Ladder operators (and their weak bi-coherent states)

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Organization of the talk

1. **Part 1, F.B., J.Phys. A, 2021:** $\mathfrak{su}(1,1)$*-type Lie algebra...*
2. *...its ladder operators...*
3. *...and the eigenvectors of the number-like operators.*
4. **Part 2 (maybe...), F.B., J.Phys.: Conference Series, 2021:** *eigenvectors and bi-coherent states outside* $\mathcal{L}^2(\mathbb{R})$*...*
5. *...when their products are in* $\mathcal{L}^1(\mathbb{R})$*...*
6. *...and when they are not: weak bicoherent states.*

Two words of history

The beginning of Susy QM: H factorizable, $H = BA = H^\dagger$, has a SUSY partner

$$H_{susy} = AB = H + [B, A].$$

There exists a well known relation between the eigenstates of H and H_{susy}: let φ be such that

$$H\varphi = E\varphi,$$

for $E \in \mathbb{R}$, its eigenvalue. Then, if $\varphi_A := A\varphi \neq 0$, then

$$H_{susy}\varphi_A = E\varphi_A.$$

The eigenvectors of H_{susy} can be deduced out of those of H... (which, by the way, have to be found! And this could be not so easy!!)

If $H \neq H^\dagger$, then the adjoint map, †, produce a second pair of operators, $H^\dagger$ and $H_{susy}^\dagger$, whose eigenvectors are again somehow related to those of H.

The best is when we can find, with a general strategy, the eigenvalues and the eigenvectors of H. Then those of $H^\dagger$, H_{susy} and $H_{susy}^\dagger$ can be obtained in an automatic way.

This is what we will show in this part of the talk.

Two more words of (recent) history

In 2018 Williams and coauthors introduced the notion of *coupled SUSY* (CSusy). Roughly speaking, CSusy arises out of two operators a and b, acting on an Hilbert space $\mathcal{H}$, and two real non-zero numbers γ, δ, with $\delta > \gamma$, satisfying the following:

$$a^\dagger a = bb^\dagger + \gamma 1\!\!1, \qquad aa^\dagger = b^\dagger b + \delta 1\!\!1.$$

Of course, we need to impose $R(a) \subset D(a^\dagger)$ and viceversa, and the same for b.. These equalities are really different. One can be satisfied while the other does not hold:

$$a = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right), \quad b = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right)e^{ix}.$$

Trivial example: bosons. Let c be an operator on $\mathcal{H}$ satisfying (in the sense on unbounded operators) the canonical commutation relation $[c, c^\dagger] = 1\!\!1$, then the equations above are satisfied taking $a = b = c$, $\delta = 1$, $\gamma = -1$.

The operators

$$\mathcal{K}_+ = \frac{1}{\delta - \gamma}a^\dagger b^\dagger, \qquad \mathcal{K}_- = \frac{1}{\delta - \gamma}ba, \qquad \mathcal{K}_0 = \frac{1}{\delta - \gamma}\left(a^\dagger a - \frac{\gamma}{2}\right),$$

satisfy the following commutation rules,

$$[\mathcal{K}_0, \mathcal{K}_\pm] = \pm\mathcal{K}_\pm, \qquad [\mathcal{K}_+, \mathcal{K}_-] = -2\mathcal{K}_0,$$

which are those of the $\mathfrak{su}(1,1)$ Lie algebra. Hence, $\mathcal{K}_\pm$ act as ladder operators, while $\mathcal{K}_0$ is some sort of Hamiltonian. Notice that $\mathcal{K}_0 = \mathcal{K}_0^\dagger$, and $\mathcal{K}_\pm^\dagger = \mathcal{K}_\mp$.

Two words of history

Main question:– how much of the ladder structure described by Williams does survive when *loosing self-adjointness*?

Technical question:– Is there any algebraic approach which *takes care* of the domain issues automatically also in presence of unbounded operators?

In other words:– is it possible to define a $*$-algebra containing also unbounded operators? ...

Two words of history

Main question:– how much of the ladder structure described by Williams does survive when *loosing self-adjointness*?

Technical question:– Is there any algebraic approach which *takes care* of the domain issues automatically also in presence of unbounded operators?

In other words:– is it possible to define a $*$-algebra containing also unbounded operators? ......**yes!**

The algebraic settings

In this talk we will use a particular unbounded operator algebra, the O^*-algebra $\mathcal{L}^\dagger(\mathcal{D})$. Other possibilities exist.

Definition:

Let $\mathcal{H}$ be a separable Hilbert space and N_0 an unbounded, densely defined, self-adjoint operator. Let $D(N_0^k)$ be the domain of the operator N_0^k, $k \geq 0$, and $\mathcal{D}$ the domain of all the powers of N_0, that is,

$$\mathcal{D} = \bigcap_{k \geq 0} D(N_0^k).$$

This set is dense in $\mathcal{H}$. We call $\mathcal{L}^\dagger(\mathcal{D})$ the $*$-algebra of all *closable operators* defined on $\mathcal{D}$ which, together with their adjoints, map $\mathcal{D}$ into itself. Here the adjoint of $X \in \mathcal{L}^\dagger(\mathcal{D})$, $X^\dagger$, is the restriction of the adjoint of X in $\mathcal{H}$ (which we also indicate with $X^\dagger$) to $\mathcal{D}$.

In $\mathcal{D}$ the topology is defined by the following N_0-depending seminorms:

$$\varphi \in \mathcal{D} \to \|\varphi\|_n \equiv \|N_0^n \varphi\|,$$

where $n \geq 0$, while the topology τ_0 in $\mathcal{L}^\dagger(\mathcal{D})$ is introduced by the seminorms

$$X \in \mathcal{L}^\dagger(\mathcal{D}) \to \|X\|^{f,k} \equiv \max\left\{\|f(N_0) X N_0^k\|, \|N_0^k X f(N_0)\|\right\},$$

where $k \geq 0$ and $f \in \mathcal{C}$, the set of all the positive, bounded and continuous functions on $\mathbb{R}_+$, which are decreasing faster than any inverse power of x: $\mathcal{L}^\dagger(\mathcal{D})[\tau_0]$ is a complete *-algebra.

The algebraic settings

Hence,

$$\forall x, y \in \mathcal{L}^\dagger(\mathcal{D}), \quad \Rightarrow \quad x^\dagger, y^\dagger, xy, yx, [x, y] \in \mathcal{L}^\dagger(\mathcal{D}).$$

Also, powers of x and y all belong to $\mathcal{L}^\dagger(\mathcal{D})$, which therefore is a good candidate to work with, also in presence of unbounded operators.

A first example:– if $N_0 = c^\dagger c$, where $c = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)$, then $\mathcal{D} = \mathcal{S}(\mathbb{R})$ and we can prove that $c, c^\dagger \in \mathcal{L}^\dagger(\mathcal{D})$. Hence $N_0 \in \mathcal{L}^\dagger(\mathcal{D})$ as well.

A second example:– let now a and b be two operators on $\mathcal{H}$, with domains $D(a)$ and $D(b)$ respectively, $a^\dagger$ and $b^\dagger$ their adjoint, and let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$ such that $a^\sharp\mathcal{D} \subseteq \mathcal{D}$ and $b^\sharp\mathcal{D} \subseteq \mathcal{D}$, where $x^\sharp$ is x or $x^\dagger$. Of course, $\mathcal{D} \subseteq D(a^\sharp)$ and $\mathcal{D} \subseteq D(b^\sharp)$.

Definition:

The operators (a, b) are $\mathcal{D}$-*pseudo-bosonic* if, for all $f \in \mathcal{D}$, we have

$$a\,b\,f - b\,a\,f = f.$$

Hence a number-like operator $N = ba$ can be defined, $N \neq N^\dagger$, with $N^\dagger$ sharing with N all its eigenvalues, $n = 0, 1, 2, 3, \ldots$. If a and b are similar to c and $c^\dagger$, via some (unbounded) operator leaving $\mathcal{D}$ stable together with its inverse, then a, b and their powers and combinations belong to $\mathcal{L}^\dagger(\mathcal{D})$.

ECSusy

Definition:

Let d, c, r and s be four elements of $\mathcal{L}^\dagger(\mathcal{D})$, for some suitable $\mathcal{D}$ dense in $\mathcal{H}$, and let γ, δ be two real numbers with $\delta > \gamma$. We say that $(d, c, r, s; \delta, \gamma)$ define an *extended coupled Susy* (ECSusy), if the following equalities are satisfied:

$$dc = rs + \gamma 1\!\!1, \qquad cd = sr + \delta 1\!\!1.$$

Let us define the following operators, still in $\mathcal{L}^\dagger(\mathcal{D})$:

$$k_+ = \frac{1}{\delta-\gamma} ds, \qquad k_- = \frac{1}{\delta-\gamma} rc, \qquad k_0 = \frac{1}{\delta-\gamma}\left(dc - \frac{\gamma}{2} 1\!\!1\right), \tag{1}$$

and their superpartners

$$l_+ = \frac{1}{\delta-\gamma} sd, \qquad l_- = \frac{1}{\delta-\gamma} cr, \qquad l_0 = \frac{1}{\delta-\gamma}\left(sr + \frac{\delta}{2} 1\!\!1\right). \tag{2}$$

They obey the following commutation relations:

$$[k_0, k_\pm] = \pm k_\pm, \qquad [k_+, k_-] = -2k_0, \tag{3}$$

$$[l_0, l_\pm] = \pm l_\pm, \qquad [l_+, l_-] = -2l_0. \tag{4}$$

These look like the commutators for $\mathcal{K}_\alpha$ before, but with a big difference: k_+ and l_+ are not the adjoint of k_- and l_-, and k_0 and l_0 are not self-adjoint.

