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EXCEPTIONAL POINTS

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IN CLOSED QUANTUM SYSTEMS

international conference *"Non-Hermitian Physics"*,
March 22nd, 2021, 14:50 - 15:30 CET, online

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.

thanks to organizers!

PLAN OF THE TALK

I. exceptional points (EPs) in physics

II. EPs in quasi-Hermitian theory

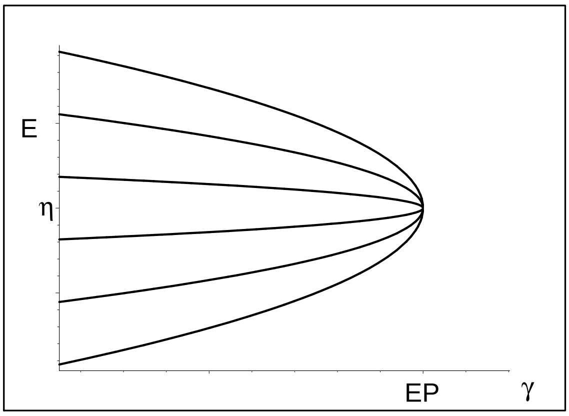
III. illustrative harmonic oscillator example

IV. finite-dimensional benchmark Hamiltonians

V. numerical localizations of EPs

VI. trajectories of access to EPs

I. exceptional points (EPs) in physics



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sample: energy eigenvalues $E = E_n(\gamma)$ merge at $\gamma = \gamma^{(EP)}$

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traditionally, **the EPs only encountered**

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. **in MATH of perturbation expansions**

$$\psi_n(g,x) = \psi_n(g_0,x) + (g-g_0)\,\psi_n^{(1)}(g,x) + (g-g_0)^2\,\psi_n^{(2)}(g,x) + \ldots$$

$$E_n(g) = E_n(g_0) + (g-g_0)\,E_n^{(1)}(g) + (g-g_0)^2\,E_n^{(2)}(g) + \ldots$$

which converge

. iff there are no EPs inside circle $|g-g_0| < R$

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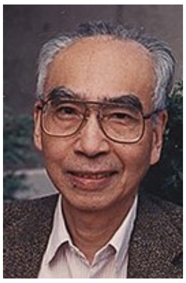
in 1966, EPs were introduced by Tosio Kato:



in PHYSICS,

in CLASSICAL OPTICS

EPs are known as

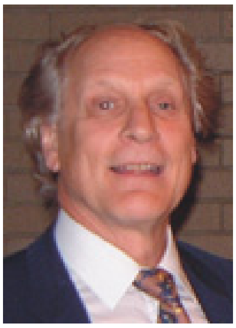
NON-HERMITIAN DEGENERACIES

see review

Czechosl. J. Phys., Vol. 54 (2004), No. 10, p. 1039.

by Michael Berry

ubiquitous EPs



helped to explain the

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CLASSICAL and QUANTUM CHAOS

and

PHASE TRANSITIONS

see review

Czechosl. J. Phys., Vol. 54 (2004), No. 10, p. 1091.

. by W. Dieter Heiss from Stellenbosch University

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NEW CHALLENGES: EPs might emerge, e.g., in cosmology:

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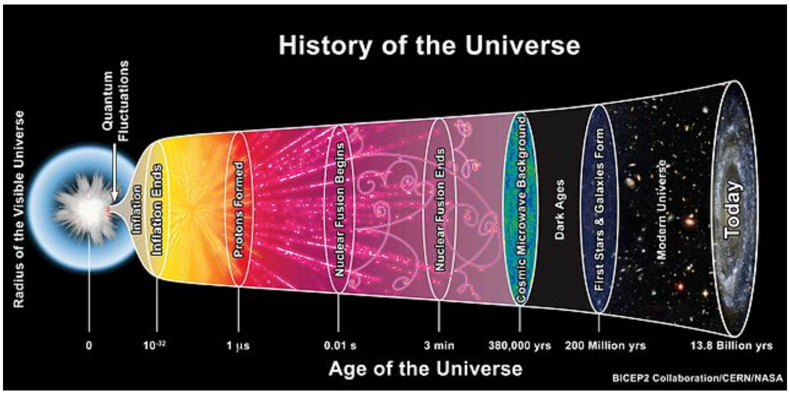

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how should one quantize the Universe shortly after BIG BANG?

