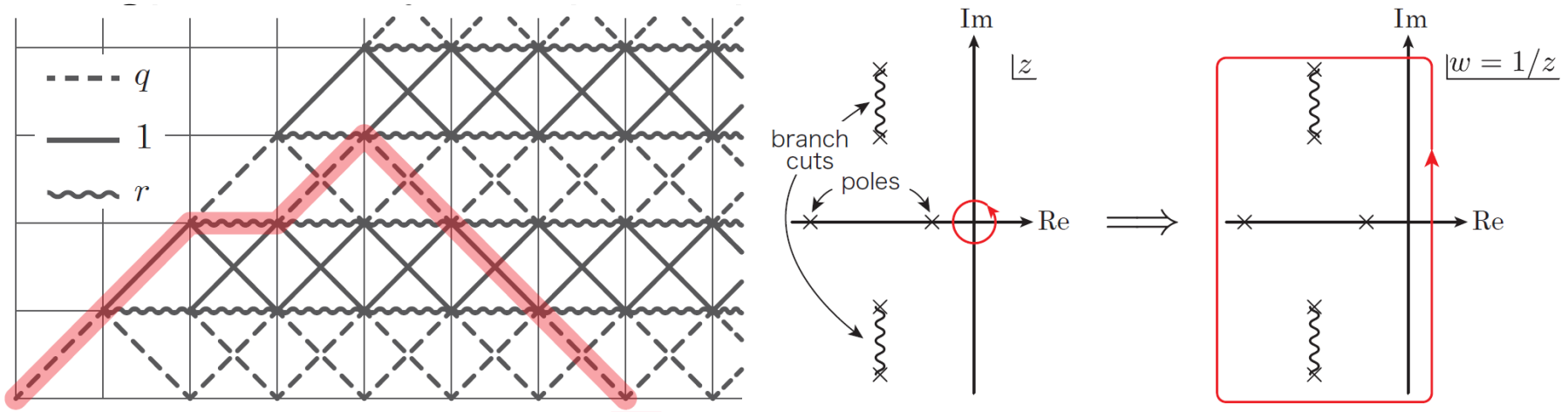
Decay is suppressed in topo. regime

- Trivial regime

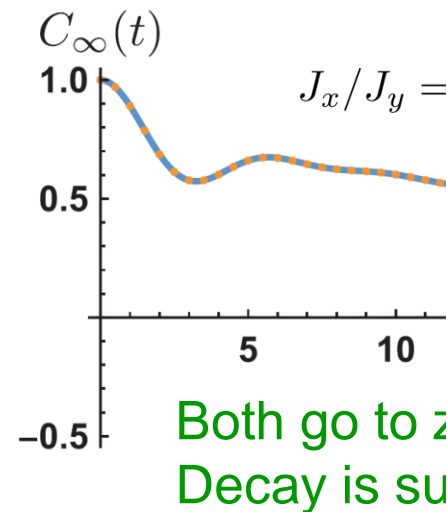




# Autocorrelator of the edge spin



- Tour de force of 19C math.
- Topological regime



$$C_\infty(t) = \begin{cases} \frac{-\eta_+^2 + 1 - q^2 + r^2}{r(\eta_+ - \eta_-)} e^{\eta_+ t} + \frac{e^{-rt}}{\pi r} \int_{1-q}^{1+q} f(y, q) \frac{(r^{-1} - r)y \cos(yt) + [y^2 - (r + \eta_+)(r + \eta_-)] \sin(yt)}{[y^2 + (r + \eta_+)^2][y^2 + (r + \eta_-)^2]} dy & (0 < q \leq 1) \\ \frac{e^{-rt}}{\pi r} \int_{q-1}^{q+1} f(y, q) \frac{(r^{-1} - r)y \cos(yt) + [y^2 - (r + \eta_+)(r + \eta_-)] \sin(yt)}{[y^2 + (r + \eta_+)^2][y^2 + (r + \eta_-)^2]} dy & (q \geq 1, 0 \leq r < 1) \\ e^{-t} \cos(\sqrt{q^2 - 1}t) + \frac{e^{-t}}{\pi} \int_{q-1}^{q+1} \frac{f(y, q)}{y^2 - q^2 + 1} \sin(yt) dy & (q \geq 1, r = 1) \\ \frac{-\eta_+^2 + 1 - q^2 + r^2}{r(\eta_+ - \eta_-)} e^{\eta_+ t} - \frac{-\eta_-^2 + 1 - q^2 + r^2}{r(\eta_+ - \eta_-)} e^{\eta_- t} + \frac{e^{-rt}}{\pi r} \int_{1-q}^{1+q} f(y, q) \frac{(r^{-1} - r)y \cos(yt) + [y^2 - (r + \eta_+)(r + \eta_-)] \sin(yt)}{[y^2 + (r + \eta_+)^2][y^2 + (r + \eta_-)^2]} dy & (q \geq 1, r > 1) \end{cases}$$

