Zero Width Resonance: A Symmetry Perspective

Prasanta K. Panigrahi
ppraasanta@iiserkol.ac.in

IISER KOLKATA

Indian Institute of Science Education and Research Kolkata
Mohanpur, Nadia - 741246, West Bengal, India

Collaborators: Aradhya Shukla
Shubhrajit Modak
Adipta Pal

Overview

- Introduction
  - $\mathcal{PT}$-Symmetry and Supersymmetric Quantum Mechanics
- Complex Scarf II Potential
  - Optical Connection and $\mathcal{PT}$-Symmetric Framework
- Zero Width Resonance
- Isospectral Deformation and Connection to KdV
- Summary and Conclusions

$\mathcal{PT}$-Symmetric Systems

- Conventional Quantum Mechanics (QM)

  Hermitian operators with real eigenvalues: $H = H^\dagger$

- In $\mathcal{PT}$-symmetric QM: Complex generalization of conventional QM

  For $x \to -x, \quad i \to -i \quad \Longrightarrow \quad \mathcal{PT}$-symmetric Hamiltonian

- Example of $\mathcal{PT}$-symmetric Hamiltonian:

$$H = p^2 + x^2(ix)^\epsilon$$

For $\epsilon \geq 0 \longrightarrow$ Real eigenvalues

$\epsilon < 0 \longrightarrow$ Complex eigenvalue

$\epsilon = 0 \longrightarrow$ Hermitian

$\mathcal{PT}$-Symmetric Systems

$$H = p^2 + x^2(ix)^\varepsilon \quad (\varepsilon \text{ real})$$

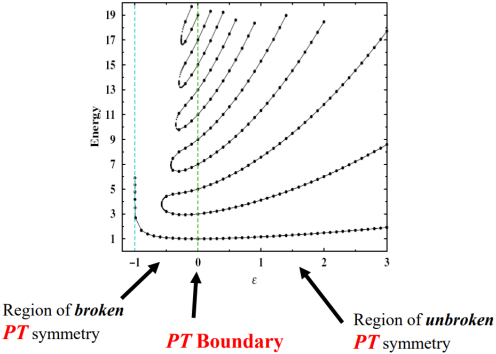

Figure 1: C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243

$\mathcal{PT}$-Symmetric Systems: Experimental Applications

PHYSICAL REVIEW LETTERS **124,** 230402 (2020)

Experimental Investigation of State Distinguishability in Parity-Time Symmetric Quantum Dynamics

Yi-Tao Wang, Zhi-Peng Li, Shang Yu, Zhi-Jin Ke, Wei Liu, Yu Meng, Yuan-Ze Yang, Jian-Shun Tang,*
Chuan-Feng Li,† and Guang-Can Guo
CAS Key Laboratory of Quantum Information, University of Science and Technology of China, Hefei 230026, People's Republic of China
and CAS Center For Excellence in Quantum Information and Quantum Physics, University of Science and Technology of China, Hefei 230026, People's Republic of China

(Received 20 October 2019; accepted 26 May 2020; published 11 June 2020)

Comprehensive study on parity-time ($\mathcal{PT}$) symmetric systems demonstrates the novel properties and innovative application of non-Hermitian physics in recent years. In the quantum regime, $\mathcal{PT}$ symmetric physics exhibits unique quantum dynamical behaviors such as spontaneous state-distinguishability oscillation. However, the construction and control of a $\mathcal{PT}$ symmetric quantum system are still challenging, that and restrict the experimental investigation of $\mathcal{PT}$ symmetric quantum nature and application. In this Letter, we propose and construct a recycling-structure $\mathcal{PT}$ symmetric quantum simulator for the first time, which can effectively simulate the discrete-time dynamical process of a $\mathcal{PT}$ symmetric quantum system in both unbroken and broken phases, to be different from our previous work [J.-S. Tang, et al., Nat. Photonics 10, 642 (2016)]. We investigate the dynamical features of quantum state distinguishability based on the $\mathcal{PT}$ symmetric simulator. Our results demonstrate the novel $\mathcal{PT}$ symmetric quantum dynamics characterized by the periodical oscillation of state distinguishability in the unbroken phase, and the monotonic decay of that in the broken phase. This work also provides a practical experimental platform for the future intensive study of $\mathcal{PT}$ symmetric quantum dynamics.