ECSusy

This gives us the possibility to introduce two other families of operators, p_α and q_α, $\alpha = 0, \pm$:

$$p_0 = k_0^\dagger, \qquad p_\pm = k_\mp^\dagger; \qquad\qquad q_0 = l_0^\dagger, \qquad q_\pm = l_\mp^\dagger.$$

They satisfy the same commutators in (3) and (4):

$$[p_0, p_\pm] = \pm p_\pm, \qquad [p_+, p_-] = -2p_0; \qquad [q_0, q_\pm] = \pm q_\pm, \qquad [q_+, q_-] = -2q_0.$$

Hence we conclude that an ECSusy produces four (in general) different triples of operators obeying the same commutators of an $\mathfrak{su}(1,1)$ Lie algebra, but with different relations under the adjoint operation.

In the next two slides we will deduce some consequences of these commutation rules.

An interlude on deformed $\mathfrak{su}(1,1)$

Let $x_\pm, x_0 \in \mathcal{L}^\dagger(\mathcal{D})$, such that

$$[x_0, x_\pm] = \pm x_\pm, \qquad [x_+, x_-] = -2x_0,$$

but with $x_+^\dagger \neq x_-$ and $x_0^\dagger \neq x_0$. We put, *with a slight abuse of notation*,

$$x^2 = x_0^2 - \frac{1}{2}(x_+x_- + x_-x_+) = x_0^2 + x_0 - x_-x_+ = x_0^2 - x_0 - x_+x_-.$$

Notice that x^2 is not really the square of an operator x (to be identified) and, moreover, x^2 is not even positive. We keep this notation since it is somehow standard. We have

$$[x^2, x_\alpha] = 0, \qquad \alpha = 0, \pm.$$

We can then look for common eigenstates of, say, x^2 and x_0. Using again the same notation adopted for ordinary $\mathfrak{su}(1,1)$, we assume the following: there exists a non zero vector $\Phi_{j,q_0} \in \mathcal{D}$ satisfying the following eigenvalue equations:

$$\begin{cases} x^2\Phi_{j,q_0} = j(j+1)\Phi_{j,q_0}, \\ x_0\Phi_{j,q_0} = q_0\Phi_{j,q_0}, \end{cases} \tag{5}$$

for some j and q_0. Of course, there is no reason a priori to assume here that j and q_0 are real or positive.

An interlude on deformed $\mathfrak{su}(1,1)$

The relevant point for us of these operators is the ladder nature of $x_\pm$. This is easily deduced:

$$\begin{cases} x^2(x_\pm\Phi_{j,q_0}) = j(j+1)(x_\pm\Phi_{j,q_0}), \\ x_0(x_\pm\Phi_{j,q_0}) = (q_0 \pm 1)(x_\pm\Phi_{j,q_0}), \end{cases} \tag{6}$$

at least if $\Phi_{j,q_0} \notin \ker(x_\pm)$. This means that x_+ is a raising while x_- is a lowering operator. Using the same standard arguments for $\mathfrak{su}(1,1)$, we can also deduce that

$$\begin{cases} x_+\Phi_{j,q_0} = (q_0 - j)\Phi_{j,q_0+1}, \\ x_-\Phi_{j,q_0} = (q_0 + j)\Phi_{j,q_0-1}, \end{cases} \tag{7}$$

which are in agreement with the fact that, as it is easy to check,

$$[x_0, x_-x_+] = [x_0, x_+x_-] = 0.$$

More results on deformed $\mathfrak{su}(1,1)$ in my paper.

Back to ECSusy

We start with the operators k_α, $\alpha = 0, \pm$. As in (5), we assume a non zero vector $\varphi_{j,q} \in \mathcal{D}$ exists, $j, q \in \mathbb{C}$, such that

$$k^2 \varphi_{j,q} = j(j+1)\varphi_{j,q}, \qquad k_0 \varphi_{j,q} = q\varphi_{j,q}. \tag{8}$$

Here $k^2 = k_0^2 + k_0 - k_- k_+$. The operators $k_\pm$ act on $\varphi_{j,q}$ as ladder operators:

$$k_+ \varphi_{j,q} = (q-j)\varphi_{j,q+1}, \qquad k_- \varphi_{j,q} = (q+j)\varphi_{j,q-1}, \tag{9}$$

for all $\varphi_{j,q} \notin \ker(k_\pm)$.

Let I_j be the set of all the $q's$ for which $\varphi_{j,q}$ is not annihilated by at least one between k_+ and k_-: if $q \in I_j$, then $\varphi_{j,q} \notin \ker(k_+)$ or $\varphi_{j,q} \notin \ker(k_-)$, or both. We put

$$\mathcal{F}_\varphi(j) := \{\varphi_{j,q}, \ \forall q \in I_j\},$$

$\mathcal{E}_j = l.s.\{\varphi_{j,q}, \ q \in I_j\}$, and $\mathcal{H}_j$ the closure of $\mathcal{E}_j$, with respect to the norm of $\mathcal{H}$. Of course, $\mathcal{H}_j \subseteq \mathcal{H}$, for each fixed j. By construction, $\mathcal{F}_\varphi(j)$ is a basis for $\mathcal{H}_j$, with an unique biorthonormal basis $\mathcal{F}_\psi(j) := \{\psi_{j,q}, \ \forall q \in I_j\}$. Then

$$\langle \varphi_{j,q}, \psi_{j,r} \rangle = \delta_{q,r}, \tag{10}$$

for all $q, r \in I_j$, and $l.s.\{\psi_{j,q}, \ q \in I_j\}$ is dense in $\mathcal{H}_j$.

Back to ECSusy

The vectors $\psi_{j,q}$ satisfy the following eigenvalue and ladder equalities:

$$p^2\psi_{j,q} = \overline{j(j+1)}\psi_{j,q}, \qquad p_0\psi_{j,q} = \overline{q}\psi_{j,q},$$

$$\begin{cases} p_+\psi_{j,q} = \overline{(q+1+j)}\psi_{j,q+1}, \\ p_-\psi_{j,q} = \overline{(q-1-j)}\psi_{j,q-1}, \end{cases} \tag{11}$$

at least if $\psi_{j,q} \notin \ker(p_\pm)$.

Remark:– These equations are different from those deduced in (7). This is because the vectors $\{\psi_{j,q}\}$ are introduced here as the only biorhonormal family to $\{\varphi_{j,q}\}$. The other possibility would be to introduce, in analogy to what we have done in (8), a family of eigenstate of p^2 and p_0, $\{\tilde{\psi}_{j,q}\}$, which, however, turns out to be biorthogonal, but not biorthonormal, to $\{\varphi_{j,q}\}$: the difference we have with these different procedures is in the normalization of the states: $\psi_{j,q}$ and $\tilde{\psi}_{j,q}$ are proportional to each other.

Summarizing, to construct the eigenvectors of p^2 and p_0 we can:–

Choice 1:– follow biorthonormality, or...

Choice 2:– ...use the commutation rules.

My choice is mixed: I use commutation rules for k_α, and biorthonormality for the other operators.

Back to ECSusy

Many intertwining relations can be deduced. These are those for the ladder operators:

$$\begin{cases} sk_+ = l_+ s, & k_+ d = dl_+ \\ ck_- = l_- c, & k_- r = rl_- \\ r^\dagger p_+ = q_+ r^\dagger, & p_+ c^\dagger = c^\dagger q_+ \\ d^\dagger p_- = q_- d^\dagger, & p_- s^\dagger = s^\dagger q_-, \end{cases}$$

while these are for the 0-operators (the *non self-adjoint Hamiltonians*)

$$\begin{cases} l_0 s = s\left(k_0 + \frac{1}{2}1\!\!1\right), & q_0 r^\dagger = r^\dagger\left(p_0 + \frac{1}{2}1\!\!1\right) \\ l_0 c = c\left(k_0 - \frac{1}{2}1\!\!1\right), & q_0 d^\dagger = d^\dagger\left(p_0 - \frac{1}{2}1\!\!1\right) \\ rl_0 = \left(k_0 + \frac{1}{2}1\!\!1\right) r, & s^\dagger q_0 = \left(p_0 + \frac{1}{2}1\!\!1\right) s^\dagger \\ dl_0 = \left(k_0 - \frac{1}{2}1\!\!1\right) d, & c^\dagger q_0 = \left(p_0 - \frac{1}{2}1\!\!1\right) c^\dagger. \end{cases} \tag{12}$$

These equalities show that the eigenvalues of l_0 differ from those of k_0 by half integers, as those of q_0 from those of p_0. Indeed we have, considering a vector $\varphi_{j,q}$ with $s\varphi_{j,q} \neq 0$ and $c\varphi_{j,q} \neq 0$, $k_0\varphi_{j,q} = q\varphi_{j,q}$, and

$$l_0\left(s\varphi_{j,q}\right) = s\left(k_0 + \frac{1}{2}1\!\!1\right)\varphi_{j,q} = \left(q + \frac{1}{2}\right)\left(s\varphi_{j,q}\right), \quad l_0\left(c\varphi_{j,q}\right) = \left(q - \frac{1}{2}\right)\left(c\varphi_{j,q}\right).$$

In other words, $s\varphi_{j,q}$ and $c\varphi_{j,q}$ are both eigenstates of l_0, with different eigenvalues. Hence they must be connected by the ladder operators $l_\pm$.