$$\eta_\pm(q, r) = \left( -1 - r^2 \pm \sqrt{(1 + r^2)^2 - 4q^2 r^2} \right) / (2r)$$

$$f(y, q) = \sqrt{[(q + 1)^2 - y^2][y^2 - (q - 1)^2]}.$$

# Outline

Introduction I

Introduction II

Model I: Quantum compass chain + dissipation

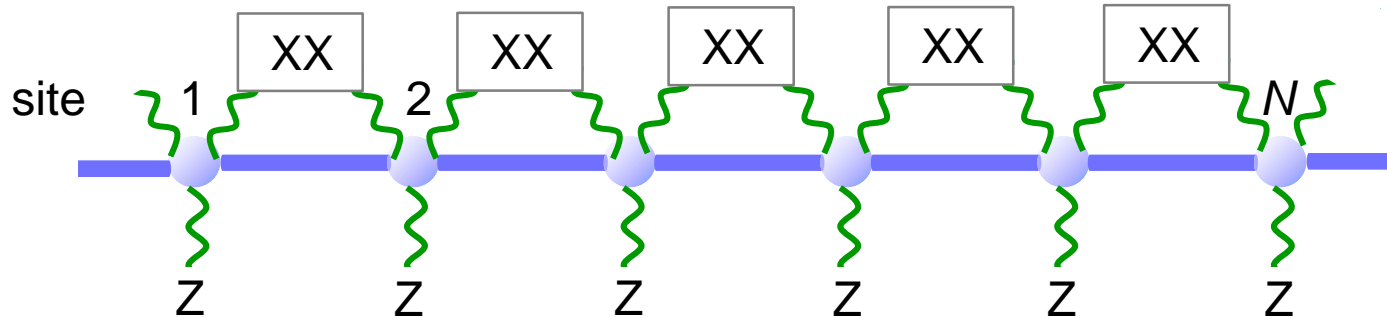
**Model II: Quantum Ising chain + dissipation**

- Hamiltonian, dissipation
- Liouvillian  $\rightarrow$  Ashkin-Teller  $\rightarrow$  Non-Hermitian XXZ
- Liouvillian gap

Summary

## Model II

### ■ Setup: spin-1/2 chain with PBC



### ■ Hamiltonian

- Quantum Ising chain

$$H = -J \sum_{j=1}^N \sigma_j^x \sigma_{j+1}^x - h \sum_{j=1}^N \sigma_j^z$$

Can be mapped to free Majorana chain. Critical at  $J=h$ .

### ■ Dissipation: dephasing + XX

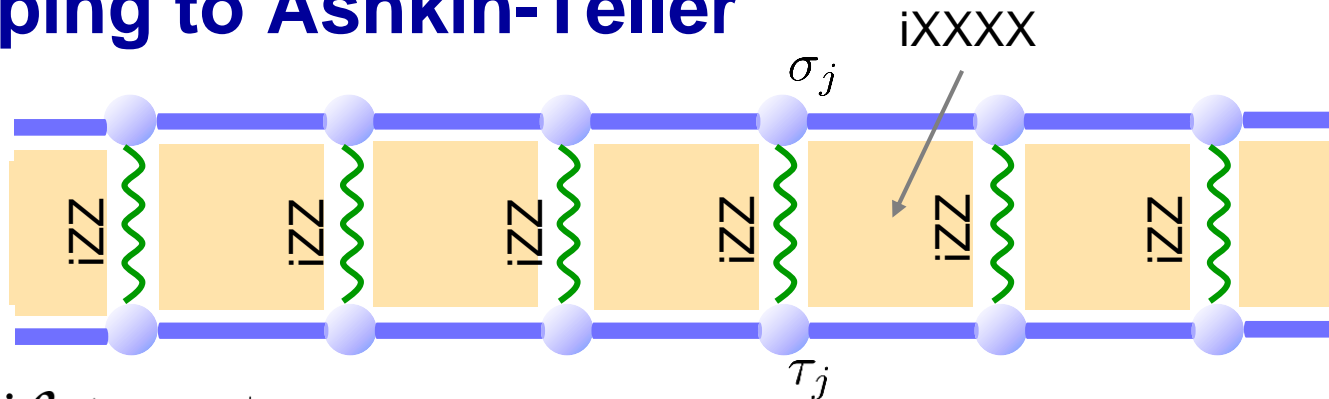
$$L_j^{(1)} = \sqrt{\Delta_1} \sigma_j^z, \quad L_j^{(2)} = \sqrt{\Delta_2} \sigma_j^x \sigma_{j+1}^x \quad (j = 1, \dots, N)$$

