Coherent state structure of pseudo-Hermitian Hamiltonian systems with position-dependent effective mass

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Abstract

The exact solvability of pseudo-Hermitian (ηH) Hamiltonians for position-dependent effective mass systems is revised. With the aid of the supersymmetric quantum mechanics formalism, a class of (ηH) Hamiltonians can be factorized. It turns out that, under a deformed algebra of generalized position and momentum, a coherent state structure of the system exists.

Hamiltonian

- **The possible states of a quantum mechanical system are described by the vectors (ψ_i) of the Hilbert-space $(\mathcal{H})$ corresponding to the system $(\mathcal{S})$.**
- **Prior to a measurement, the system remains on the superposition of all the states.**
- **Measurement processes are described by the action of operators acting on $\mathcal{H}$. The eigenvalues of the operator under process appears as the experimental outcomes with some probability.**
- **The operators associated with the energy measurements are called the Hamiltonian ($\hat{H}$) of the system.**
- **The dynamics are governed by the Schrödinger equation**

$$\hat{H}\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t). \tag{1}$$

Necessary properties of a measuring operator

- **The expectation values should be real numbers.**
- **On the other hand, it is necessary to have a sharp collapse to a state after a measurement process.**
- **Therefore, a spectral decomposition in terms of projectors and real eigen-values are necessary for a meaningful operator.**
- **Probabilities are associated with the norm of the vectors. Therefore, in order to total probability to be conserved under time-evolution, the time evolution operator should be unitary.**
- **An example:** **For a time-independent self-adjoint $\hat{H}$, all the requirements are fulfilled. In particular, the evolution operator ($\hat{\mathcal{U}}(t)$) in this case reads $\hat{\mathcal{U}}(t) = e^{-i\hat{H}t/\hbar}$.**

Role of inner products

- **Adjoint operator $\hat{\bar{\hat{\mathcal{O}}}}$ of an operator $\hat{\mathcal{O}}$ is defined by**

$$(u, \hat{\mathcal{O}} v) = (\bar{\hat{\mathcal{O}}} u, v), \tag{2}$$

where $(.,.)$ is the inner product.

- **Norm ($||w||$) of a vector w can be defined by $\sqrt{(w,w)}$.**
- **The expectation value of an operator $\hat{\mathcal{O}}$ on some state ψ is defined by the Rayleigh quotient**

$$\langle \hat{\mathcal{O}} \rangle_\psi = \frac{(\psi, \hat{\mathcal{O}} \psi)}{(\psi, \psi)}, \tag{3}$$

which is real for self-adjoint $\hat{\mathcal{O}}$ (with respect to the inner product $(.,.)$).

Role of inner products

- **For example, under the usual inner product the operator**

$$\hat{H} = -\frac{d^2}{dx^2} + x^2 - \gamma(\frac{1}{2} + x\frac{d}{dx}),\ \gamma \in \mathbb{R} \setminus \{0\}, \tag{4}$$

has the adjoint

$$\hat{H}^* = -\frac{d^2}{dx^2} + x^2 + \gamma(\frac{1}{2} + x\frac{d}{dx}). \tag{5}$$

Clearly $\hat{H} \neq \hat{H}^*$ **for** $\gamma \neq 0$**.**

- **However, the operator** (4) **is self-adjoint under the inner product**

$$\langle u, v\rangle := (u, \hat{T}v), \tag{6}$$

with

$$\hat{T} = e^{2\gamma x^2}. \tag{7}$$

Pseudo Hermitian operator

- **A linear operator $\hat{H}:\mathcal{H}\rightarrow\mathcal{H}$ is pseudo-hermitian, if an invertible hermitian operator $\hat{\eta}:\mathcal{H}\rightarrow\mathcal{H}$ exists, such that**

$$\hat{H}^{\dagger}=\hat{\eta}\hat{H}\hat{\eta}^{-1}. \tag{8}$$

- **Example: The $\mathcal{PT}$-symmetric Hamiltonians are pseudo-hermitian.**
- ηH **operators are actually hermitians in disguise.**