DOI: 10.1103/PhysRevLett.124.230402

PHYSICAL REVIEW LETTERS **123,** 193901 (2019)

Experimental Observation of *PT* Symmetry Breaking near Divergent Exceptional Points

M. Sakhdari,[1] M. Hajizadegan,[1] Q. Zhong,[2] D. N. Christodoulides,[3] R. El-Ganainy,[2,*] and P.-Y. Chen[1,†]
[1]*Department of Electrical and Computer Engineering, University of Illinois at Chicago, Chicago, Illinois 60607, USA*
[2]*Department of Physics and Henes Center for Quantum Phenomena, Michigan Technological University, Houghton, Michigan, 49931, USA*
[3]*College of Optics & Photonics-CREOL, University of Central Florida, Orlando, Florida 32816, USA*

(Received 8 July 2019; revised manuscript received 5 September 2019; published 7 November 2019)

Standard exceptional points (EPs) are non-Hermitian degeneracies that occur in open systems. At an EP, the Taylor series expansion becomes singular and fails to converge—a feature that was exploited for several applications. Here, we theoretically introduce and experimentally demonstrate a new class of parity-time symmetric systems [implemented using radio frequency (rf) circuits] that combine EPs with another type of mathematical singularity associated with the poles of complex functions. These nearly divergent exceptional points can exhibit an unprecedentedly large eigenvalue bifurcation beyond those obtained by standard EPs. Our results pave the way for building a new generation of telemetering and sensing devices with superior performance.

DOI: 10.1103/PhysRevLett.123.193901

OPTICS

Parity-time–symmetric microring lasers

Hossein Hodaei, Mohammad-Ali Miri, Matthias Heinrich,*
Demetrios N. Christodoulides, Mercedeh Khajavikhan†

The ability to control the modes oscillating within a laser resonator is of fundamental importance. In general, the presence of competing modes can be detrimental to beam quality and spectral purity, thus leading to spatial as well as temporal fluctuations in the emitted radiation. We show that by harnessing notions from parity-time (PT) symmetry, stable single–longitudinal mode operation can be readily achieved in a system of coupled microring lasers. The selective breaking of PT symmetry can be used to systematically enhance the maximum attainable output power in the desired mode. This versatile concept is inherently self-adapting and facilitates mode selectivity over a broad bandwidth without the need for other additional intricate components. Our experimental findings provide the possibility to develop synthetic optical devices and structures with enhanced functionality.

www.nature.com/scientificreports

SCIENTIFIC REPORTS

OPEN

Robustness and mode selectivity in parity-time (PT) symmetric lasers

M. H. Teimourpour[1], M. Khajavikhan[2], D. N. Christodoulides[2] & R. El-Ganainy[1]

Received: 19 May 2017
Accepted: 4 August 2017
Published online: 07 September 2017

We investigate two important aspects of PT symmetric photonic molecule lasers, namely the robustness of their single longitudinal mode operation against instabilities triggered by spectral hole burning effects, and the possibility of more versatile mode selectivity. Our results, supported by numerically integrating the nonlinear rate equations and performing linear stability analysis, reveals the following: (1) In principle a second threshold exists after which single mode operation becomes unstable, signaling multimode oscillatory dynamics, (2) For a wide range of design parameters, single mode operation of PT lasers having relatively large free spectral range (FSR) can be robust even at higher gain values, (3) PT symmetric photonic molecule lasers are more robust than their counterpart structures made of single microresonators; and (4) Extending the concept of single longitudinal mode operation based on PT symmetry in millimeter long edge emitting lasers having smaller FSR can be challenging due to instabilities induced by nonlinear modal interactions. Finally we also present a possible strategy based on loss engineering to achieve more control over the mode selectivity by suppressing the mode that has the highest gain (i.e. lies under the peak of the gain spectrum curve) and switch the lasing action to another mode.