N.B.  $(L^{(a)})^\dagger = L^{(a)}, \quad (L_j^{(a)})^2 = \Delta_a \quad (a = 1, 2)$

# Steady states

- Parity op.  $P = \sigma_1^z \sigma_2^z \cdots \sigma_N^z$  commutes with  $H$  and  $L_j^{(1)}, L_j^{(2)} \forall j$
- NESS:  $\rho_{\pm} = \frac{1 \pm P}{2^N}$  (Uniqueness can be proved.)

# Mapping to Ashkin-Teller



$$\mathcal{H} = i\mathcal{L} + \text{const.}$$

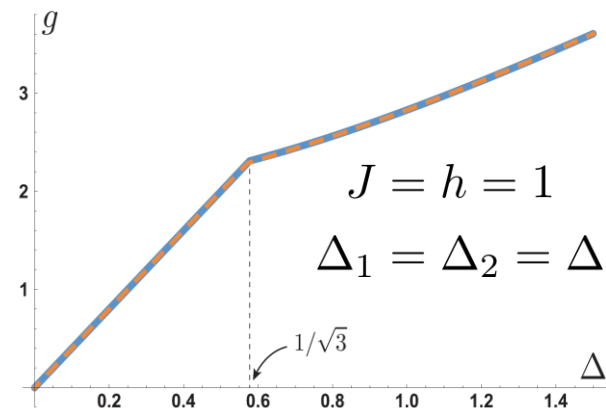
$$= - \sum_j (J\sigma_j^x \sigma_{j+1}^x + h\sigma_j^z) + \sum_j (J\tau_j^x \tau_{j+1}^x + h\tau_j^z)$$

$$+ i\Delta_1 \sum_j \sigma_j^z \tau_j^z + i\Delta_2 \sum_j \sigma_j^x \sigma_{j+1}^x \tau_j^x \tau_{j+1}^x$$

Quantum Ashkin-Teller model!

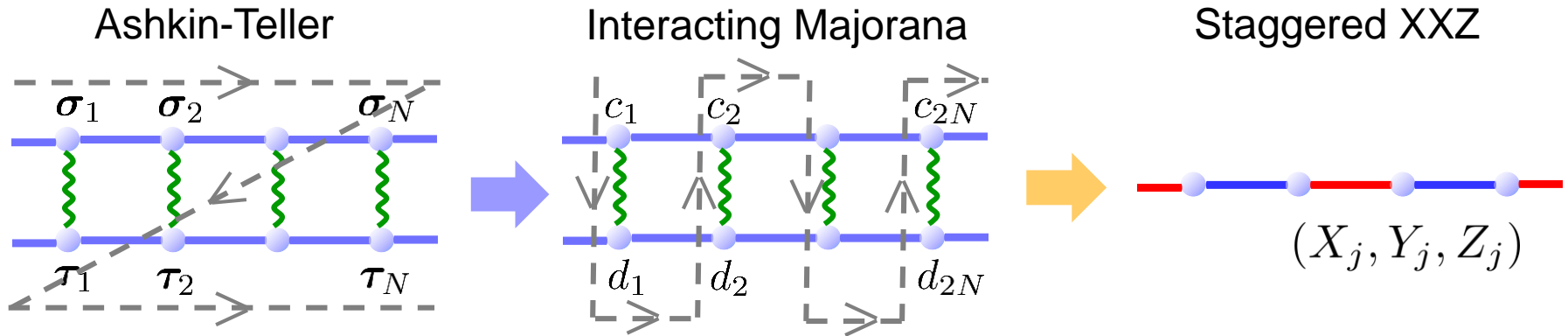
But non-Hermitian interchain coupling.

- Liouvillian gap



# From Ashkin-Teller to staggered XXZ

■ Spin  $\rightarrow$  Majorana  $\rightarrow$  Spin



Modern rephrasing of  
Kohmoto, den Nijs, Kadanoff, *PRB* **24**, 5229 (1981).

■ Non-Hermitian Hamiltonian

$$\mathcal{H} = \sum_{i=1}^N [h(X_{2i-1}X_{2i} + Y_{2i-1}Y_{2i} + i\Delta_1 Z_{2i-1}Z_{2i})$$

$$+ \sum_{i=1}^{N-1} [J(X_{2i}X_{2i+1} + Y_{2i}Y_{2i+1} + i\Delta_2 Z_{2i}Z_{2i+1})] + \mathcal{H}_{\text{boundary}}$$

$|\uparrow\rangle = |\uparrow \cdots \uparrow\rangle$  and  $|\downarrow\rangle = |\downarrow \cdots \downarrow\rangle$  correspond to 2 NESS.