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let us invert the arrow of time:

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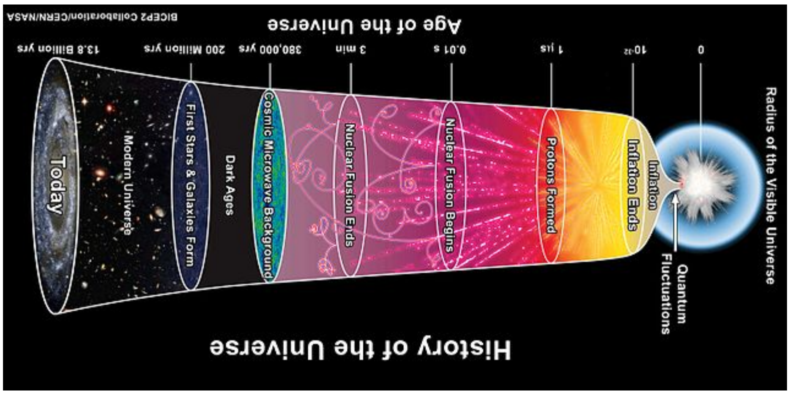

and return to a small vicinity of BIG BANG

in classical picture there exists a Big Bang singularity at $t = 0$:

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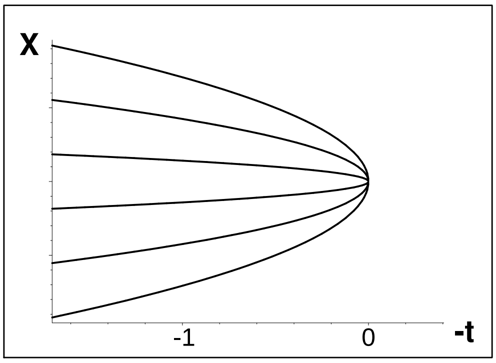
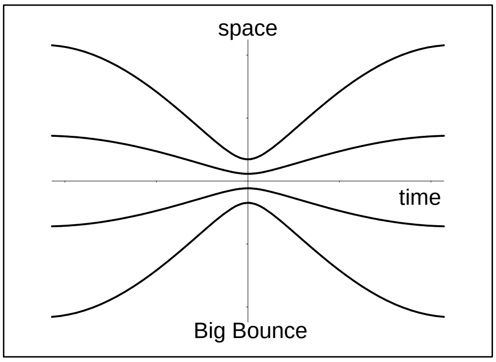
e.g., in six-grid-point non-quantum model of the Universe we have a collapse:



CW: **after quantization, Big Bang smeared to Big Bounce**



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= avoided grid point crossing = a genuine quantum phenomenon

= the EP degeneracy shifted to a complex, "unphysical" time

II. EPs in quasi-Hermitian theory

MOTIVATION:

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in unitary (a.k.a. "closed") systems

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some

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QUANTUM PHASE TRANSITIONS

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may be realized

as a fall into singularity called

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EXCEPTIONAL POINT

. **BASIC CLASSIFICATION of quantum systems:**

a. "open" (effective models)

energies = complex, **phenomena** = resonances, . . .

theory = in a Feshbach's subspace

b. "closed" (complete description)

energies = real, **phenomena** = unitary evolution , . . .

theory can be QUASI-HERMITIAN

quasi-Hermitian quantum theory

= LESS RESTRICTIVE

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∃ the proof that the Big-Bounce is not unavoidable: see

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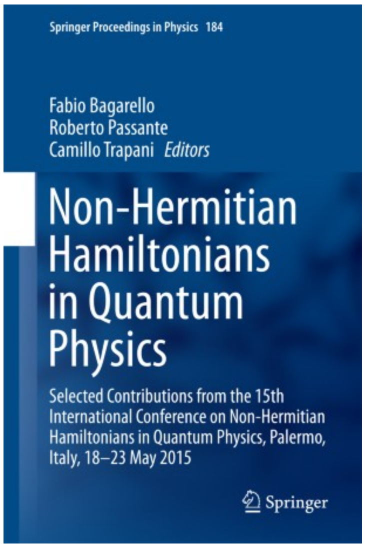
[] MZ, Quantization of Big Bang in crypto- Hermitian Heisenberg picture, chapter in this book:



A. the basics of formalism

NON-HERMITIAN H treated as HIDDENLY HERMITIAN

- **yielding an**
- **amended Schrödinger picture**
- **father founder: Freeman Dyson:** 

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$\Longrightarrow$ in such a quasi-Hermitian, "3HS" quantum theory one needs

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(1) non-Hermitian Hamiltonian H given in "working space" $\mathcal{K} = \mathcal{H}^{(False)}$

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(2) (a suitable) Hermitizing inner-product metric $\Theta = \Omega^\dagger\Omega$,

$$\langle\psi_1|\psi_2\rangle_{\mathcal{H}} = \langle\psi_1|\Theta|\psi_2\rangle_{\mathcal{K}}$$

(3) (a hidden) Hermiticity of all observables Λ defined in $\mathcal{H} = \mathcal{H}^{(Physical)}$,

$$\Lambda^\dagger\,\Theta = \Theta\,\Lambda$$

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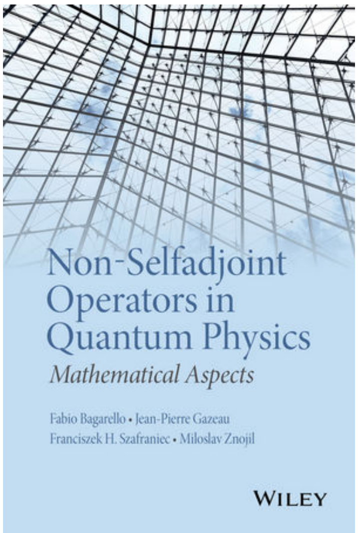
the state of art in 2015:



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B. two exactly solvable MATRIX illustrations

(1) Wheeler-DeWitt equation of canonical quantum gravity

two-by-two matrix

$$H = H^{(WDW)}(\tau) = \left[\begin{array}{cc} 0 & \exp 2\tau \\ 1 & 0 \end{array} \right] \neq H^\dagger$$

inner product $\langle \cdot | \Theta \cdot \rangle$ inside $\mathcal{D} = (-\infty, \infty)$, $E = E_\pm = \pm \exp \tau$;

the reconstruction of $\mathcal{H}$ is explicit and exhaustive,

$$H = \Theta^{-1} H^\dagger \Theta \,, \qquad \Theta = \Omega^\dagger \Omega \neq I$$

$$\Theta = \Theta^{(WDW)}(\tau, \beta) = \left[\begin{array}{cc} \exp(-\tau) & \beta \\ \beta & \exp \tau \end{array} \right] = \Theta^\dagger \,, \qquad |\beta| < 1 \,.$$

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(2) Bose-Hubbard model (Graefe et al, 2008),

the two-by-two matrix example

$$H = H^{(2)}_{(BH)}(\gamma) = \left[\begin{array}{cc} -i\gamma & 1 \\ 1 & i\gamma \end{array} \right]$$

admits the inner product $\langle \cdot | \Theta \cdot \rangle$ inside $\mathcal{D} = (-1, 1)$
supports the two boundary-point EPs $\gamma^{(EP)} = \pm 1$

$$\Theta^{(2)}(\beta) = I^{(2)} + \left[\begin{array}{cc} 0 & \beta + i\,\gamma \\ \beta - i\,\gamma & 0 \end{array} \right], \qquad -\sqrt{1-\gamma^2} < \beta < \sqrt{1-\gamma^2}\,.$$

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C. whenever $N = \infty$, <u>much more care is needed</u>

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Dieudonne 1960: early words of warning

the warnings were nontrivial:

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in 2015, for example, David Krejcirik et al

claimed that the imaginary cubic oscillator,

"the fons et origo of PT-symmetric quantum mechanics",

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exhibits spectral instability:

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"complex eigenvalues may appear very far from the unperturbed real ones despite the norm of the perturbation is arbitrarily small"

IMPORTANT: Krejcirik et al worked in open-system regime

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= in auxiliary $\mathcal{K}$ rather than in $\mathcal{H}$

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puzzle resolved: see its closed-system reinterpretation in

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[] MZ and František Růžička,

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“Nonlinearity of perturbations in PT-symmetric quantum mechanics.”

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J. Phys. Conf. Ser. 1194 (2019) 012120

IN CONTRAST, we are working here in closed-system regime:

® flowchart

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step (A):		step (B):
one picks up an unphysical but **user-friendly** Hilbert space $\mathcal{K}$ and a **non−Hermitian** H with real spectrum,	$\longrightarrow$	one constructs an eligible metric $\Theta = \Omega^\dagger\Omega$ (s. t. $H^\dagger\Theta = \Theta H$), *i.e.*, **physical** Hilbert space $\mathcal{H}$.

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III. illustrative harmonic oscillator example

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spiked $\mathcal{PT}-$symmetric ODE (i.e., $N=\infty$)

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+r^2\right]\,\psi_n^{(\sigma)}(\ell,r)=E_n^{(\sigma)}(\ell)\,\psi_n^{(\sigma)}(\ell,r)\,,\quad n=0,1,2,\ldots\,,\quad \sigma=\pm 1$$

.

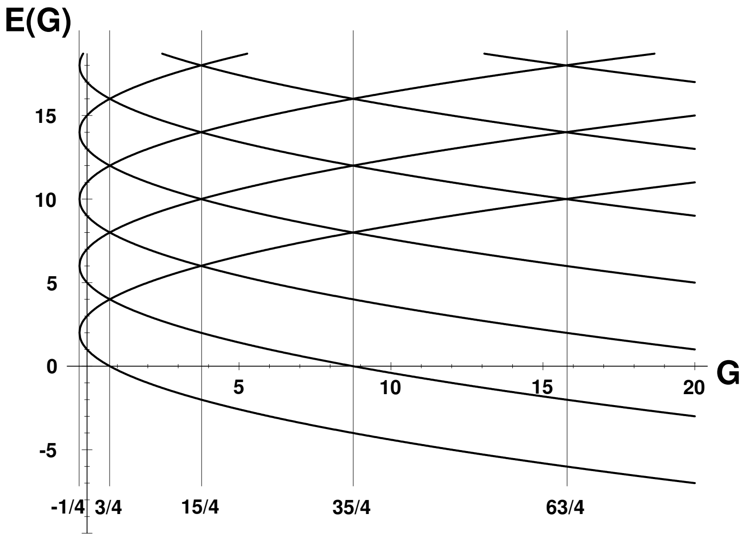
"coordinate" $r=x-\mathrm{i}\epsilon$ is complex, $\boxed{x\in\mathbb{R}}$, any parameter $\ell\in\mathbb{R}$