RESEARCH

LASERS

Supersymmetric laser arrays

Mohammad P. Hokmabadi[1], Nicholas S. Nye[1], Ramy El-Ganainy[2],
Demetrios N. Christodoulides[1], Mercedeh Khajavikhan[1]*

Scaling up the radiance of coupled laser arrays has been a long-standing challenge in photonics. In this study, we demonstrate that notions from supersymmetry—a theoretical framework developed in high-energy physics—can be strategically used in optics to address this problem. In this regard, a supersymmetric laser array is realized that is capable of emitting exclusively in its fundamental transverse mode in a stable manner. Our results not only pave the way toward devising new schemes for scaling up radiance in integrated lasers, but also, on a more fundamental level, could shed light on the intriguing synergy between non-Hermiticity and supersymmetry.

Supersymmetric Quantum Mechanics

Supersymmetric quantum mechanics (SUSY QM) interrelates the spectra of two different Hamiltonians, $H_{\pm}$,

$$H_{\pm} = -\frac{\partial^2}{\partial x^2} + V_{\pm}(x)$$

with $V_{\pm} = W^2 \pm \frac{\partial^2 W(x)}{\partial x^2}$

$H_{\pm}$ can be obtain from $H_{-} = K^{\dagger}K$ and $H_{+} = KK^{\dagger}$ where,

$$K^{\dagger} = -\frac{\partial}{\partial x} + W(x), \qquad K = \frac{\partial}{\partial x} + W(x)$$

$W(x)$: Superpotential in SUSY QM

Superpotential $W(x) = \frac{-1}{\psi_0(x)} \frac{\partial \psi_0(x)}{\partial x}$

$\psi_0(x)$ being ground-state eigenfunction of H_{-}

If two isospectral potentials satisfy the relation[1]

$$V_+(x; a_0) = V_-(x; a_1) + R(a_1)$$

where a_0 is a set of parameters $V_\pm, a_1 = f(a_0)$, and $R(a_1)$ constant then,

$$\begin{aligned} H^s &= -\frac{\partial^2}{\partial x^2} + V_-(x; a_s) + \sum_{k=1}^{s} R(a_k) \\ &= -\frac{\partial^2}{\partial x^2} + V_+(x; a_{s-1}) + \sum_{k=1}^{s-1} R(a_k) \end{aligned}$$

with ground state energy: $E_0 = \sum_{k=1}^{s} R(a_k)$

Identifying $H^1 = H_+$ and $H^0 = H_-$, spectrum for H_-

$E_n = \sum_{k=1}^{n} R(a_k)$ $V_\pm$: Shape Invariant (SI) Potentials

[1] L. Gendenshtein, Pis'ma Zh. Eksp. Teor. Fiz. 38 (1983) 299; JETP Lett. 38 (1983) 356

$\mathcal{PT}$-Symmetric Systems: Optical Connection

The propagation equation for transverse electric waves

$$\frac{d^2\mathcal{E}}{dx^2} + (k_0^2\epsilon(x) - k_y^2)\mathcal{E} = 0 \qquad \text{(Helmholtz equation)}$$

Dielectric distribution $\epsilon(x) = \epsilon_b + \alpha(x)$, as $|x| \to \infty$ $\epsilon(x) = \epsilon_b$

Propagation vector $k_y = k_0\sqrt{\epsilon_b}\sin\theta$

- The Helmholtz equation $\Longrightarrow$ Schrödinger equation

  with $E = k_0^2\epsilon_b\cos^2\theta, \qquad V(x) = k_0^2\epsilon_b - k_0^2\epsilon(x)$

$\mathcal{PT}$-Symmetric Systems: Optical Connection

- For reflectionless potential, Schrödinger-Helmholtz connection maps the quantum potential to corresponding dielectric profile

  The refractive index $\quad n^2(x) = n_b^2 - \frac{V(x)}{k_0^2}$

  In $\mathcal{PT}$-symmetric farmework: $n(x) = n^*(-x)$

- We start with Complex Scarf II potential profile[2]

$$V(x) = V_1 sech^2(\alpha x) - iV_2 sech(\alpha x)\ tanh(\alpha x)$$

$$V_1, V_2:\ \text{Real constant}$$

  - Vanishes asymptotically
  - Possible to regulate the permittivity and refractive index

[2] Z. Ahmed, Phys. Lett. A, 282 (2001) 343

$\mathcal{PT}$-Symmetric Framework

The complex superpotentials,

$$W^{\pm}_{\mathcal{PT}} = \big(A \pm iC^{\mathcal{PT}}\big)\, tanh(\alpha x) + \big(\pm C^{\mathcal{PT}} + iB\big)\, sech(\alpha x), \qquad (1)$$

both real and complex spectra.

