A family of rationally extended real and PT symmetric complex potentials

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Outline

Introduction

- Discovery of two new orthogonal polynomials (namely the exceptional Laguerre and the Jacobi polynomials) lead to the extension of known solvable potentials (both real and PT symmetric) in the form of

$$V_{ext}(x) = V_{con}(x) + V_{rat}(x), \tag{1}$$

where $V_{con}(x)$ is a known conventional potential and $V_{rat}(x)$ is the corresponding rational term i.e.

$$V_{rat}(x) = \frac{p(x)}{q(x)}. \tag{2}$$

- The energy $E_{ext,n} = E_{con,n}$ (iso-spectral) but the wave functions are completely different and obtained in the form of exceptional polynomials.

Exceptional Orthogonal Polynomials

- Similar to the usual polynomials, these polynomials[1] also satisfy a second order differential equation of the form

$$y''(x) + P_m(x)y'(x) + Q_{n,m}(x)y = 0; \quad n, m = 0, 1, 2, ... \quad (3)$$

  with m-dependent function $P_m(x)$ and m, n dependent $Q_{n,m}(x)$.
- For $m = 0$, the function $P_0(x) \to P(x)$ and $Q_{n,0}(x) \to Q_n(x)$. The corresponding solutions will be usual orthogonal polynomials.
- For $m \geq 1$, the solutions are known as X_1, $X_2 ....X_m$ exceptional orthogonal polynomials (EOPs).

[1] D. Gomez-Ullate, N. Kamran and R. Milson, *J. Math. Anal.Appl.* **359** (2009) 352; *J. Phys. A* **43** (2010) 434016; *J. Phys. A* **43** (2010) 434016; *Contemporary Mathematics* **563** 51 2012.

Exceptional Laguerre Polynomials

For this case:

$$P_m(x) = \frac{1}{x}\left((\alpha+1-x) - 2x\frac{L_{m-1}^{(\alpha)}(-x)}{L_m^{(\alpha-1)}(-x)}\right) \tag{4}$$

and

$$Q_{n,m}(x) = \frac{1}{x}\left(n - 2\alpha\frac{L_{m-1}^{(\alpha)}(-x)}{L_m^{(\alpha-1)}(-x)}\right) \tag{5}$$

Exceptional Laguerre Polynomials

Solutions:

$$\begin{aligned} y &\rightarrow \hat{L}_{n,m}^{(\alpha)}(x) \\ &= L_m^{(\alpha)}(-x)L_{n-m}^{(\alpha-1)}(x) + L_m^{(\alpha-1)}(-x)L_{n-m-1}^{(\alpha)}(x) \end{aligned} \quad (6)$$

where $L_n^{(\alpha)}(x) \rightarrow$ usual Laguerre polynomial.

Exceptional Laguerre Polynomials

Weight factor and norm:

$$\hat{W}_m^{(\alpha)}(x) = \frac{x^\alpha e^{-x}}{(L_m^{(\alpha-1)}(-x))^2} \tag{7}$$

$$\int_0^\infty \left(\hat{L}_{n,m}^{(\alpha)}(x)\right)^2 \hat{W}_m^{(\alpha)}(x)dx = \frac{(\alpha+n)\Gamma(\alpha+n-m)}{(n-m)!}, \tag{8}$$

Exceptional Jacobi Polynomials

- Here

$$P_m(x) = \left((\alpha-\beta-m+1)\frac{P_{m-1}^{(-\alpha,\beta)}(x)}{P_m^{(-\alpha-1,\beta-1)}(x)} - \left(\frac{\alpha+1}{1-x}\right) + \left(\frac{\beta+1}{1+x}\right)\right) \quad (9)$$

and

$$Q_{n,m}(x) = \frac{1}{(1-x^2)}\left(\beta(\alpha-\beta-m+1)\frac{(1-x)P_{m-1}^{(-\alpha,\beta)}(x)}{P_m^{(-\alpha-1,\beta-1)}(x)} + m(\alpha-\beta-m+1)+(n-m)(\alpha+\beta+n-m+1)\right) \quad (10)$$