. the model is exactly solvable (MZ, 1999):

$$E_n^{(\sigma)}(\ell)=4n-2\,\sigma\,\alpha+2\,,\qquad \alpha=\ell+1/2$$

(abbreviate $G=\ell(\ell+1)$ and P.T.O.)

all EPs are $\infty-$times degenerate, every EP$= \oplus EP2$



CONSEQUENCE: Hamiltonian becomes block-diagonal at EPs: e.g.,

$$\mathfrak{H}^{(HO)}(-1/4) = \left(\begin{array}{cc|cc|cc} 2 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 2 & 0 & 0 & 0 & \ldots \\ \hline 0 & 0 & 6 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 6 & 0 & \ldots \\ \hline 0 & 0 & 0 & 0 & 10 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{array} \right) + \text{ corrections}$$

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∃ a constructive proof of mathematical 3HS consistence:

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[] MZ, Sci. Reports 10(1) (2020) 18523,

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including

⇒ an exhaustive menu of correct physical Hilbert spaces,

⇒ accounting for all non-uniqueness of assignment $H \to \Theta(H)$

CONCLUSION: the HO system is truly exceptional because

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(1) the phase transitions involve, simultaneously, all levels

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(2) the HO domain $\mathcal{D}^{[HO]}$ of unitarity is "punched",

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multiply connected, with EPs $\in \partial\mathcal{D}^{[HO]}$ excluded,

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$$\boxed{\mathcal{D}^{[HO]} = \left(-\tfrac{1}{4}, \tfrac{3}{4}\right) \bigcup \left(\tfrac{3}{4}, \tfrac{15}{4}\right) \bigcup \left(\tfrac{15}{4}, \tfrac{35}{4}\right) \bigcup \ldots}$$

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ℵ recommended additional reading:

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[] MZ, "Supersymmetry and exceptional points."
SYMMETRY 12, (2020) 892
DOI: 10.3390/sym12060892 (arXiv:2005.04508)

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IV. finite-dimensional Hamiltonians with EPs

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A. real-matrix models

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[] MZ, “Quantum phase transitions
mediated by clustered non-Hermitian degeneracies.”
Physical Review E (Vol. 103, No. 3, 032120)
DOI:10.1103/PhysRevE.103.032120
(arXiv:2102.12272).

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<u>**model:**</u>, $N-$truncated $E-$shifted tridiagonal anharmonic oscillator (TAO)

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$$H^{(N)}_{(\text{TAO})}(\lambda) = \left[\begin{array}{ccccc} 1-N & b_1(\lambda) & 0 & \dots & 0 \\ -\,b_1(\lambda) & 3-N & \ddots & \ddots & \vdots \\ 0 & -b_2(\lambda) & \ddots & b_2(\lambda) & 0 \\ \vdots & \ddots & \ddots & N-3 & b_1(\lambda) \\ 0 & \dots & 0 & -b_1(\lambda) & N-1 \end{array} \right] .$$

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antisymmetry = maximal non-Hermiticity

$\mathcal{PT}-$**symmetry** = symmetry with respect to the second diagonal

non-diagonalizable EPN limits

$$H^{(2)}_{(\mathrm{TAO})}(\lambda^{(EP2)}) = \left[\begin{array}{cc} -1 & 1 \\ -1 & 1 \end{array} \right], \qquad H^{(3)}_{(\mathrm{TAO})}(\lambda^{(EP3)}) = \left[\begin{array}{ccc} -2 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 2 \end{array} \right],$$

$$H^{(4)}_{(\mathrm{TAO})}(\lambda^{(EP4)}) = \left[\begin{array}{cccc} -3 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & -1 & 2 & 0 \\ 0 & -2 & 1 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 \end{array} \right],$$

etc

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EPN degeneracy (non-clustered)

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eigenvalues

$$\lim_{\lambda \to \lambda^{(EPN)}} E_n(\lambda) = \eta \,, \qquad n = 1, 2, \ldots, N \,.$$

eigenvectors

$$\lim_{\lambda \to \lambda^{(EPN)}} |\psi_n^{(N)}(\lambda)\rangle = |\chi^{(N)}(\lambda)\rangle \,, \qquad , \qquad n = 1, 2, \ldots, N \,.$$

canonical (Jordan) form of a non-diagonalizable matrix

$$J^{(\mathrm{N})}(\eta) = \begin{bmatrix} \eta & 1 & 0 & \dots & 0 \\ 0 & \eta & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 0 & \eta & 1 \\ 0 & \dots & 0 & 0 & \eta \end{bmatrix} .$$