A detailed example: $\mathcal{D}$-pseudo bosons

Let a and b be two operators on $\mathcal{H}$, $a^\dagger$ and $b^\dagger$ their adjoint, and let $\mathcal{D}$, dense in $\mathcal{H}$, be such that $a^\sharp\mathcal{D} \subseteq \mathcal{D}$ and $b^\sharp\mathcal{D} \subseteq \mathcal{D}$, $(x^\sharp = x, x^\dagger)$. In general $\mathcal{D} \subseteq D(a^\sharp)$ and $\mathcal{D} \subseteq D(b^\sharp)$.

Definition 1:

The operators (a, b) are $\mathcal{D}$-pseudo bosonic ($\mathcal{D}$-pb) if, for all $f \in \mathcal{D}$, we have

$$a\,b\,f - b\,a\,f = f. \tag{13}$$

($[a, b] = 1\!\!1$, for simplicity). [If $b = a^\dagger$ then we recover the CCR].

We now assume that

Assumption $\mathcal{D}$-pb 1.– there exists a non-zero $\varphi_0 \in \mathcal{D}$ such that $a\varphi_0 = 0$,

Assumption $\mathcal{D}$-pb 2.– there exists a non-zero $\Psi_0 \in \mathcal{D}$ such that $b^\dagger\Psi_0 = 0$.

Remark:– If $a = \frac{d}{dx}$ and $b = x$ these assumptions are not satisfied in $\mathcal{L}^2(\mathbb{R})$.

Now, if (a, b) satisfy Definition 1, then $\varphi_0 \in D^\infty(b)$ and $\Psi_0 \in D^\infty(a^\dagger)$. Hence...

A detailed example: $\mathcal{D}$-pseudo bosons

$$\varphi_n := \frac{1}{\sqrt{n!}}\, b^n \varphi_0, \qquad \Psi_n := \frac{1}{\sqrt{n!}}\, {a^\dagger}^n \Psi_0, \qquad (14)$$

$n \geq 0$, can be defined and they all belong to $\mathcal{D}$. We introduce $\mathcal{F}_\Psi = \{\Psi_n,\ n \geq 0\}$ and $\mathcal{F}_\varphi = \{\varphi_n,\ n \geq 0\}$. Of course, both φ_n and Ψ_n belong to the domains of $a^\sharp$, $b^\sharp$ and $N^\sharp$ (here $N = ba$).

The following lowering and raising relations hold:

$$\begin{cases} b\,\varphi_n = \sqrt{n+1}\varphi_{n+1}, & n \geq 0, \\ a\,\varphi_0 = 0, \quad a\varphi_n = \sqrt{n}\,\varphi_{n-1}, & n \geq 1, \\ a^\dagger \Psi_n = \sqrt{n+1}\Psi_{n+1}, & n \geq 0, \\ b^\dagger \Psi_0 = 0, \quad b^\dagger \Psi_n = \sqrt{n}\,\Psi_{n-1}, & n \geq 1, \end{cases} \qquad (15)$$

as well as the following eigenvalue equations:

$$N\varphi_n = n\varphi_n, \quad N^\dagger \Psi_n = n\Psi_n, \quad n \geq 0.$$

A consequence: if $\langle \varphi_0, \Psi_0 \rangle = 1$, then

$$\langle \varphi_n, \Psi_m \rangle = \delta_{n,m}, \qquad (16)$$

for all $n, m \geq 0$.

Assumption $\mathcal{D}$-pb 3.– $\mathcal{F}_\varphi$ is a basis for $\mathcal{H}$. (iff $\mathcal{F}_\Psi$ is a basis for $\mathcal{H}$). Or...

A detailed example: $\mathcal{D}$-pseudo bosons

Assumption $\mathcal{D}$-pbw 3.– $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are $\mathcal{G}$-quasi bases for $\mathcal{H}$, for some $\mathcal{G}$ dense in $\mathcal{H}$ ($\mathcal{G}=\mathcal{D}$, in some cases).

This means that, $\forall\, f,g\in\mathcal{G}$,

$$\sum \langle f,\varphi_n\rangle \langle \Psi_n,g\rangle = \sum \langle f,\Psi_n\rangle \langle \varphi_n,g\rangle = \langle f,g\rangle .$$

And now, we take $c=r=a$, $d=s=b$, $\delta=-\gamma=1$. Hence $\{k_\alpha,l_\alpha,p_\alpha,q_\alpha\}$ become

$$k_+=l_+=\frac{1}{2}b^2,\qquad k_-=l_-=\frac{1}{2}a^2,\qquad k_0=l_0=\frac{1}{2}\left(N+\frac{1}{2}1\!\!1\right), \tag{17}$$

where $N=ba$, and

$$p_+=q_+=\frac{1}{2}a^{\dagger\,2},\qquad p_-=q_-=\frac{1}{2}b^{\dagger\,2},\qquad p_0=q_0=\frac{1}{2}\left(N^\dagger+\frac{1}{2}1\!\!1\right). \tag{18}$$

It is clear that the four original families collapse to two. Moreover:

$$k^2=p^2=-\frac{3}{16}1\!\!1, \tag{19}$$

which, of course, commute with all the other operators, as expected. Notice that k^2 and p^2 are not positive operators.

A detailed example: $\mathcal{D}$-pseudo bosons

Formula (5) is based on the assumption that an eigenstate of x^2 and x_0 exists: this is the vacuum of a, φ_0. In fact, we have

$$k^2\varphi_0 = -\frac{3}{16}\varphi_0, \qquad k_0\varphi_0 = \frac{1}{4}\varphi_0.$$

Hence, comparing these with (5), we have $q_0 = \frac{1}{4}$ and $j(j+1) = -\frac{3}{16}$, that is $j = -\frac{1}{4}$ or $j = -\frac{3}{4}$. Formula (9) and $k_-\varphi_0 = 0$, imply that $j = -\frac{1}{4}$. Hence we call

$$\varphi_{-\frac{1}{4},\frac{1}{4}} := \varphi_0, \tag{20}$$

so that the spectrum of k_0 is bounded below ($\varphi_{-\frac{1}{4},\frac{1}{4}} \in \ker(k_-)$). Acting on $\varphi_{-\frac{1}{4},\frac{1}{4}}$ with $(k_+)^m$ we get

$$\varphi_{-\frac{1}{4},m+\frac{1}{4}} = \frac{\sqrt{(2m)!}}{(2m-1)!!}\varphi_{2m}, \tag{21}$$

where φ_{2m} are the eigenstates of $N = ba$, $0!! = (-1)!! = 1$ and $(2m-1)!! = 1\cdot 3\cdot\dots\cdot(2m-1)$. We find

$$k_0\varphi_{-\frac{1}{4},m+\frac{1}{4}} = \left(m+\frac{1}{4}\right)\varphi_{-\frac{1}{4},m+\frac{1}{4}}, \tag{22}$$

$$k_+\varphi_{-\frac{1}{4},m+\frac{1}{4}} = \left(m+\frac{1}{2}\right)\varphi_{-\frac{1}{4},m+\frac{5}{4}}, \qquad k_-\varphi_{-\frac{1}{4},m+\frac{1}{4}} = m\,\varphi_{-\frac{1}{4},m-\frac{3}{4}}. \tag{23}$$

In particular, this last equality is true only if $m \geq 1$. If $m = 0$ we have $k_-\varphi_{-\frac{1}{4},\frac{1}{4}} = k_-\varphi_0 = 0$, as already noticed.

A detailed example: $\mathcal{D}$-pseudo bosons

We can now define the set of linearly independent vectors

$$\mathcal{F}_\varphi^{(e)}\left(\frac{1}{4}\right)=\left\{\varphi_{-\frac{1}{4},m+\frac{1}{4}},\, m=0,1,2,3,\ldots\right\},$$

and the Hilbert space $\mathcal{H}^{(e)}_{-\frac{1}{4}}$, constructed by taking the closure of the linear span of its vectors. Here the suffix *e* stands for *even*, since only the vectors φ_{2m} belong to $\mathcal{F}_\varphi^{(e)}\left(\frac{1}{4}\right)$. On the other hand, since $\mathcal{H}\ni\varphi_{2m+1}\notin\mathcal{H}^{(e)}_{-\frac{1}{4}}$, we have $\mathcal{H}^{(e)}_{-\frac{1}{4}}\subset\mathcal{H}$. Hence, the set $\mathcal{F}_\varphi^{(e)}\left(\frac{1}{4}\right)$ cannot be complete in $\mathcal{H}$, and, therefore, a basis for $\mathcal{H}$. Nevertheless, by construction, $\mathcal{H}^{(e)}_{-\frac{1}{4}}$ is an Hilbert space by itself, and $\mathcal{F}_\varphi^{(e)}\left(\frac{1}{4}\right)$ is a basis for it. Then an unique biorthonormal basis $\mathcal{F}_\psi^{(e)}\left(\frac{1}{4}\right)=\left\{\psi_{-\frac{1}{4},m+\frac{1}{4}},\, m=0,1,2,3,\ldots\right\}$ exists:

$$\left\langle\varphi_{-\frac{1}{4},m+\frac{1}{4}},\psi_{-\frac{1}{4},l+\frac{1}{4}}\right\rangle=\delta_{m,l}, \tag{24}$$

where the scalar product is the one in $\mathcal{H}$, and, for each $f\in\mathcal{H}^{(e)}_{-\frac{1}{4}}$,

$$f=\sum_{m=0}^{\infty}\left\langle\varphi_{-\frac{1}{4},m+\frac{1}{4}},f\right\rangle\psi_{-\frac{1}{4},m+\frac{1}{4}}=\sum_{m=0}^{\infty}\left\langle\psi_{-\frac{1}{4},m+\frac{1}{4}},f\right\rangle\varphi_{-\frac{1}{4},m+\frac{1}{4}}.$$