Exceptional Jacobi Polynomials

Solutions:

$$\begin{aligned} y &\rightarrow \hat{P}_{n,m}^{(\alpha,\beta)}(x) \\ &= (-1)^m\left[\frac{(1+\alpha+\beta+j)}{2(1+\alpha+j)}(x-1)P_m^{(-\alpha-1,\beta-1)}(x)\right. \\ &\times P_{j-1}^{(\alpha+2,\beta)}(x)+\frac{(1+\alpha-m)}{(\alpha+1+j)}P_m^{(-2-\alpha,\beta)}(x) \\ &\times \left. P_j^{(\alpha+1,\beta-1)}(x)\right]; \quad j=n-m\geq 0. \end{aligned} \tag{11}$$

where $P_n^{(\alpha,\beta)}(x) \rightarrow$ usual Jacobi polynomial.

Exceptional Jacobi Polynomials

Weight factor and Norm:

$$\hat{W}_m^{\alpha,\beta} = \frac{(1-x)^\alpha (1+x)^\beta}{[P_m^{(-\alpha-1,\beta-1)}(x)]^2} \tag{12}$$

$$\int_{-1}^{1} [\hat{P}_{n,m}^{(\alpha,\beta)}(x)]^2 \hat{W}_m^{\alpha,\beta} dx = C_{n,m}^{(\alpha,\beta)}, \tag{13}$$

with

$$C_{n,m}^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}(1+\alpha+n-2m)(\beta+n)\Gamma(\alpha+2+n-m)\Gamma(\beta+n-m)}{(n-m)!(\alpha+1+n-m)^2(\alpha+\beta+2n-2m+1)\Gamma(\alpha+\beta+n-m+1)}$$

Rationally extended potentials

- These potentials are obtained by different approaches such as SUSY or Pre-potential approach, Darböux Backlünd transformation (DBT), Group theoretic or potential algebra approach, Point canonical transformation (PCT) approach etc.
- Rational extension of most of the SIP potentials are available.
- Solutions of some of the extended potentials are not in the exact form of EOPs (e.g. Morse, Rosen-Morse, Coulomb etc).
- **Extension of some of the real potentials are not possible (e.g Real Scarf-II), but PT symmetric extensions are possible.**

Extended Real potentials

- **Radial oscillator[2]: The X_1 case**

$$\begin{aligned} V_{ext}(r) &= V_{con}(r) + \frac{p(r)}{q(r)} \\ &= \frac{1}{4}\omega^2 r^2 + \frac{l(l+1)}{r^2} + \frac{4\omega}{\omega r^2 + 2l + 1} - \frac{8\omega(2l+1)}{(\omega r^2 + 2l + 1)^2} \end{aligned} \tag{14}$$

- The wavefunction

$$\psi_{n,ext}(r) \propto \frac{r^{\ell+1} \exp\left(-\frac{\omega r^2}{4}\right)}{L_1^{(\ell-\frac{1}{2})}(-\frac{\omega r^2}{2})} \hat{L}_{n+1}^{(\ell+\frac{1}{2})}(\frac{\omega r^2}{2}), \tag{15}$$

[2] C. Quesne, *J.Phys.A* **41** (2008) 392001; B. Bagchi, C. Quesne and R. Roychoudhary, *Pramana J. Phys.* **73**(2009) 337.

Extended Real potentials

- **Radial oscillator: The X_m case**[3]

$$V^m_{ext}(r) = V_{con}(r) - \omega^2 r^2 \frac{L^{(\ell+\frac{3}{2})}_{m-2}(-\frac{\omega r^2}{2})}{L^{(\ell-\frac{1}{2})}_{m}(-\frac{\omega r^2}{2})} + \omega(\omega r^2 + 2\ell - 1)\frac{L^{(\ell+\frac{1}{2})}_{m-1}(-\frac{\omega r^2}{2})}{L^{(\ell-\frac{1}{2})}_{m}(-\frac{\omega r^2}{2})} + 2\omega^2 r^2 \left(\frac{L^{(\ell+\frac{1}{2})}_{m-1}(-\frac{\omega r^2}{2})}{L^{(\ell-\frac{1}{2})}_{m}(-\frac{\omega r^2}{2})}\right)^2 - 2m\omega. \qquad (16)$$

and $$\psi^m_{n,ext}(r) \propto \frac{r^{\ell+1}\exp\left(-\frac{\omega r^2}{4}\right)}{L^{(\ell-\frac{1}{2})}_{m}(-\frac{\omega r^2}{2})}\hat{L}^{(\ell+\frac{1}{2})}_{n+m}(\frac{\omega r^2}{2}), \qquad (17)$$

[3] S. Odake and R. Sasaki, *Phys. Lett. B*, **684**.