- Under certain parametrization, the superpotentials leading to real eigenvalues and corresponding wavefunctions
- For a different parametrization, two different superpotentials yield the same potential, leading to complex conjugate eigenvalues
  - The energy spectra are separated by a bifurcation.
  - Parametric condition for phase-transition from real to complex conjugate spectrum

Spontaneous Breaking of $\mathcal{PT}$-Symmetry!

$\mathcal{PT}$-Symmetric Framework

Using SUSY quantum mechanics, lead to potential

$$V(x) = -\left[(A \pm iC^{\mathcal{PT}})(A \pm iC^{\mathcal{PT}} + \alpha) - (\pm C^{\mathcal{PT}} + iB)^2\right] sech^2(\alpha x) - i(\pm iC^{\mathcal{PT}} - B)\left[2(A \pm iC^{\mathcal{PT}}) + \alpha)\right] sech(\alpha x) tanh(\alpha x), \quad (2)$$

- To be $\mathcal{PT}$-symmetric, the coefficient of the first term must be real and that of the second term must be purely imaginary
- $\mathcal{PT}$-symmetric condition gives a unique relationship[3]

$$C^{\mathcal{PT}}\left[2(A - B) + \alpha\right] = 0$$

[3] K. Abhinav and P. K. Panigrahi, Annals of Phys., 325 (2010) 1198

$\mathcal{PT}$-Symmetric Phase

- For $C^{\mathcal{PT}} = 0$ Case

  $\mathcal{PT}$-symmetric potential

$$V_{\mathcal{PT}}(x) = -\left[A(A+\alpha)+B^2\right] sech^2(\alpha x) + iB(2A+\alpha) sech(\alpha x) tanh(\alpha x), \quad (3)$$

  has superpotential,

$$W_1(x) = A\ tanh(\alpha x) + iB\ sech(\alpha x). \quad (4)$$

- Parametric freedom: $A \leftrightarrow B - \frac{\alpha}{2}$ leads to

$$W_2(x) = (B - \frac{\alpha}{2}) tanh(\alpha x) + i(A + \frac{\alpha}{2}) sech(\alpha x). \quad (5)$$

  Corresponding real energy eigenvalues,

$$\boxed{E_n^1 = -(A - n\alpha)^2,} \qquad \boxed{E_n^2 = -(B - \alpha/2 - n\alpha)^2}$$

Domain I $(A < 0,\ B > \alpha/2)$

Shape Invariance: $A \to -A, B \to -B$ and $A \to A - \alpha,\ B \to B$

$$E^1_{n,l} = -(A + n\alpha)^2 \qquad\qquad E^2_{n,l} : \text{Unchanged}$$

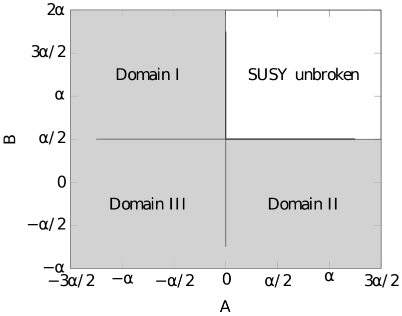

Figure 2: The shaded regions I, II and III are broken SUSY areas in the parameter space

$\mathcal{PT}$-Symmetric Phase

Domain II $(A > 0,\ B < \alpha/2)$

SI: $B - \alpha/2 \to -(B - \alpha/2),\ \ A + \alpha/2 \to -(A - \alpha/2)$

$$B - \alpha/2 \to B - \alpha/2 - \alpha \quad A - \alpha/2 \to A - \alpha/2,$$

$$\boxed{E^2_{n,II} = -(B - \alpha/2 + n\alpha)^2} \qquad \boxed{\mathrm{E}^1_{n,II} : \text{Unchanged}}$$