A detailed example: $\mathcal{D}$-pseudo bosons

From (21) and (15) it is clear that the vectors of this biorthonormal basis are the following:

$$\psi_{-\frac{1}{4},m+\frac{1}{4}}=\frac{(2m-1)!!}{\sqrt{(2m)!}}\psi_{2m}. \tag{25}$$

Formulas (11) can now be explicitly checked, and we get

$$p^2\psi_{-\frac{1}{4},m+\frac{1}{4}}=-\frac{3}{16}\psi_{-\frac{1}{4},m+\frac{1}{4}}, \qquad p_0\psi_{-\frac{1}{4},m+\frac{1}{4}}=\left(m+\frac{1}{4}\right)\psi_{-\frac{1}{4},m+\frac{1}{4}}, \tag{26}$$

together with

$$p_+\psi_{-\frac{1}{4},m+\frac{1}{4}}=(m+1)\,\psi_{-\frac{1}{4},m+\frac{5}{4}}, \qquad p_-\psi_{-\frac{1}{4},m+\frac{1}{4}}=\left(m-\frac{1}{2}\right)\psi_{-\frac{1}{4},m-\frac{3}{4}}. \tag{27}$$

We stress again that the difference between these ladder equations and those in (23) arises because, while the $\varphi_{-\frac{1}{4},m+\frac{1}{4}}$'s are introduced using directly the deformed $\mathfrak{su}(1,1)$ algebra, the $\psi_{-\frac{1}{4},m+\frac{1}{4}}$'s are just the unique basis which is biorthonormal to $\mathcal{F}_\varphi^{(e)}\left(\frac{1}{4}\right)$. However, these vectors are still eigenstates of p^2 and p_0, and obey interesting ladder equations with respect to $p_\pm$, even if slightly different from those in (7).

A detailed example: $\mathcal{D}$-pseudo bosons

More explicitly:– if we repeat for the operators p_α what we have done for the k_α, we get a different (but related) set of vectors. In fact:

in analogy with (20), we put $\tilde{\psi}_{-\frac{1}{4},\frac{1}{4}} = \psi_0$, since $b^\dagger\psi_0 = 0$. Then, using the first equation in (7) (rather than the second in (11)), $p_+\tilde{\psi}_{j,q} = (q-j)\tilde{\psi}_{j,q+1}$, we deduce that

$$\tilde{\psi}_{-\frac{1}{4},m+\frac{1}{4}} = \frac{(2m-1)!!}{\sqrt{(2m)!}}\,\psi_{2m} = \frac{(2m)!}{((2m-1)!!)^2}\psi_{-\frac{1}{4},m+\frac{1}{4}},$$

which shows the difference in the normalizations between the $\tilde{\psi}_{-\frac{1}{4},m+\frac{1}{4}}$ and the $\psi_{-\frac{1}{4},m+\frac{1}{4}}$. Now, while $\tilde{\psi}_{-\frac{1}{4},m+\frac{1}{4}}$ satisfies the analogous of formulas (7), $\psi_{-\frac{1}{4},m+\frac{1}{4}}$ satisfies (27), which is slightly different. On the other hand, while this last vector satisfies (24), $\tilde{\psi}_{-\frac{1}{4},m+\frac{1}{4}}$ does not.

In conclusion, we prefer to keep biorthonormality of the sets we work with, rather than using (7) several times.

The odd sector

But...what about the odd indexes?

The odd sector

But...what about the odd indexes? These are connected with the SUSY partner of k_0 which essentially differs from k_0 for an addictive constant.
Let

$$a\varphi_{-\frac{1}{4},m+\frac{1}{4}} = \sqrt{2m}\,\frac{\sqrt{(2m)!}}{(2m-1)!!}\,\varphi_{2m-1}, \qquad b\varphi_{-\frac{1}{4},m+\frac{1}{4}} = \frac{\sqrt{(2m+1)!}}{(2m-1)!!}\,\varphi_{2m+1}, \tag{28}$$

with the agreement that $\varphi_{-1}=0$. Let us now define

$$\varphi_{-\frac{1}{4},m+\frac{3}{4}} := b\varphi_{-\frac{1}{4},m+\frac{1}{4}} = \frac{\sqrt{(2m+1)!}}{(2m-1)!!}\,\varphi_{2m+1}, \tag{29}$$

for all $m \geq 0$. The reason for calling this vector in this way is because $\varphi_{-\frac{1}{4},m+\frac{3}{4}}$ is an eigenstate of k_0 with eigenvalue $m+\frac{3}{4}$:

$$k_0\varphi_{-\frac{1}{4},m+\frac{3}{4}} = \left(m+\frac{3}{4}\right)\varphi_{-\frac{1}{4},m+\frac{3}{4}}, \tag{30}$$

The odd sector

These vectors satisfy the following raising and lowering relations:

$$k_+\varphi_{-\frac{1}{4},m+\frac{3}{4}} = \left(m+\frac{1}{2}\right)\varphi_{-\frac{1}{4},m+\frac{7}{4}}, \qquad k_-\varphi_{-\frac{1}{4},m+\frac{3}{4}} = m\,\frac{2m+1}{2m-1}\,\varphi_{-\frac{1}{4},m-\frac{1}{4}}, \tag{31}$$

with the agreement that $\varphi_{-\frac{1}{4},-\frac{1}{4}} = 0$.

An useful comment:– It is clear that, in the same way in which a and b map $\varphi_{-\frac{1}{4},m+\frac{1}{4}}$ into some $\varphi_{-\frac{1}{4},l+\frac{3}{4}}$, they also map these last vectors into the first ones (see the figure, in a moment...).

The odd sector

In analogy with what we have done before, we put

$$\mathcal{F}_\varphi^{(o)}\left(\frac{1}{4}\right)=\left\{\varphi_{-\frac{1}{4},m+\frac{3}{4}},\, m=0,1,2,3,\dots\right\},$$

where o stands for *odd*, and the related Hilbert space $\mathcal{H}^{(o)}_{-\frac{1}{4}}$: $\mathcal{H}^{(e)}_{-\frac{1}{4}}\cap\mathcal{H}^{(o)}_{-\frac{1}{4}}=\emptyset$, and $\mathcal{F}_\varphi\left(\frac{1}{4}\right):=\mathcal{F}_\varphi^{(e)}\left(\frac{1}{4}\right)\cup\mathcal{F}_\varphi^{(o)}\left(\frac{1}{4}\right)$ is complete in $\mathcal{H}$, at least if the set $\mathcal{F}_\varphi$ is complete. This is true, e.g., if the $\mathcal{D}$-PBs are *regular*, since $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are b.o. Riesz bases. Now, since $\mathcal{F}_\varphi^{(o)}\left(\frac{1}{4}\right)$ is a basis for $\mathcal{H}^{(o)}_{-\frac{1}{4}}$, we can introduce an unique b.o. basis $\mathcal{F}_\psi^{(o)}\left(\frac{1}{4}\right)=\{\psi_{-\frac{1}{4},m+\frac{3}{4}},\, m=0,1,2,3,\dots\}$, whose vectors can be easily identified using (29) and (15). We have

$$\psi_{-\frac{1}{4},m+\frac{3}{4}}=\frac{(2m-1)!!}{\sqrt{(2m+1)!}}\,\psi_{2m+1}=\frac{1}{2m+1}a^\dagger\psi_{-\frac{1}{4},m+\frac{1}{4}}. \qquad (32)$$

In fact, with this choice,

$$\langle\varphi_{-\frac{1}{4},m+\frac{3}{4}},\psi_{-\frac{1}{4},l+\frac{3}{4}}\rangle=\delta_{m,l}, \qquad (33)$$

where the scalar product is the one in $\mathcal{H}$, and, for each $f\in\mathcal{H}^{(o)}_{-\frac{1}{4}}$,

$$f=\sum_{m=0}^\infty\langle\varphi_{-\frac{1}{4},m+\frac{3}{4}},f\rangle\,\psi_{-\frac{1}{4},m+\frac{3}{4}}=\sum_{m=0}^\infty\langle\psi_{-\frac{1}{4},m+\frac{3}{4}},f\rangle\,\varphi_{-\frac{1}{4},m+\frac{3}{4}}.$$

The odd sector

Repeating then what we have done for $\mathcal{H}^{(e)}$, we can define the set $\mathcal{F}_\psi^{(o)}\left(\frac{1}{4}\right) = \{\psi_{-\frac{1}{4},m+\frac{3}{4}},\, m = 0,1,2,3,\ldots\}$, and observe that $\mathcal{F}_\psi\left(\frac{1}{4}\right) := \mathcal{F}_\psi^{(e)}\left(\frac{1}{4}\right) \cup \mathcal{F}_\psi^{(o)}\left(\frac{1}{4}\right)$ is complete in $\mathcal{H}$, or it is even a Riesz basis for $\mathcal{H}$, depending on the nature of the $\mathcal{D}$-PBs we are considering. More in detail, if we now introduce the families $\mathcal{F}_\Phi = \{\Phi_k,\, k \geq 0\}$ and $\mathcal{F}_\xi = \{\xi_k,\, k \geq 0\}$, where

$$\Phi_k = \begin{cases} \varphi_{-\frac{1}{4},j+\frac{1}{4}}, & \text{if } k = 2j, \\ \varphi_{-\frac{1}{4},j+\frac{3}{4}}, & \text{if } k = 2j+1, \end{cases} \quad \text{and} \quad \xi_k = \begin{cases} \psi_{-\frac{1}{4},j+\frac{1}{4}}, & \text{if } k = 2j, \\ \psi_{-\frac{1}{4},j+\frac{3}{4}}, & \text{if } k = 2j+1, \end{cases}$$

$k \geq 0$, we can check that

$$\langle \Phi_k, \xi_l \rangle = \delta_{k,l},$$

and that, $\forall f, g \in \mathcal{D}$,

$$\sum_{k=0}^{\infty} \langle f, \Phi_k \rangle \langle \xi_k, g \rangle = \sum_{k=0}^{\infty} \langle f, \varphi_k \rangle \langle \psi_k, g \rangle, \qquad \sum_{k=0}^{\infty} \langle f, \xi_k \rangle \langle \Phi_k, g \rangle = \sum_{k=0}^{\infty} \langle f, \psi_k \rangle \langle \varphi_k, g \rangle.$$

These equalities imply that $\mathcal{F}_\Phi$ and $\mathcal{F}_\xi$ are b.o., and that they are $\mathcal{D}$-quasi bases if and only if $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are $\mathcal{D}$-quasi bases.