Extended Real potentials

- **Scarf-I: The X_m case**

$$V_{con}(x) = [(A-1)A + B^2]\sec^2 x - B(2A-1)\sec x \tan x - A^2,$$
$$-\pi/2 < x < \pi/2; \quad 0 < B < A-1, \quad (18)$$

$$\begin{aligned} V_{m,rat}(x) &= (2B-m-1)[2A-1+(-2B+1)\sin x] \\ &\times \left(\frac{P_{m-1}^{(-\alpha,\beta)}(\sin x)}{P_m^{(-\alpha-1,\beta-1)}(\sin x)}\right) + \frac{(-2B-m+1)^2}{2}\cos^2 x \\ &\times \left(\frac{P_{m-1}^{(-\alpha,\beta)}(\sin x)}{P_m^{(-\alpha-1,\beta-1)}(\sin x)}\right)^2 - 2m(-2B-m-1), \end{aligned}$$

$$\psi_{n,ext}^{m}(x) \propto \frac{(1-\sin x)^{\frac{(A-B)}{2}}(1+\sin x)^{\frac{(A+B)}{2}}}{P_m^{(-\alpha-1,\beta-1)}(\sin x)}\hat{P}_{n+m}^{(\alpha,\beta)}(\sin x) \quad (19)$$

Extended Real potentials

- **Generalized Pöschl-Teller: The X_m case**

$$V_{con}(x) = [(A+1)A + B^2]\text{cosech}^2 x - B(2A+1)\text{cosech}x \coth x + A^2$$
$$0 \leq x \leq \infty, B > A+1 > 1$$

$$\begin{aligned} V_{m,rat}(x) &= -(2B-m+1)[2A+1-(2B+1)\cosh x] \\ &\times \left(\frac{P_{m-1}^{(-\alpha,\beta)}(\cosh x)}{P_m^{(-\alpha-1,\beta-1)}(\cosh x)}\right) + \frac{(2B-m+1)^2 \sinh^2 x}{2} \\ &\times \left(\frac{P_{m-1}^{(-\alpha,\beta)}(\cosh x)}{P_m^{(-\alpha-1,\beta-1)}(\cosh x)}\right)^2 + 2m(-2B-m-1) \end{aligned}$$

$$\psi_{n,ext}^{m}(x) \propto \frac{(\cosh x-1)^{\frac{(B-A)}{2}}(\cosh x+1)^{-\frac{(B+A)}{2}}}{P_m^{(-\alpha-1,\beta-1)}(\cosh x)} \hat{P}_{n+m}^{(\alpha,\beta)}(\cosh x) \quad (20)$$

Extended PT symmetric complex potentials

- PT symmetric Scarf II potential

  The well known complex and PT symmetric conventional Scarf II potential is given by

  $$V_{con}(x) = (-B^2 - A(A+1))\text{sech}^2 x + iB(2A+1)\text{sech} x \tanh x; A > 0$$

  The bound state energy eigenvalues and the eigenfunctions are

  $$E_n = -(A-n)^2; \qquad n = 0, 1, 2....n_{\text{max}} < A, \tag{22}$$

  $$\psi_n(x) \propto (\text{sech} x)^A \exp(-iB \tan^{-1}(\sinh x)) P_n^{(\alpha,\beta)}(i \sinh x), \tag{23}$$

  with $\quad -\infty < x < \infty$.