Schrödinger-like equation

$$H^{(N)}_{(TAO)}(\lambda^{(EPN)})\, Q^{(N)} = Q^{(N)}\, J^{(\mathrm{N})}(\eta)$$

transition matrices $Q^{(N)}$ available in closed form

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B. clustered non-Hermitian degeneracies

subscripts n decomposed into K non-overlapping subsets S_k,

$$\lim_{\lambda \to \lambda^{(EPN)}} |\psi_{n_k}^{(N)}(\lambda)\rangle = |\chi_k^{(N)}(\lambda)\rangle \,, \qquad n_k \in S_k \,, \qquad k = 1, 2, \ldots, K \,.$$

general anharmonic oscillator (GAO) Hamiltonians

$$H^{(N)}_{(\mathrm{GAO})}(\lambda) = \left[\begin{array}{cccccc} 1-N & b_1(\lambda) & c_1(\lambda) & d_1(\lambda) & \ldots & \omega_1(\lambda) \\ -\,b_1(\lambda) & 3-N & b_2(\lambda) & c_2(\lambda) & \ddots & \vdots \\ -\,c_1(\lambda) & \ddots & \ddots & \ddots & \ddots & d_1(\lambda) \\ -\,d_1(\lambda) & \ddots & -b_3(\lambda) & N-5 & b_2(\lambda) & c_1(\lambda) \\ \vdots & \ddots & -c_2(\lambda) & -b_2(\lambda) & N-3 & b_1(\lambda) \\ -\,\omega_1(\lambda) & \ldots & -d_1(\lambda) & -c_1(\lambda) & -b_1(\lambda) & N-1 \end{array} \right]$$

general Schrödinger-like equation

$$H_{(GAO)}^{(N)}(\lambda^{(EPN)})\,Q^{(N)} = Q^{(N)}\,\mathcal{J}^{[\mathcal{R}(N)]}(\eta)$$

where $\mathcal{R}(N)$ marks one of partitions of $N = N_1 + N_2 + \ldots + N_K$

such that $N_1 \geq N_2 \geq \ldots \geq N_K \geq 2$ (OEIS A002865)

K = clusterization index *alias* geometric multiplicity

$$\mathcal{J}^{[\mathcal{R}(N)]}(\eta) = J^{(N_1)}(\eta) \oplus J^{(N_2)}(\eta) \oplus \ldots \oplus J^{(N_K)}(\eta)\,.$$

EXAMPLES:

unique ($K = 1$) partitions $\mathcal{R}(2) = 2$ and $\mathcal{R}(3) = 3$,

nontriviality: $\mathcal{R}_1(4) = 4$ (i.e., $K = 1$) or $\mathcal{R}_2(4) = 2 + 2$ (i.e., $K = 2$)

$$\mathcal{J}^{[2+2]}(\eta) = \left[\begin{array}{cc|cc} \eta & 1 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ \hline 0 & 0 & \eta & 1 \\ 0 & 0 & 0 & \eta \end{array}\right], \quad \mathcal{J}^{[3+2]}(\eta) = \left[\begin{array}{ccc|cc} \eta & 1 & 0 & 0 & 0 \\ 0 & \eta & 1 & 0 & 0 \\ 0 & 0 & \eta & 0 & 0 \\ \hline 0 & 0 & 0 & \eta & 1 \\ 0 & 0 & 0 & 0 & \eta \end{array}\right] \quad \text{etc}\,.$$

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C. pentadiagonal Hamiltonians

one-parametric model

$$H^{(\mathrm{toy})}(\lambda) = \begin{bmatrix} 1 & 0 & \sqrt{3}g & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & \sqrt{2}g & 0 & 0 & 0 \\ -\sqrt{3}g & 0 & 5 & 0 & 2g & 0 & 0 \\ 0 & -\sqrt{2}g & 0 & 7 & 0 & \sqrt{2}g & 0 \\ 0 & 0 & -2g & 0 & 9 & 0 & \sqrt{3}g \\ 0 & 0 & 0 & -\sqrt{2}g & 0 & 11 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{3}g & 0 & 13 \end{bmatrix}.$$

Schrödinger equation

$$H^{(\text{toy})}(g)\,|\psi_n(g)\rangle = E_n(g)\,|\psi_n(g)\rangle$$

solvable exactly,

$$E_0(g) = 7\,, \qquad E_{\pm 1}(g) = 7 \pm \sqrt{4-g^2}$$

$$E_{\pm 2}(g) = 7 \pm 2\,\sqrt{4-g^2}\,, \qquad E_{\pm 3}(g) = 7 \pm 3\,\sqrt{4-g^2}\,.$$