Domain III $(A < 0,\ B < \alpha/2)$

Using both the above SIs

$$\boxed{E^1_{n,III} = -(A + n\alpha)^2} \qquad \boxed{E^2_{n,III} = -(B - \alpha/2 + n\alpha)^2}$$

$\mathcal{PT}$-Broken Phase

- For $C^{\mathcal{PT}} \neq 0$ Case $\boxed{A = B - \alpha/2}$

$$W^{\pm}_{\mathcal{PT}} = \Big(A \pm iC^{\mathcal{PT}}\Big) tanh(\alpha x) + \Big(\pm C^{\mathcal{PT}} + i(A+\alpha/2)\Big) sech(\alpha x) \quad (6)$$

$$\begin{aligned} V^{C}_{\mathcal{PT}}(x) = & -\left[2A(A+\alpha) - 2C^{\mathcal{PT}2} + \frac{\alpha^2}{4}\right] sech^2(\alpha x) \\ & + i\left[2A(A+\alpha) + 2C^{\mathcal{PT}2} + \frac{\alpha^2}{2}\right] sech(\alpha x) tanh(\alpha x), \end{aligned} \quad (7)$$

which satisfy the condition[4] $|V_2| > V_1 + \frac{1}{4}$

V_1, V_2 : coefficients of real and imaginary part of $V^{C}_{\mathcal{PT}}$

- The superpotentials lead to the same PT-symmetric complex potential with energy spectra,

$$E^{\pm}_{n} = -(n\alpha - A \pm iC^{\mathcal{PT}})^2,$$

[4] Z. Ahmed, Phys. Lett. A 282 (2001) 343,
K. Abhinav and P. K. Panigrahi, Annals of Phys., 325 (2010) 1198

$\mathcal{PT}$-Broken Phase

Corresponding eigenfunctions

$$\psi_n^{\pm}(x) \propto \Big(sech(\alpha x)\Big)^{\frac{1}{\alpha}(A+\frac{\alpha}{2}\pm iC^{\mathcal{PT}})} \exp\Big[-\frac{i}{\alpha}(A+\frac{\alpha}{2}\mp iC^{\mathcal{PT}})$$

$$tan^{-1}(sinh(\alpha x))\Big] P_n^{\mp\frac{2iC^{\mathcal{PT}}}{\alpha},-\frac{2A}{\alpha}-1}[i\ sinh(\alpha x)].$$

$P_n^{\mp\frac{2iC^{\mathcal{PT}}}{\alpha},\frac{2A}{\alpha}+1}[i\,\sinh(\alpha x)]$: n^{th} order Jacobi polynomial

Scattering by $\mathcal{PT}$-Symmetric Potential

For $\mathcal{PT}$-symmetric potential, the continuity equation is

$$\frac{-i\hbar}{2m}\frac{\partial}{\partial x}\left[\psi(x,t)\frac{\partial}{\partial x}\psi^{\star}(-x,t)-\psi^{\star}(-x,t)\frac{\partial}{\partial x}\psi(x,t)\right]=\frac{\partial}{\partial x}\left(\psi^{\star}(-x,t)\psi(x,t)\right)$$

New definition for Flux Conservation

- The scalar emerging from above equation correlation function between two parity-opposite spatial points
- Correlation function is a conserved scalar of the $\mathcal{PT}$-symmetric system[5]

[5] K. Abhinav, A. Jayannavar and P.K. Panigrahi, Annals of Phys. 331, (2013) 110

Scattering by $\mathcal{PT}$-Symmetric Potential

$\mathcal{PT}$-symmetric scalar product of $\phi(x,t)$ and $\psi(x,t)$ is

$$\begin{aligned}\int_{-\infty}^{\infty}\phi(x,t)\mathcal{PT}\psi(x,t)dx &= \int_{-\infty}^{\infty}\phi(x,t)\psi^{\star}(-x,t)dx \\ &= \int_{-\infty}^{\infty}\psi^{\star}(x,t)\phi(-x,t)dx \\ &= \int_{-\infty}^{\infty}\psi^{\star}(x,t)\mathcal{P}\phi(x,t)dx\end{aligned}$$

last expression is Dirac–von Neumann scalar product[6]