Extended PT symmetric complex potentials

- *PT* symmetric Scarf II potential: Scattering sates

$$t^{con}_{left}(k) = \frac{\Gamma(-A-ik)\Gamma(1+A-ik)\Gamma(\frac{1}{2}-B-ik)\Gamma(\frac{1}{2}+B-ik)}{\Gamma(-ik)\Gamma(1+ik)\Gamma^2(\frac{1}{2}-ik)}, \tag{24}$$

$$r^{con}_{left}(k) = t^{con}_{left}(k) \times i\left[\frac{\cos\pi A \sin\pi B}{\cosh\pi k} + \frac{\sin\pi A\cos\pi B}{\sinh\pi k}\right] \tag{25}$$

Extended PT symmetric complex potentials

- *PT* symmetric Scarf II potential: Scattering sates

$$t^{usual}_{right}(k) = t^{usual}_{left}(k)$$

$$r^{usual}_{right}(k) = t^{usual}_{right}(k) \times i\left[-\frac{\cos(\pi A)\sin(\pi B)}{\cosh(\pi k)} + \frac{\cos(\pi B)\sin(\pi A)}{\sinh(\pi k)}\right]. \tag{26}$$

Extended PT symmetric complex potentials

- The extended PT symmetric Scarf-II (The X_m case)*(B. Midya and B. Roy, J. Phys. A **46** (2013) 175201.)* is given by

$$
\begin{aligned}
V_{m,ext}(x) &= V_{con}(x) + 2m(2B-m+1) + (2B-m+1) \\
&\times [(-2A-1)+(2B+1)i\sinh x]\frac{P_{m-1}^{(-\alpha,\beta)}(i\sinh x)}{P_m^{(-\alpha-1,\beta-1)}(i\sinh x)} \\
&- \frac{(2B-m+1)^2\cosh^2 x}{2}\left(\frac{P_{m-1}^{(-\alpha,\beta)}(i\sinh x)}{P_m^{(-\alpha-1,\beta-1)}(i\sinh x)}\right)^2 \quad (27)
\end{aligned}
$$

The bound state energy eigenvalues are same but the eigenfunctions are different i.e

$$
\psi_{n,m}(x) \propto \frac{(\mathrm{sech} x)^A \exp\left\{-iB\tan^{-1}(\sinh x)\right\}}{P_m^{(-\alpha-1,\beta-1)}(i\sinh x)}\hat{P}_{n+m}^{(\alpha,\beta)}(i\sinh x). \quad (28)
$$

Extended PT symmetric complex potentials

- The reflection and transmission amplitudes (obviously m dependent) are *(N. Kumari, R. K. Yadav, A. Khare, B. Bagchi, B. P. Mandal, Annals of Physics* ***373*** *(2016) 163.*)

$$t_{left}(k,m) = t_{left}^{usual}(k)\times\left(\frac{[B^2-(ik-\frac{1}{2})^2]+(B-ik+\frac{1}{2})(1-m)}{[B^2-(ik+\frac{1}{2})^2]+(B+ik+\frac{1}{2})(1-m)}\right)$$

$$r_{left}(k,m) = r_{left}^{usual}(k)\times\left(\frac{[B^2-(ik-\frac{1}{2})^2]+(B-ik+\frac{1}{2})(1-m)}{[B^2-(ik+\frac{1}{2})^2]+(B+ik+\frac{1}{2})(1-m)}\right)$$

Similarly, $t_{right}(k,m)$ and $r_{right}(k,m)$ for the X_m case are

$$t_{right}(k,m) = t_{left}(k,m)$$

$$r_{right}(k,m) = r_{right}^{usual}(k)\times\left(\frac{[B^2-(ik-\frac{1}{2})^2]+(B-ik+\frac{1}{2})(1-m)}{[B^2-(ik+\frac{1}{2})^2]+(B+ik+\frac{1}{2})(1-m)}\right) \quad (30)$$

Three-body problems

- Calogero[4] in 1969 solved a problem of three particles interacting pairwise by inverse square potential in addition to the harmonic potential ($\hbar = 2m = 1$)

$$H = -\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + V_C, \tag{31}$$

with

$$V_C = V_H + V_I, \tag{32}$$

$$V_H = \frac{\omega^2}{8} \sum_{i<j} (x_i - x_j)^2; \quad V_I = g \sum_{i<j} (x_i - x_j)^{-2}. \tag{33}$$

[4] F. Calogero, *J. Math. Phys.* **10** (1969) 2191.