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strong-coupling dynamical regime

redefine $g = \widetilde{g}(\kappa) = 2\,(1-\kappa^2)$

$$\widetilde{E}_0(\kappa) = 7\,, \qquad \widetilde{E}_{\pm 1}(\kappa) = 7 \pm 2\,\sqrt{-\kappa^4 + 2\,\kappa^2} \sim 7 \pm 2\,\sqrt{2}\,\kappa + \mathcal{O}(\kappa^3)\,,$$
$$\widetilde{E}_{\pm 2}(\kappa) = 7 \pm 4\,\sqrt{-\kappa^4 + 2\,\kappa^2}\,, \qquad \widetilde{E}_{\pm 3}(\kappa) = 7 \pm 6\,\sqrt{-\kappa^4 + 2\,\kappa^2}\,.$$

EPN limit, $N = 7$

$g \to g^{(EP7)} = 2$

$$H^{(\mathrm{toy})}(\lambda^{(EP7)}) = H^{[odd]}_{(EP7)} \oplus H^{[even]}_{(EP7)}$$

$$H^{[odd]}_{(EP7)} = \begin{bmatrix} 1 & 2\sqrt{3} & 0 & 0 \\ -2\sqrt{3} & 5 & 4 & 0 \\ 0 & -4 & 9 & 2\sqrt{3} \\ 0 & 0 & -2\sqrt{3} & 13 \end{bmatrix}, \quad H^{[even]}_{(EP7)} = \begin{bmatrix} 3 & 2\sqrt{2} & 0 \\ -2\sqrt{2} & 7 & 2\sqrt{2} \\ 0 & -2\sqrt{2} & 11 \end{bmatrix}.$$

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canonical form of the Hamiltonian

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$$\mathcal{J}^{(4+3)}(\eta) = \left[\begin{array}{cccc|ccc} \eta & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \eta & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \eta \end{array}\right] = \begin{bmatrix} \eta & 1 & 0 & 0 \\ 0 & \eta & 1 & 0 \\ 0 & 0 & \eta & 1 \\ 0 & 0 & 0 & \eta \end{bmatrix} \oplus \begin{bmatrix} \eta & 1 & 0 \\ 0 & \eta & 1 \\ 0 & 0 & \eta \end{bmatrix} .$$

the role of unperturbed basis is relegated to

transition-matrix

$$Q^{(\mathrm{toy})} = \begin{bmatrix} -48 & 24 & -6 & 1 & 0 & 0 & 0 \\ 0 & 8 & -4 & 1 & 8 & -4 & 1 \\ -48\sqrt{3} & 16\sqrt{3} & -2\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 8\sqrt{2} & -2\sqrt{2} & 0 & 8\sqrt{2} & -2\sqrt{2} & 0 \\ -48\sqrt{3} & 8\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 8 & 0 & 0 \\ -48 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

pentadiagonal multiparametric

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$$H^{(N)}_{(\text{pent.special})}(\lambda) = \left[\begin{array}{c|c|c|c|c|c} 1-N & 0 & c_1(\lambda) & 0 & \dots & 0 \\ \hline 0 & 3-N & 0 & \ddots & \ddots & \vdots \\ \hline -c_1(\lambda) & 0 & \ddots & \ddots & c_2(\lambda) & 0 \\ \hline 0 & \ddots & \ddots & N-5 & 0 & c_1(\lambda) \\ \hline \vdots & \ddots & -c_2(\lambda) & 0 & N-3 & 0 \\ \hline 0 & \dots & 0 & -c_1(\lambda) & 0 & N-1 \end{array}\right]$$

.

abbreviated as $\boxed{\text{1-N,3-N,}\dots\text{,N-1}}$

direct-sum decomposed

$$\boxed{\text{1-N,3-N,}\ldots\text{,N-1}} = \boxed{\text{1-N,5-N,9-N, }\ldots} \oplus \boxed{\text{3-N,7-N,11-N,}\ldots}\ .$$

$$H^{(N)}_{(\text{component one})}(\lambda) = \begin{bmatrix} 1-N & c_1(\lambda) & 0 & \ldots \\ -c_1(\lambda) & 5-N & c_3(\lambda) & \ddots \\ 0 & -c_3(\lambda) & 9-N & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$H^{(N)}_{(\text{component two})}(\lambda) = \begin{bmatrix} 3-N & c_2(\lambda) & 0 & \ldots \\ -c_2(\lambda) & 7-N & c_4(\lambda) & \ddots \\ 0 & -c_4(\lambda) & 11-N & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Lemma 1 *At the even matrix dimension $N = 2J$, the decomposition of the pentadiagonal sparse-matrix model into its tridiagonal TAO components only supports the two $K = 1$ EP limits with different respective energies $\eta = \pm 1$. At any one of them, the EP confluence is incomplete, involving just J levels.*

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Proof. $\boxed{\text{1-N,3-N,...,N-1}} = \boxed{\text{1-2J,3-2J,...,2J-1}}$ has central interval $(-1, 1)$ which is too short; the resulting direct sum will be centrally asymmetric

$$\boxed{\text{1-N,3-N,...,N-1}} = \boxed{\text{1-2J,5-2J, ...,2J-3}} \oplus \boxed{\text{3-2J,7-2J,...,2J-1}},$$