Some formulas

$\varphi_{-\frac{1}{4},m+\frac{1}{4}} = \frac{\sqrt{(2m)!}}{(2m-1)!!}\varphi_{2m}$	$\varphi_{-\frac{1}{4},m+\frac{3}{4}} = b\varphi_{-\frac{1}{4},m+\frac{1}{4}} = \frac{\sqrt{(2m+1)!}}{(2m-1)!!}\varphi_{2m+1}$
$\psi_{-\frac{1}{4},m+\frac{1}{4}} = \frac{(2m-1)!!}{\sqrt{(2m)!}}\psi_{2m}$	$\psi_{-\frac{1}{4},m+\frac{3}{4}} = \frac{a^\dagger}{2m+1}\psi_{-\frac{1}{4},m+\frac{1}{4}} = \frac{(2m-1)!!}{\sqrt{(2m+1)!}}\psi_{2m+1}$
$a\varphi_{-\frac{1}{4},m+\frac{1}{4}} = \sqrt{2m}\frac{\sqrt{(2m)!}}{(2m-1)!!}\varphi_{2m-1} = \frac{2m}{2m-1}\varphi_{-\frac{1}{4},m-\frac{1}{4}}$	$a\varphi_{-\frac{1}{4},m+\frac{3}{4}} = \sqrt{2m+1}\frac{\sqrt{(2m+1)!}}{(2m-1)!!}\varphi_{2m} = (2m+1)\varphi_{-\frac{1}{4},m+\frac{1}{4}}$
$b\varphi_{-\frac{1}{4},m+\frac{1}{4}} = \frac{\sqrt{(2m+1)!}}{(2m-1)!!}\varphi_{2m+1} = \varphi_{-\frac{1}{4},m+\frac{3}{4}}$	$b\varphi_{-\frac{1}{4},m+\frac{3}{4}} = \frac{\sqrt{(2m+2)!}}{(2m-1)!!}\varphi_{2m+2} = (2m+1)\varphi_{-\frac{1}{4},m+\frac{5}{4}}$
$a^\dagger\psi_{-\frac{1}{4},m+\frac{1}{4}} = \frac{\sqrt{(2m+1)!}}{(2m+1)!!}\psi_{2m+1} = (2m+1)\psi_{-\frac{1}{4},m+\frac{3}{4}}$	$a^\dagger\psi_{-\frac{1}{4},m+\frac{3}{4}} = \sqrt{2m+2}\frac{(2m-1)!!}{\sqrt{(2m+1)!}}\psi_{2m+2} = \frac{2m+2}{2m+1}\varphi_{-\frac{1}{4},m+\frac{5}{4}}$
$b^\dagger\psi_{-\frac{1}{4},m+\frac{1}{4}} = \frac{(2m-1)!!}{\sqrt{(2m-1)!}}\psi_{2m+1} = (2m-1)\psi_{-\frac{1}{4},m-\frac{1}{4}}$	$b^\dagger\psi_{-\frac{1}{4},m+\frac{3}{4}} = \frac{(2m-1)!!}{\sqrt{(2m)!}}\psi_{2m} = \psi_{-\frac{1}{4},m+\frac{1}{4}}$

Figure: Formulas involving ladder operators

Some more formulas

$k_0\varphi_{-\frac{1}{4},m+\frac{1}{4}} = \left(m+\frac{1}{4}\right)\varphi_{-\frac{1}{4},m+\frac{1}{4}}$	$k_+\varphi_{-\frac{1}{4},m+\frac{1}{4}} = \left(m+\frac{1}{2}\right)\varphi_{-\frac{1}{4},m+\frac{5}{4}}$	$k_-\varphi_{-\frac{1}{4},m+\frac{1}{4}} = m\,\varphi_{-\frac{1}{4},m-\frac{3}{4}}$
$p_0\psi_{-\frac{1}{4},m+\frac{1}{4}} = \left(m+\frac{1}{4}\right)\psi_{-\frac{1}{4},m+\frac{1}{4}}$	$p_+\psi_{-\frac{1}{4},m+\frac{1}{4}} = (m+1)\,\psi_{-\frac{1}{4},m+\frac{5}{4}}$	$p_-\psi_{-\frac{1}{4},m+\frac{1}{4}} = \left(m-\frac{1}{2}\right)\psi_{-\frac{1}{4},m-\frac{3}{4}}$
$k_0\varphi_{-\frac{1}{4},m+\frac{3}{4}} = \left(m+\frac{3}{4}\right)\varphi_{-\frac{1}{4},m+\frac{3}{4}}$	$k_+\varphi_{-\frac{1}{4},m+\frac{3}{4}} = \left(m+\frac{1}{2}\right)\varphi_{-\frac{1}{4},m+\frac{7}{4}}$	$k_-\varphi_{-\frac{1}{4},m+\frac{3}{4}} = m\,\frac{2m+1}{2m-1}\,\varphi_{-\frac{1}{4},m-\frac{1}{4}}$
$p_0\psi_{-\frac{1}{4},m+\frac{3}{4}} = \left(m+\frac{3}{4}\right)\psi_{-\frac{1}{4},m+\frac{3}{4}}$	$p_+\psi_{-\frac{1}{4},m+\frac{3}{4}} = (m+1)\,\frac{2m+3}{2m+1}\psi_{-\frac{1}{4},m+\frac{7}{4}}$	$p_-\psi_{-\frac{1}{4},m+\frac{3}{4}} = \left(m-\frac{1}{2}\right)\psi_{-\frac{1}{4},m-\frac{1}{4}}$

Figure: Formulas involving number operators

Deforming the deformed!

Let a and b be $\mathcal{D}$-pseudo-bosonic operators in $\mathcal{L}^\dagger(\mathcal{D})$, for some suitable $\mathcal{D}$. Let now $S, T \in \mathcal{L}^\dagger(\mathcal{D})$ be two invertible operators, with $S^{-1}, T^{-1} \in \mathcal{L}^\dagger(\mathcal{D})$. In the following we will assume that ${T^{-1}}^\dagger = {T^\dagger}^{-1}$ and ${S^{-1}}^\dagger = {S^\dagger}^{-1}$. Conditions for these to be satisfied are discussed in literature. They are trivially true for bounded operators. If we define now

$$c = SaT^{-1}, \qquad s = SbT^{-1}, \qquad d = TbS^{-1}, \qquad r = TaS^{-1},$$

then these operators, which are all in $\mathcal{L}^\dagger(\mathcal{D})$, satisfy an ECSUSY with $\delta = -\gamma = 1$. Moreover

$$\tilde{k}_\alpha = Tk_\alpha T^{-1}, \qquad \tilde{l}_\alpha = Sk_\alpha S^{-1}, \qquad \tilde{p}_\alpha = {T^{-1}}^\dagger p_\alpha T^\dagger, \qquad \tilde{q}_\alpha = {S^{-1}}^\dagger p_\alpha S^\dagger,$$

where $\alpha = 0, \pm$ and where the *un-tilted* operators k_α and p_α are those in (17) and (18).

Since $\varphi_{-\frac{1}{4},m+\frac{1}{4}}, \psi_{-\frac{1}{4},m+\frac{1}{4}}, \varphi_{-\frac{1}{4},m+\frac{3}{4}}, \psi_{-\frac{1}{4},m+\frac{3}{4}} \in \mathcal{D}$, for all $m = 0, 1, 2, 3, \ldots$, it follows that the following vectors are in $\mathcal{D}$ as well:

$$\tilde{\varphi}_{-\frac{1}{4},m+\frac{1}{4}} = T\varphi_{-\frac{1}{4},m+\frac{1}{4}}; \qquad \tilde{\psi}_{-\frac{1}{4},m+\frac{1}{4}} = {T^{-1}}^\dagger \psi_{-\frac{1}{4},m+\frac{1}{4}};$$

and

$$\tilde{\chi}_{-\frac{1}{4},m+\frac{3}{4}} = S\varphi_{-\frac{1}{4},m+\frac{3}{4}}; \qquad \tilde{\eta}_{-\frac{1}{4},m+\frac{3}{4}} = {S^{-1}}^\dagger \psi_{-\frac{1}{4},m+\frac{3}{4}}.$$

Deforming the deformed!