Three-body problem

- Solved by defining

$$\rho^2 = \frac{1}{3}[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2], \qquad (34)$$

and using the Jacobi co-ordinates

$$R = \frac{1}{3}(x_1 + x_2 + x_3),$$

with

$$x = \frac{(x_1 - x_2)}{\sqrt{2}}, \qquad y = \frac{(x_1 + x_2 - 2x_3)}{\sqrt{6}}. \qquad (35)$$

- In polar co-ordinates

$$x = \rho \sin\phi, \qquad y = \rho\cos\phi; \qquad 0 \le \rho \le \infty, \quad 0 \le \phi \le 2\pi, \qquad (36)$$

Three-body problem

- We can easily show

$$(x_1 - x_2) = \sqrt{2}\rho \sin\phi,$$
$$(x_2 - x_3) = \sqrt{2}\rho \sin(\phi + 2\pi/3),$$
$$(x_3 - x_1) = \sqrt{2}\rho \sin(\phi + 4\pi/3). \quad (37)$$

- Thus, the Schrödinger equation $H\Psi = E\Psi$ is now solvable by separation of variables as

$$\Psi(\rho, \phi) = R(\rho)\Phi(\phi) \quad (38)$$

with
$R(\rho) \longrightarrow f(\rho)F(g(\rho))$ and $\Phi(\phi) \longrightarrow f(\phi)F(g(\phi))$.

The Calogero-Wolfes type 3-body problems (contd...)

- In 1974 Wolfes[5] showed that a three-body problem

$$V_W(g) = g[(x_1+x_2-2x_3)^{-2}+(x_2+x_3-2x_1)^{-2}+(x_3+x_1-2x_2)^{-2}], \quad (39)$$

  is also solvable when it is added to V_C with or without the inverse square potential V_I.
- Later on, Khare and Bhaduri[6] defined different Wolfes type of interaction terms V_{int} and obtained the exact solutions of a number of three body potentials in one dimension.
- Thus, we have the solution of $V = V_C + V_W + V_{int}$.

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[5] J. Wolfes *J. Math. Phys.* **15** (1974) 1420.
[6] A. Khare and R. K. Bhaduri, *J. Phys. A* **27** (1994) 2213.

The Calogero-Wolfes type 3-body problems (contd...)

- Examples of V_{int}:

$$(a)\quad \frac{3f_1}{2\sqrt{2}\rho}\left[\frac{(x_1-x_2)}{(x_1+x_2-2x_3)^2}+\mathrm{c.p}\right].$$

$$(b)\quad \frac{-f_1}{(\sqrt{6})\rho}\left[\frac{(x_1+x_2-2x_3)}{(x_1-x_2)^2}+c.p\right].$$

$$(c)\quad \frac{\sqrt{3}}{2\rho^2}f_1\left[\frac{(x_1+x_2-2x_3)}{(x_1-x_2)}+\mathrm{c.p}\right];\quad (f_1\rightarrow if_1)$$

$$(d)\quad \frac{-3\sqrt{3}}{2\rho^2}f_1\left[\frac{(x_1-x_2)}{(x_1+x_2-2x_3)}+c.p\right];\quad (f_1\rightarrow if_1)$$

Note: The solutions of $V=V_C+V_W+V_{int}$ are obtained in the form of product of classical Laguerre and Jacobi orthogonal polynomials.

Rationally extended 3-body problems

- We extended the potential of the form[7]

$$V = V_C + V_W(g) + V_{int} + V_{rat}^{(1)}(\mathbf{x}) + V_{rat}^{(2)}(\mathbf{x}), \tag{40}$$

- The exact forms of $V_{rat}^{(1)}(\mathbf{x})$ and $V_{rat}^{(2)}(\mathbf{x})$ are

$$V_{rat}^{(1)}(\mathbf{x}) = \frac{a\sum_{i<j}(x_i - x_j)^2 + c_1}{\left(b\sum_{i<j}(x_i - x_j)^2 + c_2\right)^2}, \quad \text{and} \tag{41}$$

$$V_{rat}^{(2)}(\mathbf{x}) = \frac{\delta}{\rho^2}\left[\frac{k_1}{(k_2 + k_3\xi(\mathbf{x}))} - \frac{k_4}{(k_2 + k_3\xi(\mathbf{x}))^2}\right]. \tag{42}$$

[7] N. Kumari, R.K.Yadav, A. Khare, B.P.Mandal, *Annals of Physics, 385 (2017) 57.*

RE 3-body potentials (contd...)