□

Lemma 2 *In the EPN limit the energies of the tridiagonal-matrix components the direct sum coincide at odd $N = 2J + 1$. The EPN limit has the geometric multiplicity two.*

Proof. We have

$$\boxed{\text{1-N,3-N,}\ldots\text{,N-1}} = \boxed{\text{-2J,2-2J,}\ldots\text{,2J}} = \boxed{\text{-2J,4-2J, }\ldots\text{,2J}} \oplus \boxed{\text{2-2J,6-2J,}\ldots\text{,2J-2}}$$

and $\eta_\pm = 0$ due to the central symmetry. The $K = 2$ clusterization takes place. □

general pentadiagonal models

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$$H^{(N)}_{\text{(pentadiagonal)}}(\lambda) = \begin{bmatrix} 1-N & b_1(\lambda) & c_1(\lambda) & 0 & \dots & 0 \\ -\,b_1(\lambda) & 3-N & b_2(\lambda) & \ddots & \ddots & \vdots \\ -\,c_1(\lambda) & -b_2(\lambda) & \ddots & \ddots & c_2(\lambda) & 0 \\ 0 & \ddots & \ddots & N-5 & b_2(\lambda) & c_1(\lambda) \\ \vdots & \ddots & -c_2(\lambda) & -b_2(\lambda) & N-3 & b_1(\lambda) \\ 0 & \dots & 0 & -c_1(\lambda) & -b_1(\lambda) & N-1 \end{bmatrix}$$

with small $b_n(\lambda)$ are tractable by perturbation techniques,

$$H^{(N)}_{\text{(pentadiagonal)}}(\lambda) = H^{(N)}_{\text{(pent.special)}}(\lambda) + \text{small perturbations}$$

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D. general-matrix GAO Hamiltonians

TAO-component separation from GAO:

$$H^{(N)}_{(\text{spec.partit.})}(\lambda) = \left[\begin{array}{c|cccc|c} 1-N & 0 & 0 & \dots & 0 & \omega_1(\lambda) \\ \hline 0 & 3-N & b_2(\lambda) & \dots & z_2(\lambda) & 0 \\ 0 & -b_2(\lambda) & \ddots & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & N-5 & b_2(\lambda) & \vdots \\ 0 & -z_2(\lambda) & \dots & -b_2(\lambda) & N-3 & 0 \\ \hline -\omega_1(\lambda) & 0 & 0 & \dots & 0 & N-1 \end{array}\right]$$

.

$$\text{decomposition } \boxed{\text{1-N,3-N,...,N-1}} = \boxed{\text{1-N,N-1}} \oplus \boxed{\text{3-N,5-N,...,N-3}},$$

$$H^{(N)}_{(\text{spec.partit.})}(\lambda) = \left[(N-1) \times H^{(2)}_{(\text{toy})}(\lambda)\right] \oplus H^{(N-2)}_{(\text{GAO})}(\lambda)$$

the complete TAO decompositions:

$$H^{(\mathrm{N})}_{(GAO)}(\lambda^{(EPN)}) = \widetilde{H^{(N_1)}}(\lambda^{(EPN)}) \oplus \widetilde{H^{(N_2)}}(\lambda^{(EPN)}) \oplus \ldots \oplus \widetilde{H^{(N_K)}}(\lambda^{(EPN)})$$

$$\widetilde{H^{(N_j)}}(\lambda^{(EPN)}) = c_j\, H^{(N_j)}_{(TAO)}(\lambda^{(EPN)})\,, \qquad j = 1, 2, \ldots, K$$

$$H^{(\mathrm{N})}_{(GAO)}(\lambda) = \widetilde{H^{(N_1)}}(\lambda) \oplus \widetilde{H^{(N_2)}}(\lambda) \oplus \ldots \oplus \widetilde{H^{(N_K)}}(\lambda) \;\; + \text{small corrections}$$

weights = λ−independent,

$$\widetilde{H^{(N_j)}}(\lambda) = c_j\, H^{(N_j)}_{(TAO)}(\lambda)\,, \qquad j = 1, 2, \ldots, K\,.$$

equidistant spectrum required in the $\lambda \to 0$ limit:

mimicking the truncated harmonic oscillator:

the sample of the alternative TAO-direct-sum $K > 1$ decompositions:

GAO label -5,-3,-1,1,3,5					
K	$\mathcal{R}(6)$	j	N_j	c_j	TAO$_j$ label
2	4+2	1	4	1	-3,-1,1,3
		2	2	5	-5,5
3	2+2+2	1	2	1	-1,1
		2	2	3	-3,3
		3	2	5	-5,5

systematics of the TAO components

$$\boxed{(1 - N_j)\, c_j, (3 - N_j)\, c_j, \ldots, (N_j - 3)\, c_j, (N_j - 1)\, c_j}\,.$$

$N = 2$: no anomalous degeneracy

$H^{(2)}_{(\mathrm{TAO})}(\lambda^{(EPN)})$, boxed symbol $\boxed{\text{-1,1}}$

the number $a(N)$ of eligible scenarios is one, $a(2) = 1$

the geometric multiplicity of the spectrum is $K = 1$.