# Liouvillian gap on self-dual line

## ■ Reducible to uniform XXZ

$$J = h(= 1), \quad \Delta_1 = \Delta_2 = \Delta$$

• Hamiltonian 
$$\mathcal{H} = \sum_{j=1}^N (X_j X_{j+1} + Y_j Y_{j+1} + i\Delta Z_j Z_{j+1}) + \mathcal{H}_{\text{boundary}}$$

• Boundary term

$$\mathcal{H}_{\text{boundary}} = (-1)^N Q_X (X_{2N} X_1 + Q_Z Y_{2N} Y_1) + i\Delta Q_Z Z_{2N} Z_1$$

$$Q_Z = \prod_j Z_j, \quad Q_X = \prod_j X_j \quad \text{commute with } \mathcal{H}.$$

## ■ Different sectors Alcaraz *et al.*, *Ann. Phys.* **182**, 280 (1988).

(i) $Q_Z = +1, \quad (-1)^N Q_X = +1$	Periodic	}	U(1) symm.
(ii) $Q_Z = +1, \quad (-1)^N Q_X = -1$	Anti-periodic		
(iii) $Q_Z = -1, \quad (-1)^N Q_X = +1$	Anti-diagonal twisted	}	<del>U(1) symm.</del>
(iv) $Q_Z = -1, \quad (-1)^N Q_X = -1$			

# Liouvillian gap in sectors (i) and (ii)

## ■ Explicit 1st decay mode

- Singular state  $|\chi^{(i)}\rangle = \frac{1}{\sqrt{2N}} \sum_{j=1}^{2N} (-1)^{j-1} \sigma_j^- \sigma_{j+1}^- |\uparrow\uparrow\rangle$

Avdeev, Vladimirov, *Theor. Math. Phys.* **69**, 1071 (1986)

Essler, Korepin, Schoutens, *JPA* **25**, 4115 (1992)

- Eigenstate of  $\mathcal{H}$   $\frac{1 + (-1)^N Q_X}{\sqrt{2}} |\chi^{(i)}\rangle$

with eigenvalue  $(2N - 4)\Delta i$       Similar eigenstate in (ii)

## ■ Exact gap

$$g_{(i)} = g_{(ii)} = 4\Delta$$

One can prove independently that the lower bound for  $g$  is  $4\Delta$ .

→ The above state saturates the bound!

# Gap in sectors (iii) and (iv)

## ■ Off-diagonal Bethe ansatz

- Qiao *et. al.*, *NJP* **20**, 073046 (2018)

- Maximum modulus eigenvalue  $\eta = i\Delta$

$$E = -2i \sinh \eta \sum_{j=1}^{2N} \left[ \cot \left( u_j + \frac{i\eta}{2} \right) - \cot \left( u_j - \frac{i\eta}{2} \right) \right] + 2N \sinh \eta + 2 \sinh \eta,$$

- Bethe roots

$$\mathcal{H}(\theta) = \sum (X_j X_{j+1} + Y_j Y_{j+1} + \Delta e^{i\theta} Z_j Z_{j+1})$$