[6] C.M. Bender, D.C. Brody and S.F. Jones Phys. Rev. Lett., 89 (2002) 270401; Amer. J. Phys., 71 (2003), 1095

Scattering by $\mathcal{PT}$-Symmetric Potential

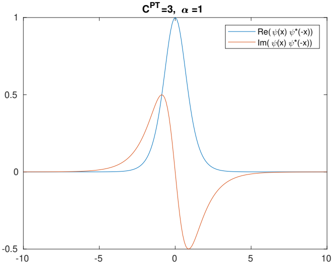

Figure 3: Ground state correlation function for $\psi^{\pm}_{n,SS}(x)$

Scattering by $\mathcal{PT}$-Symmetric Potential

Asymptotic analysis leads to the transmission and reflection coefficients[7],

$$\mathcal{T}\left(\kappa, \frac{A}{\alpha}, \frac{iB}{\alpha}\right) = \frac{\Gamma[-\frac{A}{\alpha} - \frac{i\kappa}{\alpha}]\Gamma[1 + \frac{A}{\alpha} - \frac{i\kappa}{\alpha}]\,\Gamma[\frac{1}{2} - \frac{B}{\alpha} - \frac{i\kappa}{\alpha}]\Gamma[\frac{1}{2} + \frac{B}{\alpha} - \frac{i\kappa}{\alpha}]}{\Gamma[-\frac{ik}{\alpha}]\,\Gamma[1 + \frac{ik}{\alpha}]\,\Gamma^2[\frac{1}{2} - \frac{ik}{\alpha}]},$$

$$\mathcal{R}\left(\kappa, \frac{A}{\alpha}, \frac{iB}{\alpha}\right) = i\,\mathcal{T}\left(\kappa, \frac{A}{\alpha}, \frac{iB}{\alpha}\right)\left[\frac{cos(\frac{\pi A}{\alpha})\,sin(\frac{\pi B}{\alpha})}{cosh\frac{\pi\kappa}{\alpha}} - \frac{sinh(\frac{\pi A}{\alpha})\,cosh(\frac{\pi B}{\alpha})}{sinh\frac{\pi\kappa}{\alpha}}\right],$$

where $\kappa = \frac{\alpha}{i}(n - A/\alpha)$

Dirac–von-Neumann scalar product

$$|\mathcal{R}|^2 + |\mathcal{T}|^2 = 1 + \left[\frac{cos^2(\pi A/\alpha)sin^2(\pi B/\alpha)sinh^2(\pi k/\alpha) + sin(2\pi A/\alpha)sin(2\pi B/\alpha)sinh(2\pi k/\alpha)}{sinh^2(\pi k/\alpha)sin^2(\pi A/\alpha)cos^2(\pi B/\alpha))cosh^2(\pi k/\alpha) - cos^2(\pi A/\alpha)sin^2(\pi B/\alpha)sinh^2(\pi k/\alpha)}\right]$$

- Flux is NOT conserved, due to imaginariy part, causing absorption and emmision[8]
- The deviation term is Non-Zero whereas it vanishes for $B \to \pm iB$

---

[7] K. Abhinav, A. Jayannavar and P. K. Panigrahi, Annals of Phys., 331 (2013) 110

[8] A. Mostafazadeh, Pramana,73 (2009) 269

Zero Width Resonance

- For $A = n\,\alpha$: Abrupt transition from low amplitude constant function to high amplitude oscillations[9]

No net gain or loss $\implies$ Zero Width Resonance!