They are eigenstates respectively of $\tilde{k}_0$ and $\tilde{p}_0$, with eigenvalue $m+\frac{1}{4}$, and of $\tilde{l}_0$ and $\tilde{q}_0$, with eigenvalue $m+\frac{3}{4}$. Moreover, they satisfy the following ladder equations:

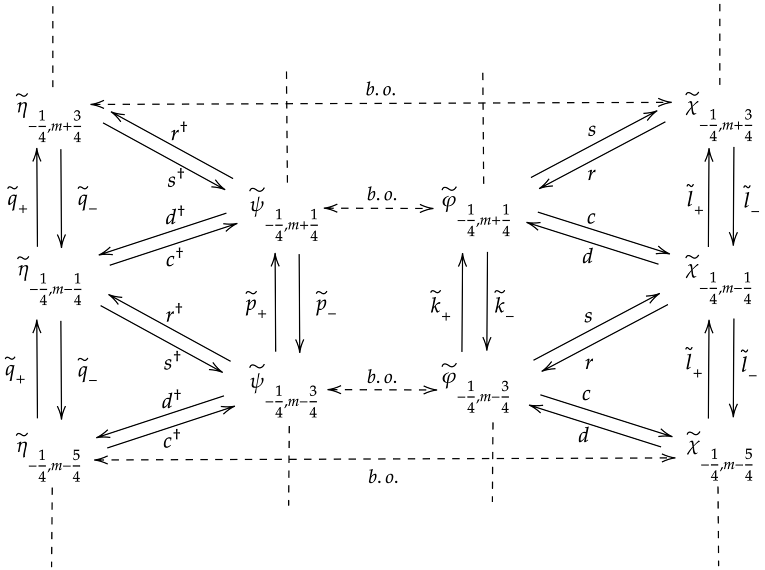
$$\left\{\begin{array}{ll}
\tilde{k}_+\tilde{\varphi}_{-\frac{1}{4},m+\frac{1}{4}}=\left(m+\frac{1}{2}\right)\tilde{\varphi}_{-\frac{1}{4},m+\frac{5}{4}}, & \tilde{k}_-\tilde{\varphi}_{-\frac{1}{4},m+\frac{1}{4}}=m\tilde{\varphi}_{-\frac{1}{4},m-\frac{3}{4}},\\
\tilde{p}_+\tilde{\psi}_{-\frac{1}{4},m+\frac{1}{4}}=(m+1)\,\tilde{\psi}_{-\frac{1}{4},m+\frac{5}{4}}, & \tilde{p}_-\tilde{\psi}_{-\frac{1}{4},m+\frac{1}{4}}=\left(m-\frac{1}{2}\right)\tilde{\psi}_{-\frac{1}{4},m-\frac{3}{4}},\\
\tilde{l}_+\tilde{\chi}_{-\frac{1}{4},m+\frac{3}{4}}=\left(m+\frac{1}{2}\right)\tilde{\chi}_{-\frac{1}{4},m+\frac{7}{4}}, & \tilde{l}_-\tilde{\chi}_{-\frac{1}{4},m+\frac{3}{4}}=m\,\frac{2m+1}{2m-1}\tilde{\chi}_{-\frac{1}{4},m-\frac{1}{4}},\\
\tilde{q}_+\tilde{\eta}_{-\frac{1}{4},m+\frac{3}{4}}=(m+1)\,\frac{2m+3}{2m+1}\tilde{\eta}_{-\frac{1}{4},m+\frac{7}{4}}, & \tilde{q}_-\tilde{\eta}_{-\frac{1}{4},m+\frac{3}{4}}=\left(m-\frac{1}{2}\right)\tilde{\eta}_{-\frac{1}{4},m-\frac{1}{4}},
\end{array}\right.$$

for every m for which the lowering operators do not destroy the state. Also, they are biorthonormal in pairs, meaning that

$$\langle\tilde{\varphi}_{-\frac{1}{4},m+\frac{1}{4}},\tilde{\psi}_{-\frac{1}{4},l+\frac{1}{4}}\rangle=\langle\tilde{\chi}_{-\frac{1}{4},m+\frac{3}{4}},\tilde{\eta}_{-\frac{1}{4},l+\frac{3}{4}}\rangle=\delta_{m,l}, \tag{34}$$

for all $m,l\in\mathbb{N}_0$, while, if S and T are not chosen in some special way, we get, for instance, $\langle\tilde{\varphi}_{-\frac{1}{4},m+\frac{1}{4}},\tilde{\eta}_{-\frac{1}{4},l+\frac{3}{4}}\rangle\neq 0$.

Deforming the deformed!



Is $\mathcal{L}^2$ the only "good" space for Quantum Mechanics?

....and I am not speaking of *changing the metric!!*

More exactly:– are all physical systems described only in Hilbert (or Krein) spaces?

This is relevant mainly when dealing with unbounded operators. Otherwise Hilbert spaces work well (in my knowledge)!

Some results exist having *rigged Hilbert spaces* as their essential structure: R. De La Madrid, C. Trapani, J.-P. Antoine, A. Bohm, M. Gadella, M. del Olmo,...

Other results exist in a pure *distributional domain*: the axiomatic (Wightman) approach to quantum field theory, Calcada et al,....

Recently, I introduced the concept of weak pseudo-bosons for dealing with the position and the momentum operators, and for analysing the damped quantum harmonic oscillator:

$\hat{x}$ and $\hat{D} = \frac{d}{dx}$ obey $[\hat{D}, \hat{x}] = \mathbb{1}$ and they act as ladder operators between *distributions*, and not *functions*. In particular, the vacuum of $\hat{x}$ is $\varphi_0(x) = \delta(x)$, and acting on this with powers of $\hat{D}$ produces other distributions, $\delta^{(n)}(x)$, which are biorthogonal (in a distributional sense), with respect to the family $x^n = (\hat{x})^n \Psi_0(x)$, where $\hat{D}\Psi_0(x) = 0$. Similar results can be deduced for the DQHO.

A concrete example

Let $\mathcal{H}=\mathcal{L}^2(\mathbb{R})$, with the usual scalar product, and let

$$A=\frac{d}{dx}+w_A(x), \qquad B=-\frac{d}{dx}+w_B(x). \tag{35}$$

Here $w_A(x), w_B(x)\in C^\infty$, and $w_A(x)\neq\overline{w_B(x)}$. Then $B^\dagger\neq A$. We call these functions superpotentials: they produce two supersymmetric Hamiltonians $H_1=BA$ and $H_2=AB$, whose eigenvectors are related as in usual SUSY QM. The fact that H_1 and H_2 are SUSY partners is discussed in [Part 1]. Here this aspect is not relevant, since with our constraint on the superpotentials (see below), we have $H_2=H_1+1\!\!1$. Now, since

$$H_1=BA=-\frac{d^2}{dx^2}+q_1(x)\frac{d}{dx}+V_1(x), \qquad H_2=AB=-\frac{d^2}{dx^2}+q_1(x)\frac{d}{dx}+V_2(x),$$

where

$$q_1(x)=w_B(x)-w_A(x),\; V_1(x)=w_A(x)w_B(x)-w_A'(x),\; V_2(x)=w_A(x)w_B(x)+w_B'(x),$$

we find that

$$[A,B]=H_2-H_1=V_2(x)-V_1(x)=w_A'(x)+w_B'(x). \tag{36}$$

Hence, to have $[A,B]=1\!\!1$, we must have $w_A(x)+w_B(x)=x+k$, for a generic k which we fix in $\mathbb{R}$. In this case $w_A(x)$ and $w_B(x)$ are called pseudo-bosonic superpotentials: PBSs.

A concrete example

The vacua of A and $B^\dagger$ are the following:

$$\varphi_0(x) = N_\varphi \, \exp\left\{-s_A(x)\right\}, \qquad \Psi_0(x) = N_\Psi \, \exp\left\{-\overline{s_B(x)}\right\},$$

where

$$s_A(x) = \int w_A(x)dx, \qquad s_B(x) = \int w_B(x)dx,$$

and N_φ and N_Ψ are two normalization constants to be computed.

We are not assuming here $\varphi_0(x), \Psi_0(x) \in \mathcal{L}^2(\mathbb{R})$. Still, they are both C^∞ functions, so that

$$\varphi_n(x) = \frac{1}{\sqrt{n!}} B^n \varphi_0(x), \qquad \psi_n(x) = \frac{1}{\sqrt{n!}} {A^\dagger}^n \psi_0(x),$$

$n = 1, 2, 3, \ldots$. are C^∞ functions, too, for all choices of C^∞ PBSs and $\forall n \geq 0$.

A concrete example

Proposition:–

For any choice of C^∞ PBSs we have

$$\frac{\varphi_n(x)}{\varphi_0(x)} = \frac{\psi_n(x)}{\psi_0(x)} = p_n(x,k),$$

for all $n \geq 0$, where $p_n(x,k)$ is independent of $w_A(x)$ and $w_B(x)$ and is defined recursively as follows:

$$p_0(x,k) = 1, \qquad p_n(x,k) = \frac{1}{\sqrt{n}}\left(p_{n-1}(x,k)(x+k) - p'_{n-1}(x,k)\right), \; n \geq 1.$$

More explicitly, we get

$$p_n(x,k) = \frac{1}{\sqrt{2^n\, n!}}\, H_n\left(\frac{x+k}{\sqrt{2}}\right).$$

Furthermore $\varphi_n(x)\,\overline{\Psi_m(x)} \in \mathcal{L}^1(\mathbb{R})$ and $\langle \Psi_m, \varphi_n \rangle = \delta_{n,m}$, for all $n, m \geq 0$, and

$$N\varphi_n(x) = n\varphi_n(x), \qquad N^\dagger \Psi_n(x) = n\Psi_n(x),$$

where $N = BA$ and $N^\dagger = A^\dagger B^\dagger$.