- Different forms[8] of ξ:

$$
\begin{aligned}
&(1.a) \quad \frac{\sqrt{2}}{\rho^2}(x_1 - x_2)(x_2 - x_3)(x_3 - x_1) \\
&(1.b) \quad -\frac{3}{2\sqrt{2}\rho}\left(\sum_{i<j}(x_i - x_j)^{-1}\right)^{-1} \\
&(2.a) \quad \frac{3}{2\rho^2}\left[(x_1 + x_2 - 2x_3)^{-1} + c.p\right]^{-2} \\
&(2.b) \quad \left(\frac{2}{3\sqrt{6}}\right)^2 \frac{1}{\rho^6}[(x_1 + x_2 - 2x_3)(x_2 + x_3 - 2x_1)(x_1 + x_3 - 2x_3)]^2
\end{aligned}
\tag{43}
$$

[8] N. Kumari, R.K.Yadav, A. Khare, B.P.Mandal, *Annals of Physics, 385 (2017) 57.*

RE 3-body potentials (contd...)

$$
\begin{aligned}
&(3.a) \quad \frac{3}{\sqrt{6}}\frac{1}{\rho}\left[(x_1+x_2-2x_3)^{-1}+c.p\right]^{-1}. \\
&(3.b) \quad -\frac{4}{6\sqrt{6}\rho^3}(x_1+x_2-2x_3)(x_2+x_3-2x_1)(x_1+x_3-2x_2). \\
&(4) \quad \frac{1}{\sqrt{3}}\left[\frac{(x_1-x_2)}{(x_1+x_2-2x_3)}+c.p\right] \qquad (44)
\end{aligned}
$$

RE 3-body complex potential

- Let us consider a complex potential V of the form[9]

$$\begin{aligned} V &= V_H + V_I + V_{int} + V_{rat}^{(1)} + V_{rat}^{(2)}, \\ &= \frac{\omega^2}{8}\sum_{i<j}(x_i - x_j)^2 + g\sum_{i<j}(x_i - x_j)^{-2} \\ &+ \frac{\sqrt{3}}{2\rho^2} i f_1 \left[\frac{(x_1 + x_2 - 2x_3)}{(x_1 - x_2)} + \text{c.p}\right] + V_{rat}^{(1)} + V_{rat}^{(2)}. \end{aligned} \tag{45}$$

- Keeping $V_{rat}^{(1)}$ same but different $V_{rat}^{(2)}$ by defining new form of function ξ i.e.

$$\xi = \frac{1}{3\sqrt{3}}\left(\frac{x_1 + x_2 - 2x_3}{x_1 - x_2} + \text{c.p}\right), \tag{46}$$

[9] N. Kumari, R.K.Yadav, A. Khare, B.P.Mandal, *Annals of Physics, 385 (2017) 57.*

RE 3-body complex potential (contd....)

- We get, the ϕ-dependent potential

$$V_{ext}(\phi) = V_{Con}(\phi) + V_{rat}^{(2)}(\phi), \tag{47}$$

with the equivalent conventional PT symmetric trigonometric Eckart potential

$$V_{Con}(\phi) = \frac{9}{2} g cosec^2(3\phi) + \frac{9}{2} i f_1 \cot(3\phi), \tag{48}$$

and the rational term

$$V_{rat}^{(2)}(\phi) = \delta\left[\frac{k_1}{(k_2 + k_3\cot(3\phi))} + \frac{k_4}{(k_2 + k_3\cot(3\phi))^2}\right]. \tag{49}$$

RE 3-body complex potential (contd....)

- The potential $V(\phi)$ is equivalent to the rationally extended PT symmetric[10] complex trigonometric Eckart potential [11]

$$V(\phi) = A(A-3)cosec^2(3\phi) + 2iB\cot(3\phi) + \frac{9}{A^2(A-3)^2}\left[\frac{-4iB[A^2(A-3)^2 - B^2]}{(iB + A(A-3)\cot(3\phi))} + \frac{2[A^2(A-3)^2 - B^2]^2}{(iB + A(A-3)\cot(3\phi))^2}\right], \quad (50)$$

[10] By P (i.e. parity) we mean here $\phi \to \pi - \phi$ while by T (i.e. time reversal) we mean $t \to -t$ and $i \to -i$.