- Coefficients V_1, V_2 satisfy the Zero Width Resonance Condition[10]

$$V_1 + |V_2| = 4n^2 + 4n + 3/4 \ \ (n = 0, 1, 2, ...)$$

- Energy Eigenvalue $E_{SS} = {C^{\mathcal{PT}}}^2$: Real and Positive

And corresponding eigenfunctions

$$\psi^{\pm}_{n,SS}(x) \propto \Big(sech(\alpha x)\Big)^{n+\frac{1}{2}\pm i\frac{C^{\mathcal{PT}}}{\alpha}} \exp\Big[(-i(n+\frac{1}{2}) \mp \frac{C^{\mathcal{PT}}}{\alpha}) \tan^{-1}(sinh(\alpha x)\Big] P_n^{\mp\frac{2iC^{\mathcal{PT}}}{\alpha},-2n-1}[i\ sinh(\alpha x)]. \quad (8)$$

[9] A. Pal, S. Modak and P. K. Panigrahi, arXiv:1803.06476v2 [quant-ph]
[10] Z. Ahmed, J. Phys. a: Math. Theor. 42 (2009) 472005

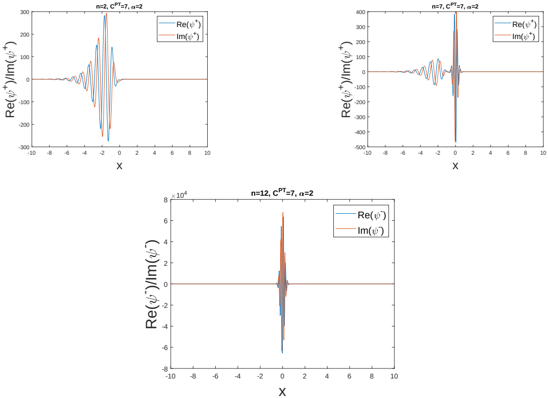

Figure 4: Competing effects between A and $C^{\mathcal{PT}}$. When $A > C^{\mathcal{PT}}$ localization at $x = 0$.

The Wigner function in phase space is

$$W(x,p) = \int_{-\infty}^{\infty} dy \left[e^{-ipy}\ \psi^{\pm\star}_{n,SS}\left(x - \frac{y}{2}\right)\ \psi^{\pm}_{n,SS}\left(x + \frac{y}{2}\right)\right]$$

For ground state

$$W(x,p) = \int_{-\infty}^{\infty} dy\ e^{-ipy} \left[\Big(sech\alpha(x - y/2)\Big)^{n+\frac{1}{2}\mp i\frac{C^{\mathcal{PT}}}{\alpha}}\right.$$
$$\exp\Big[(i(n+\frac{1}{2}) \mp \frac{C^{\mathcal{PT}}}{\alpha})tan^{-1}(sinh\alpha(x - y/2)\Big]$$
$$\Big[\Big(sech\alpha(x + y/2)\Big)^{n+\frac{1}{2}\pm i\frac{C^{\mathcal{PT}}}{\alpha}} \exp\Big[(-i(n+\frac{1}{2}) \mp \frac{C^{\mathcal{PT}}}{\alpha})$$
$$\left. tan^{-1}(sinh\alpha(x + y/2)\Big]\right.$$

Zero Width Resonance

Wigner Function

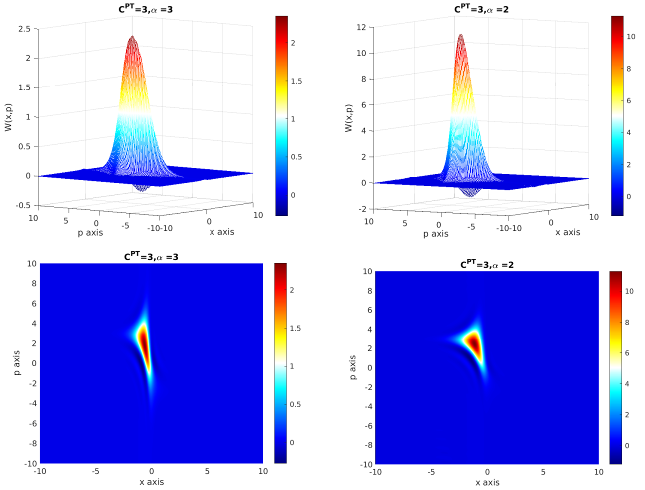

Figure 5: Ground state Wigner function for $\psi^{\pm}_{n,SS}(x)$

Isospectral Deformation and Connection with KdV

Isospectral deformation $\implies$ The two different superpotentials leading to same potential superpotentials

$$\tilde{W}(x) = W(x) + v(x), \tag{9}$$

$\tilde{W}(x)$ and $W(x)$ will provide the Same $V_-(x)$ when

$$\frac{dv(x)}{dx} - 2W(x)v(x) - v^2(x) = 0 \quad \text{[Bernoulli equation]} \tag{10}$$