A concrete example

In principle, without extra assumptions on the PBSs, we don't know if the functions of $\mathcal{F}_\varphi=\{\varphi_n(x)\}$ and $\mathcal{F}_\Psi=\{\Psi_n(x)\}$ are square-integrable or not. Hence, it makes no sense to check if they are biorthogonal bases, or if they are complete, in $\mathcal{L}^2(\mathbb{R})$, in general. However, some info on this side can also be deduced....
We can rewrite $\varphi_n(x)$ and $\Psi_n(x)$ as follows:

$$\varphi_n(x)=c_n(x)\rho_A(x),\qquad \Psi_n(x)=c_n(x)\rho_B(x), \tag{37}$$

where

$$c_n(x)=\frac{1}{2^{1/4}}\,e_n\left(\frac{x+k}{\sqrt{2}}\right),\qquad \text{with}\qquad e_n(x)=\frac{1}{\sqrt{2^n n!\sqrt{\pi}}}H_n(x)\,e^{-\frac{x^2}{2}}, \tag{38}$$

is the well known n-th eigenstate of the harmonic oscillator, and

$$\rho_A(x)=N_\varphi(2\pi)^{1/4}e^{\frac{1}{2}\left(\frac{x+k}{\sqrt{2}}\right)^2}e^{-s_A(x)},\qquad \rho_B(x)=N_\Psi(2\pi)^{1/4}e^{\frac{1}{2}\left(\frac{x+k}{\sqrt{2}}\right)^2}e^{-\overline{s_B(x)}},$$

which are independent of n. Notice that $\rho_A(x)$ and $\rho_B(x)$ satisfy

$$\rho_A(x)\,\overline{\rho_B(x)}=1, \tag{39}$$

Is $\mathcal{L}^2$ the only "good" space for bi-coherent states?

First of all: let's work in $\mathcal{H}$:–

Let us consider two biorthogonal $\mathcal{G}$ -quasi bases, $\mathcal{F}_{\tilde\varphi}=\{\tilde\varphi_n\in\mathcal{H},\, n\geq 0\}$ and $\mathcal{F}_{\tilde\Psi}=\{\tilde\Psi_n\in\mathcal{H},\, n\geq 0\}$, $\mathcal{G}$ some dense subset of $\mathcal{H}$. Consider an increasing sequence of real numbers α_n satisfying the inequalities $0=\alpha_0<\alpha_1<\alpha_2<\ldots$, and let

$$\overline{\alpha}=\sup_n \alpha_n.$$

Consider two operators, A and $B^\dagger$, acting as lowering operators on $\mathcal{F}_{\tilde\varphi}$ and $\mathcal{F}_{\tilde\Psi}$ in the following way:

$$A\,\tilde\varphi_n=\alpha_n\tilde\varphi_{n-1},\qquad B^\dagger\,\tilde\Psi_n=\alpha_n\tilde\Psi_{n-1}, \tag{40}$$

for all $n\geq 1$, with $A\,\tilde\varphi_0=B^\dagger\,\tilde\Psi_0=0$. Then, putting

$$\alpha_0!=1,\qquad \alpha_k!=\alpha_1\alpha_2\cdots\alpha_k,\quad k\geq 1,$$

the following theorem holds:

Is $\mathcal{L}^2$ the only "good" space for coherent states?

Theorem, pt 1:

Assume that four strictly positive constants A_φ, A_Ψ, r_φ and r_Ψ exist, together with two strictly positive sequences $M_n(\varphi)$ and $M_n(\Psi)$, for which

$$\lim_{n\to\infty}\frac{M_n(\varphi)}{M_{n+1}(\varphi)}=M(\varphi),\qquad \lim_{n\to\infty}\frac{M_n(\Psi)}{M_{n+1}(\Psi)}=M(\Psi), \tag{41}$$

where $M(\varphi)$ and $M(\Psi)$ could be infinity, and such that, for all $n\geq 0$,

$$\|\tilde\varphi_n\|\leq A_\varphi\, r_\varphi^n M_n(\varphi),\qquad \|\tilde\Psi_n\|\leq A_\Psi\, r_\Psi^n M_n(\Psi). \tag{42}$$

Then the following series:

$$N(|z|)=\left(\sum_{k=0}^\infty \frac{|z|^{2k}}{(\alpha_k!)^2}\right)^{-1/2}, \tag{43}$$

$$\varphi(z)=N(|z|)\sum_{k=0}^\infty\frac{z^k}{\alpha_k!}\tilde\varphi_k,\qquad \Psi(z)=N(|z|)\sum_{k=0}^\infty\frac{z^k}{\alpha_k!}\tilde\Psi_k, \tag{44}$$

are all convergent inside the circle $C_\rho(0)$ in $\mathbb{C}$ centered in the origin of the complex plane and of radius $\rho=\overline{\alpha}\,\min\left(1,\frac{M(\varphi)}{r_\varphi},\frac{M(\Psi)}{r_\Psi}\right)$.

Is $\mathcal{L}^2$ the only "good" space for coherent states?

Theorem, pt 2:

Moreover, for all $z \in C_\rho(0)$,

$$A\varphi(z) = z\varphi(z), \qquad B^\dagger \Psi(z) = z\Psi(z). \tag{45}$$

Suppose further that a measure $d\lambda(r)$ does exist such that

$$\int_0^\rho d\lambda(r)\, r^{2k} = \frac{(\alpha_k!)^2}{2\pi}, \tag{46}$$

for all $k \geq 0$. Then, putting $z = re^{i\theta}$ and calling $d\nu(z,\overline{z}) = N(r)^{-2} d\lambda(r) d\theta$, we have

$$\int_{C_\rho(0)} \langle f, \Psi(z)\rangle \langle \varphi(z), g\rangle \, d\nu(z,\overline{z}) = \int_{C_\rho(0)} \langle f, \varphi(z)\rangle \langle \Psi(z), g\rangle \, d\nu(z,\overline{z}) = \langle f, g\rangle , \tag{47}$$

for all $f, g \in \mathcal{G}$.

Is $\mathcal{L}^2$ the only "good" space for coherent states?

$\mathcal{L}^2(\mathbb{R})$ is enough: case 1:–

Going back to ours PBSs, we see that if $\rho_A(x), \rho_B(x) \in \mathcal{L}^\infty(\mathbb{R})$, then

$$\|\varphi_n\| = \|c_n\, \rho_A\| \leq \|\rho_A\|_\infty \|c_n\| = \|\rho_A\|_\infty, \qquad \|\Psi_n\| = \|c_n\, \rho_B\| \leq \|\rho_B\|_\infty,$$

for all $n \geq 0$. Moreover, since $\alpha_n = \sqrt{n}$, $\overline{\alpha} = \infty$. Hence the theorem above holds with the following choice:

$$A_\varphi = \|\rho_A\|_\infty,\ A_\Psi = \|\rho_B\|_\infty,\ r_\varphi = r_\Psi = M_n(\varphi) = M_n(\Psi) = 1,$$

$\forall n \geq 0$. Then $\rho = \infty$, and the series in (43) and (44) converge in all the complex plane. For instance, let us take

$$s_A(x) = \frac{x^2}{4} + \frac{kx}{2} + \Phi(x), \qquad s_B(x) = \frac{x^2}{4} + \frac{kx}{2} - \Phi(x),$$

where $\Phi(x)$ is any real C^∞ function bounded from below and from above, i.e. when there exist m, M such that $-\infty < m \leq \Phi(x) \leq M < \infty$, a.e. in $\mathbb{R}$. In fact, we get

$$\|\rho_A\|_\infty = N_\varphi (2\pi)^{1/4} e^{k^2/4-m}, \qquad \|\rho_B\|_\infty = N_\Psi (2\pi)^{1/4} e^{k^2/4+M},$$

while the PBSs are $w_A(x) = \frac{x}{2} + k + \Phi'(x)$ and $w_B(x) = \frac{x}{2} + k - \Phi'(x)$.

Is $\mathcal{L}^2$ the only "good" space for coherent states?

$\mathcal{L}^2(\mathbb{R})$ is enough: case 2:–

Let us take $s_A(x)=\frac{x^2}{4}$ and $s_B(x)=\frac{x^2}{4}+kx$. In this case

$$\varphi_n(x)=\frac{N_\varphi}{\sqrt{2^n\,n!}}\,H_n\left(\frac{x+k}{\sqrt{2}}\right)\,e^{-x^2/4},\qquad \Psi_n(x)=\frac{N_\Psi}{\sqrt{2^n\,n!}}\,H_n\left(\frac{x+k}{\sqrt{2}}\right)\,e^{-x^2/4-kx},$$

which are both square-integrable, but with diverging norms:

$$\|\varphi_n\|\simeq\frac{|N_\varphi|}{(2|k|)^{1/4}}\,e^{-k^2/4}\,\frac{e^{|k|\sqrt{n}}}{n^{1/8}},\qquad \|\Psi_n\|\simeq\frac{|N_\Psi|}{(2|k|)^{1/4}}\,e^{3k^2/4}\,\frac{e^{|k|\sqrt{n}}}{n^{1/8}},$$

where $\simeq$ stands for *except for corrections* $O(n^{-1/2})$. Still we can use our Theorem, taking (for $k\neq 0$, the only relevant case)

$$A_\varphi=\frac{|N_\varphi|}{(2|k|)^{1/4}}\,e^{-k^2/4},\quad A_\Psi=\frac{|N_\Psi|}{(2|k|)^{1/4}}\,e^{3k^2/4},$$

$$r_\varphi=r_\Psi=e^{|k|},\qquad M_n(\varphi)=M_n(\Psi)=\frac{1}{n^{1/8}}.$$

Is $\mathcal{L}^2$ the only "good" space for coherent states?