[11] Which is easily obtained by complex co-ordinate transformation $x \to ix$ of the rationally extended hyperbolic Eckart potential **[C. Quesne,** ***SIGMA*** **8 (2012) 080.]**

RE 3-body complex potential (contd....)

- The associated wavefunction $\Phi_\ell(\phi)$ is given by

$$\Phi_\ell(\phi) \propto \frac{(z-1)^{\frac{\alpha_\ell}{2}}(z+1)^{\frac{\beta_\ell}{2}}}{(iB + A(A-3)\cot(3\phi))} y_\ell^{(A/3,B/3)}(z), \qquad (51)$$

with $z = i\xi = i\cot(3\phi)$.

- with the polynomial function

$$\begin{aligned} y_\ell^{(A/3,B/3)}(z) &= \frac{2(\ell+\alpha_\ell)(\ell+\beta_\ell)}{(2\ell+\alpha_\ell+\beta_\ell)} q_1^{(A/3,B/3)}(z) P_{\ell-1}^{(\alpha_\ell,\beta_\ell)}(z) \\ &- \frac{2(1+\alpha_1)(1+\beta_1)}{(2+\alpha_1+\beta_1)} P_\ell^{(\alpha_\ell,\beta_\ell)}(z), \qquad (52) \end{aligned}$$

- where $q_1^{(A/3,B/3)}(z) = P_{p=1}^{(\alpha_p,\beta_p)}$.

RE 3-body complex potential (contd....)

- The parameters α_ℓ and β_ℓ in terms of A and B are given by

$$\alpha_\ell = -(A/3 - 1 + \ell) + \frac{B/9}{(A/3 - 1 + \ell)};$$
$$\beta_\ell = -(A/3 - 1 + \ell) - \frac{B/9}{(A/3 - 1 + \ell)}. \qquad (53)$$

- The energy eigenvalues are same as given by Eq. (**??**), where λ_l is given by

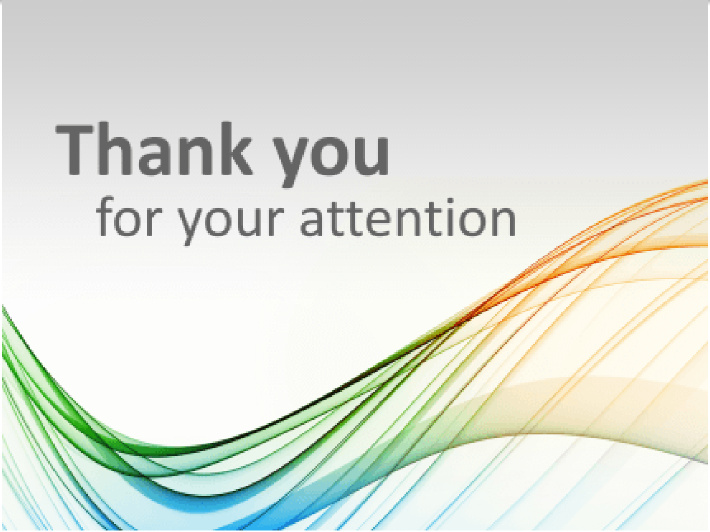
$$\lambda_\ell^2 = 9(\ell + a - \frac{1}{2}) - \frac{9f_1^2}{16(\ell + a - \frac{1}{2})^2}; \quad \ell = 0, 1, 2, ... \qquad (54)$$

Summary

- We have considered the differential equations related with EOPs and discussed the solutions (X_1 and X_m) in brief.
- The X_m case is more general and for $m = 0$, we recover the usual cases.
- Some examples (real and PT symmetric complex) related to theses EOPs are considered and discussed their solutions
- The scattering state solutions for extended PT symmetric complex Scarf-II potentials discussed and shown that the handedness property by this extended potential is also satisfied.

Summary

- New exactly solvable rationally extended Calogero-Wolfes type three body real and complex PT symmetric problems are constructed by adding new types of rational interaction terms.
- The solutions of these extended 3-body problems are obtained as a product of the X_1 Laguerre times X_1 Jacobi EOPs.


Thank you
for your attention