- The deformed potentials satisfy the KdV equation and the corresponding superpotentials obey the mKdV equation

Isospectral Deformation and Connection with KdV

For $\mathcal{PT}-$symmetric potential

$$V_{\mathcal{PT}}(x) = -\left[A(A+\alpha)+B^2\right] sech^2(\alpha x) + iB(2A+\alpha) sech(\alpha x) tanh(\alpha x),$$

The difference between the superpotentials

$$v(x) = -2A\ tanh(\alpha x) + i\ 2A\ sech(\alpha x). \tag{11}$$

which satisfies the mKdV equation

- Two distinct mKdV equations whose solutions are related through $v \to iv$, where v is the mKdV variable.
- The Miura transformation for the solution of KdV, $u = v^2 \pm v_x$, then implies that, $u = -v^2 \pm iv_x$, is also a solution[11]

[11] S. Modak, A. P. Singh and Panigrahi, P. K. Panigrahi
Eur. Phys. J. B , 89 (v) 149

Hysteresis Approach

To get the correct ground state, we apply hysteresis approach which gives hyperbola in the parameter space A and B

$$\frac{A^2}{K^2} - \frac{(B - \alpha/2)^2}{K^2} = 1,$$

- Ground state energy is $-A^2$ with Superpotential given in Eqn. (4)

$$\frac{(B - \alpha/2)^2}{K_1^2} - \frac{A^2}{K_1^2} = 1$$

- $-(B - \alpha/2)^2$ Ground state energy and Superpotential is in Eqn. (5)

Hysteresis Approach

- **Region I** Ground state $-A^2$ with superpotential W_1
- **Region II** $-(B - \frac{\alpha}{2})^2$ is ground state with superpotential W_2

  (a) correspond to $A = B - \alpha/2$ asymptotic condition for zero width resonance

  (b) is the asymptote $A = -B + \alpha/2$ for region of real energy eigenvalues and it relates the superpotentials in Eq. (4) and Eq. (5) as **isospectrally deformed** counterparts.
- For co-ordinates $(\alpha/2, 0)$ and $(-\alpha/2, \alpha)$ isospectral deformation and its associated potential satisfy the **mKdV equation and KdV equation** respectively.

Hysteresis Approach

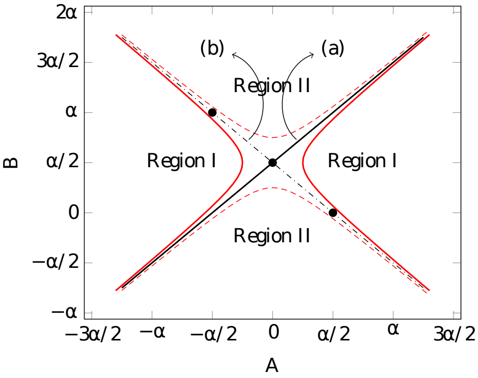

Figure 6: Regions I and II correspond to the specific ground states based on the parameters involved (a) corresponds to the condition for exceptional point and zero width resonance while that with negative slope (b) corresponds to isospectral deformation

Summary and Conclusions

- The formation of exceptional point as well as existence of both bound and zero-width resonance states in the broken $\mathcal{PT}$ scenario have been demonstrated.

- Breaking of SUSY and $\mathcal{PT}$-symmetry are found to exist in mutually exclusive domains with the $\mathcal{PT}$-symmetry boundaries showing hysteresis.

- The unusual nature of eigenfunctions at zero width resonances give rise to wave packet localization for higher modes.

- This is interesting in the sense that it can be utilized to build novel devices to store electric field energy, without loss.

Thanks!