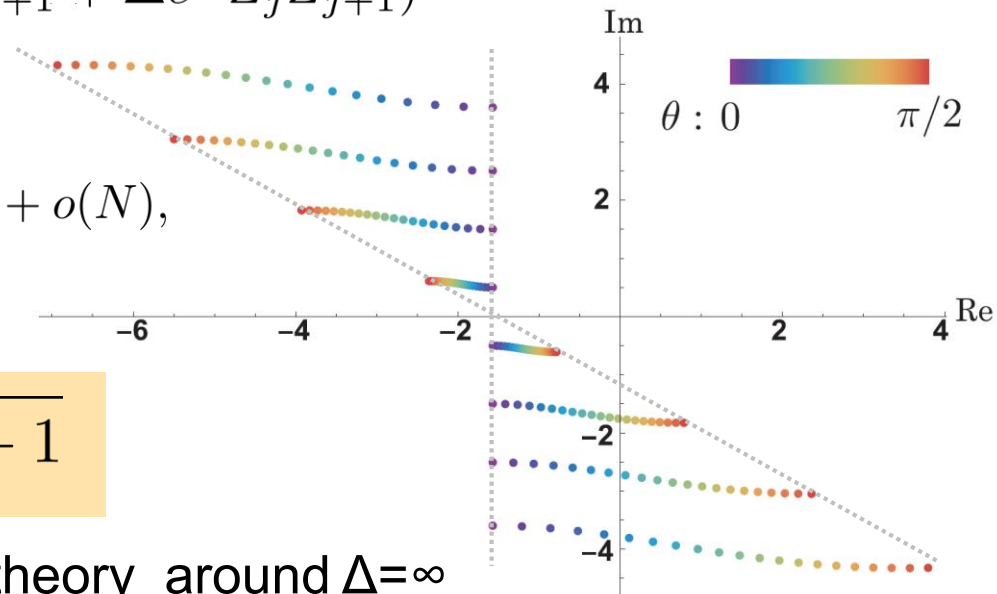
Tilted string

$$u_j = -\frac{\pi}{2} + \left( \frac{2N+1}{2} - j \right) i\eta + o(N),$$

## ■ Exact gap

$$g_{(\text{iii})} = g_{(\text{iv})} = 2\sqrt{\Delta^2 + 1}$$

Consistent with perturbation theory around  $\Delta = \infty$



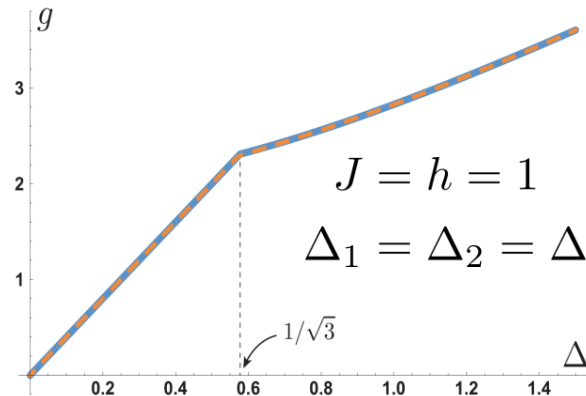
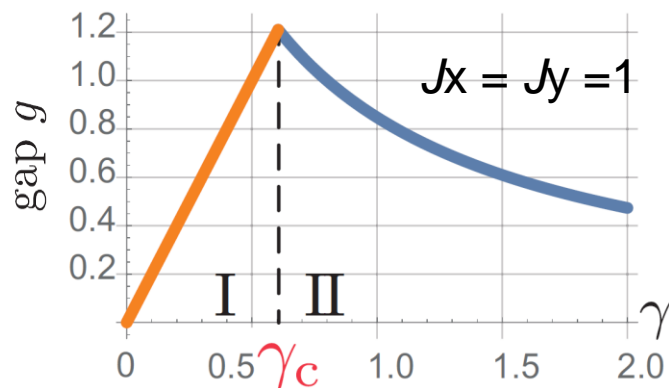


# Summary

	Hamiltonian	Dissipation	Liouvillian (after vec.)
I	$\sigma_{2j-1}^x \sigma_{2j}^x, \sigma_{2j}^y \sigma_{2j+1}^y$	$\sigma_j^z$ (dephasing)	Non-Hermitian Kitaev Ladder
II	$\sigma_j^z, \sigma_j^x \sigma_{j+1}^x$	$\sigma_j^z, \sigma_j^x \sigma_{j+1}^x$	Non-Hermitian Ashkin-Teller ( $\rightarrow$ XXZ)

- Mapping to integrable non-Hermitian Hamiltonians
- NESS = completely mixed state in each parity sector
- Exact formulas for Liouvillian gap, transition at 1st decay mode

N. Shibata and H.K: *PRB* **99**, 174303 (2019), *PRB* **99**, 224432 (2019)



# Boundary dephasing

Hamiltonian	Dissipation	Liouvillian (after vec.)
$\sigma_j^z, \sigma_j^x \sigma_{j+1}^x$	$\sigma_1^z, \sigma_N^z$ only at boundaries	SSH chain with boundary potentials

- Liouvillian gap:  $g \propto 1/N^3$  for large system size  $N$

N. Shibata and H.K: *PTEP* **2020** (2020)

## Future directions

- Complete list of integrable (lattice) Liouvillians?
- Adding purely driving term to U(1) invariant models?  
 Nakagawa, Kawakami, Ueda, PRL **126**, 110404 (2021)  $L_j = \sqrt{2\gamma} c_{j,\downarrow} c_{j,\uparrow}$   
 Buca *et al.*, NJP **22**, 123040 (2020).  $L_j = \sqrt{\gamma} \sigma_j^-$
- Continuum limit? Anything to do with (B)CFT?  
 Quantum Ising chain with dephasing
- What about higher dimensions?