Hence $M(\varphi) = M(\Psi) = 1$, $\rho = \infty$, and the bi-coherent states are well defined in all the complex plane, belong to $\mathcal{L}^2(\mathbb{R})$, and satisfies the usual eigenvalue equations and the weak resolution of the identity (47). In fact, taking

$$d\lambda(r) = \frac{1}{\pi} e^{-r^2} r\, dr, \qquad (48)$$

condition (1) is satisfied:

$$\int_0^\infty d\lambda(r) r^{2k} = \frac{k!}{2\pi}.$$

Then (2) follows from the fact that $(\mathcal{F}_\varphi, \mathcal{F}_\Psi)$ are $\mathcal{E}$-quasi bases for all possible choices of PBSs. Here

$$\mathcal{E} = \left\{h(x) \in \mathcal{L}^2(\mathbb{R}) :\, h(x)\rho_j(x) \in \mathcal{L}^2(\mathbb{R}),\, j = A, B\right\},$$

which is dense in $\mathcal{L}^2(\mathbb{R})$, since it contains $D(\mathbb{R})$.

Working outside $\mathcal{L}^2(\mathbb{R})$

$\mathcal{L}^2(\mathbb{R})$ is not enough

From our previous analysis it is easy to understand that we can somehow force the system to live in $\mathcal{L}^2(\mathbb{R})$, ...

Working outside $\mathcal{L}^2(\mathbb{R})$

$\mathcal{L}^2(\mathbb{R})$ is not enough

From our previous analysis it is easy to understand that we can somehow force the system to live in $\mathcal{L}^2(\mathbb{R})$, ...but not so much!

We could introduce a metric in $\mathcal{L}^2(\mathbb{R})$ which makes some non square-integrable function, square-integrable (by, of course, changing its meaning!), but then the functions in its original biorthogonal set are no longer biorthogonal, or o.n. Moreover, the adjoint maps changes, and this change depends on which metric we adopt: we gain on one side, but we lose on the other!

We use an alternative approach, learning from our PBSs.

Let $\mathcal{F}_c = \{c_n(x),\, n \geq 0\}$ be an o.n. basis in $\mathcal{L}^2(\mathbb{R})$, and let $\rho_f(x)$ and $\rho_g(x)$ be two Lebesgue-measurable functions such that, calling

$$f_n(x) = c_n(x)\,\rho_f(x), \qquad g_n(x) = c_n(x)\,\rho_g(x), \tag{49}$$

we have $f_n(x)\,g_m(x) \in \mathcal{L}^1(\mathbb{R})$, for all $n, m \geq 0$.

Working outside $\mathcal{L}^2(\mathbb{R})$

This implies that, despite of $f_n(x)$ or $g_n(x)$ being square-integrable or not, the form $\langle f_n, g_m\rangle$ is always well defined. With a slight abuse of language, we still call $\langle f_n, g_m\rangle$ the scalar product between $f_n(x)$ and $g_m(x)$. If we take $\rho_f(x) = \overline{\rho_g(x)}^{-1}$, then $\mathcal{F}_f = \{f_n(x)\}$ and $\mathcal{F}_g = \{g_n(x)\}$ are *biorthonormal*:

$$\langle f_n, g_m\rangle = \delta_{n,m}. \tag{50}$$

We next define

$$\mathcal{V} = \left\{v(x) \in \mathcal{L}^2(\mathbb{R}) : v(x)\rho_j(x) \in \mathcal{L}^2(\mathbb{R}),\ j = f, g\right\} \tag{51}$$

If $\rho_f(x)$ and $\rho_g(x)$ are C^∞ functions, then $\mathcal{D}(\mathbb{R}) \subseteq \mathcal{V}$, which is therefore dense in $\mathcal{L}^2(\mathbb{R})$. In our condition $(\mathcal{F}_f, \mathcal{F}_g)$ are $\mathcal{V}$-quasi bases,

$$\langle v, w\rangle = \sum_{n\geq 0}\langle v, f_n\rangle\langle g_n, w\rangle = \sum_{n\geq 0}\langle v, g_n\rangle\langle f_n, w\rangle, \tag{52}$$

for all $v(x), w(x) \in \mathcal{V}$, so that they are both complete in $\mathcal{V}$.

But, what about bi-coherent states?

Working outside $\mathcal{L}^2(\mathbb{R})$

It is clear that the assumption on $\|f_n\|$ and $\|g_n\|$, $\|\tilde{\varphi}_n\| \leq A_\varphi\, r_\varphi^n M_n(\varphi)$ and $\|\tilde{\Psi}_n\| \leq A_\Psi\, r_\Psi^n M_n(\Psi)$, make no sense now. This is because we can easily have $\|f_n\| = \infty$ or $\|g_n\| = \infty$.
However, if we take $v \in \mathcal{V}$, and consider the following series

$$S_{f,v}(z) = \sum_{n\geq 0} \frac{z^n}{\alpha_n!}\langle f_n, v\rangle, \qquad S_{g,v}(z) = \sum_{n\geq 0} \frac{z^n}{\alpha_n!}\langle g_n, v\rangle, \tag{53}$$

then, both these series converge, for all $v \in \mathcal{V}$, inside $C_{\overline{\alpha}}(0)$. Hence we can introduce

$$F(z)[v] = \langle f(z), v\rangle = N(|z|) \sum_{n\geq 0} \frac{\overline{z}^n}{\alpha_n!}\langle f_n, v\rangle,$$

$$G(z)[v] = \langle g(z), v\rangle = N(|z|) \sum_{n\geq 0} \frac{\overline{z}^n}{\alpha_n!}\langle g_n, v\rangle,$$

which can be used to define $f(z)$ and $g(z)$ *in a weak sense* (like in distribution theory).

We need a topology on $\mathcal{V}$: we say that a sequence $\{v_n(x)\}$ in $\mathcal{V}$ is $\tau_\mathcal{V}$-convergent to a certain $v(x) \in \mathcal{L}^2(\mathbb{R})$ if $\{v_n(x)\}$ converges to $v(x)$ in the norm $\|.\|$, and if $\{\rho_j(x)\, v_n(x)\}$, $j = f, g$, are Cauchy sequences in $\|.\|$ and converges to $\rho_j(x)v(x)$. Then $v(x) \in \mathcal{V}$. Hence, $\mathcal{V}$ is closed in $\tau_\mathcal{V}$.

Working outside $\mathcal{L}^2(\mathbb{R})$

Now, if we call $\mathcal{V}'$ the set of all the continuous functionals on $\mathcal{V}$, and if we consider

$$\mathcal{V}_0 = \left\{ w(x) \in \mathcal{V} : \, w(x) \in D(a^\dagger) \cap D(b), \quad a^\dagger w(x), bw(x) \in \mathcal{V} \right\},$$

Proposition:

1. $F(z)$ and $G(z)$ both belong to $\mathcal{V}'$.
2. for all $v(x) \in \mathcal{V}_0$ we have, for all $z \in C_{\overline{\alpha}}(0)$,

$$\langle v, af(z) \rangle = z \langle v, f(z) \rangle, \qquad \langle v, b^\dagger g(z) \rangle = z \langle v, g(z) \rangle. \tag{54}$$

3. Suppose that a measure $d\lambda(r)$ does exist such that

$$\int_0^{\overline{\alpha}} d\lambda(r) r^{2k} = \frac{(\alpha_k!)^2}{2\pi}, \tag{55}$$

for all $k \geq 0$. Then, putting $z = re^{i\theta}$ and calling $d\nu(z, \overline{z}) = N(r)^{-2} d\lambda(r) d\theta$, we have

$$\int_{C_{\overline{\alpha}}(0)} \langle v, f(z) \rangle \langle g(z), w \rangle \, d\nu(z, \overline{z}) = \int_{C_{\overline{\alpha}}(0)} \langle v, g(z) \rangle \langle f(z), w \rangle \, d\nu(z, \overline{z}) = \langle v, w \rangle, \tag{56}$$

for all $v, w \in \mathcal{V}$.

$F(z)$ and $G(z)$ are **weak bi-coherent states**.

Working outside $\mathcal{L}^2(\mathbb{R})$

Suppose now to apply these results to our PBSs, identifying $\varphi_n(x)$, $\rho_A(x)$, $\Psi_n(x)$ and $\rho_B(x)$ respectively with $f_n(x)$, $\rho_f(x)$, $g_n(x)$ and $\rho_g(x)$.

Since A and B are pseudo-bosonic operators, $\alpha_n = \sqrt{n}$,and $\overline{\alpha} = \infty$. Condition on the measure $d\lambda(r)$ reads therefore

$$\int_0^\infty d\lambda(r) r^{2k} = \frac{k!}{2\pi},$$

which is solved by $d\lambda(r) = \frac{1}{\pi} e^{-r^2} r\, dr$. This means that the forms $F(z)[v]$ and $G(z)[v]$ are well defined for all $v \in \mathcal{V}$ and for all $z \in \mathbb{C}$.

Moreover, $\mathcal{D}(\mathbb{R}) \subseteq \mathcal{V}_0 \subseteq \mathcal{V}$. Hence $\mathcal{V}_0$ and $\mathcal{V}$ are in $\mathcal{L}^2(\mathbb{R})$:

we are indeed working with *large* sets!

Conclusion:– a distributional approach can be relevant not only for eigenstates of number-like (non selfadjoint) operators, but also for bi-coherent states for some physical system.

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....the end!!!