$N = 3$: no anomalous degeneracy, either

$a(3) = 1$ and $K = 1$, symbol $\boxed{\text{-2,0,2}}$

the simplest $K = 2$ anomaly: $N = 4$, $a(4) = 2$

.

$$\boxed{\text{-3,-1,1,3}} = \boxed{\text{-1,1}} \oplus \boxed{\text{-3,3}}\,, \qquad K = 2\,.$$

$$H^{(4)}_{(K=2)}(\lambda^{(EP4)}) = \left[\begin{array}{cccc} -3 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -3 & 0 & 0 & 3 \end{array}\right] .$$

$N=5$, $a(5)=3$, two $K=2$ options: nine- and pentadiagonal

.

(a) $\boxed{\text{-4,-2,0,2,4}} = \boxed{\text{-2,0,2}} \oplus \boxed{\text{-4,4}}$ and (b) $\boxed{\text{-4,-2,0,2,4}} = \boxed{\text{-4,0,4}} \oplus \boxed{\text{-2,2}}$

$$H^{(5)}_{(K=2,a)}(\lambda^{(EP5)}) = \left[\begin{array}{ccccc} -4 & 0 & 0 & 0 & 4 \\ 0 & -2 & \sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} & 2 & 0 \\ -4 & 0 & 0 & 0 & 4 \end{array}\right]$$

$$H^{(5)}_{(K=2,b)}(\lambda^{(EP5)}) = \left[\begin{array}{ccccc} -4 & 0 & 2\sqrt{2} & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 & 2\sqrt{2} \\ 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & -2\sqrt{2} & 0 & 4 \end{array}\right].$$

$N = 6$, $a(6) = 3$ **and the first occurrence of** $K = 3$

.

$\boxed{\text{-5,-3,-1,1,3,5}} = \boxed{\text{-3,-1,1,3}} \oplus \boxed{\text{-5,5}}$
and $\boxed{\text{-5,-3,-1,1,3,5}} = \boxed{\text{-1,1}} \oplus \boxed{\text{-3,3}} \oplus \boxed{\text{-5,5}}$:

$$H^{(6)}_{(K=3)}(\lambda^{(EP6)}) = \begin{bmatrix} -5 & 0 & 0 & 0 & 0 & 5 \\ 0 & -3 & 0 & 0 & 3 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 3 & 0 \\ -5 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

$N = 7$**:** $a(7) = 6$

.

$$\boxed{\text{-6,-4,-2,0,2,4,6}} = \boxed{\text{-4,-2,0,2,4}} \oplus \boxed{\text{-6,6}}\,, \qquad K = 2\,,$$
$$\boxed{\text{-6,-4,-2,0,2,4,6}} = \boxed{\text{-2,0,2}} \oplus \boxed{\text{-4,4}} \oplus \boxed{\text{-6,6}}\,, \qquad K = 3\,,$$
$$\boxed{\text{-6,-4,-2,0,2,4,6}} = \boxed{\text{-4,0,4}} \oplus \boxed{\text{-2,2}} \oplus \boxed{\text{-6,6}}\,, \qquad K = 3\,,$$
$$\boxed{\text{-6,-4,-2,0,2,4,6}} = \boxed{\text{-4,0,4}} \oplus \boxed{\text{-6,-2,2,6}}\,, \qquad K = 2\,,$$
$$\boxed{\text{-6,-4,-2,0,2,4,6}} = \boxed{\text{-6,0,6}} \oplus \boxed{\text{-2,2}} \oplus \boxed{\text{-4,4}}\,, \qquad K = 3\,.$$

decrease of $a(N)$ at $N = 8$

$a(8) = 4$

$$\boxed{\text{-7,-5,-3,-1,1,3,5,7}} = \boxed{\text{-5,-3,-1,1,3,5}} \oplus \boxed{\text{-7,7}}\,, \qquad K = 2\,,$$

$$\boxed{\text{-7,-5,-3,-1,1,3,5,7}} = \boxed{\text{-3,-1,1,3}} \oplus \boxed{\text{-5,5}} \oplus \boxed{\text{-7,7}}\,, \qquad K = 3\,,$$

$$\boxed{\text{-7,-5,-3,-1,1,3,5,7}} = \boxed{\text{-1,1}} \oplus \boxed{\text{-3,3}} \oplus \boxed{\text{-5,5}}\,, \oplus \boxed{\text{-7,7}}\,, \qquad K = 4\,.$$

Discussion

- last example is typical: fifteen-diagonal but very sparse matrix $H^{(8)}_{(K=4)}(\lambda^{(EP8)})$ with bi-diagonal structure.
- conjecture: a triviality of the geometric multiplicity $K = 1$ correlated with (explained by) the tridiagonality of Hamiltonian matrices
- summary: we constructed the GAO $K > 1$ benchmark models via the TAO-direct-sum ansatzs.

thanks for your attention