Position dependent effective mass Hamiltonian (Toy model)

- **We start with the Hamiltonian in $\{|x\rangle\}$ representation as**

$$\hat{H} = -\frac{\hbar^2}{2}\frac{d}{dx}\left(\frac{1}{M(x)}\frac{d}{dx}\right) - \nu\left(\nu_0 + x\frac{d}{dx}\right) + V_0(x), \tag{9}$$

where the parameters $\nu, \nu_0 \in \mathbb{R}$.

- **Under the usual inner product the adjoint of $\hat{H}$ is given by**

$$\hat{H}^\dagger = -\frac{\hbar^2}{2}\frac{d}{dx}\left(\frac{1}{M(x)}\frac{d}{dx}\right) + \nu\left((1-\nu_0) + x\frac{d}{dx}\right) + V_0(x). \tag{10}$$

- **Clearly $\hat{H} \neq \hat{H}^\dagger$, unless $\nu = 0$.**
- **In order to the perturbation not able to completely destroy the original behavior of our system, we restrict that ν is sufficiently small $(|\nu| << 1)$.**

Pseudo metric operator associated with PDEM hamiltonian

- **Let the Hamiltonian $\hat{H}$ tranforms under $\hat{\eta}$ as follows.**

$$\hat{H}_0 = \hat{\eta}\hat{H}\hat{\eta}^{-1}. \tag{11}$$

- **Let us consider the ansatz**

$$\hat{\eta} = e^{\Lambda(x)}. \tag{12}$$

- **Using (12) in (11), we get**

$$\hat{H}_0 = -\frac{\hbar^2}{2}\frac{d}{dx}\left(\frac{1}{M(x)}\frac{d}{dx}\right) - \nu\left(\nu_0 + x\frac{d}{dx}\right) + V_0(x) + \hbar^2\frac{\Lambda'(x)}{M(x)}\frac{d}{dx} + \Lambda_1(x). \tag{13}$$

Pseudo metric operator associated with PDEM hamiltonian

- $$\Lambda_1(x) = \left(\nu x - \frac{\hbar M'}{2M^2}\right)\Lambda' - \frac{\hbar^2}{2M}(\Lambda')^2. \tag{14}$$
- **Since, we wish to construct a choice for Λ so that $\hat{H}_0$ becomes self-adjoint, we choose**
$$\Lambda'(x) = \frac{\nu}{\hbar^2} x M(x). \tag{15}$$
- **That means, we shall use**
$$\Lambda(x) = \lambda_0 + \frac{\nu}{\hbar^2}\int xM(x)dx. \tag{16}$$
The integration constant λ_0 can be set to any value including zero.

Equivalent hermitian Hamiltonian

- **Now the choice of $\Lambda(x)$ in (15), reduces $\hat{H}_0$ as**

$$\hat{H}_1 = -\frac{\hbar^2}{2}\frac{d}{dx}\left(\frac{1}{M(x)}\frac{d}{dx}\right) - \nu\nu_0 + V_1(x). \tag{17}$$

- **Where**

$$V_1(x) = V_0(x) + \frac{\nu^2}{2\hbar^2}M(x)x^2 - \frac{\nu}{2\hbar}\frac{M'(x)}{M(x)}x. \tag{18}$$

- **For constant mass case, $\hat{H}_1$ is an one dimensional (1-D) harmonic oscillator (HO) placed in an external potential $V_0(x)$. Therefore (17) can be regarded as a 1-D HO with position dependent effective mass (PDEM) under the potential $V_0(x)$.**

Eigenstate of the equivalent Hamiltonian

- **The time-independent Schrödinger equation for position dependent effective mass $M(x)$ reads**

$$-\frac{\hbar^2}{2}\frac{d}{dx}\left(\frac{1}{M(x)}\frac{d}{dx}\tilde{\psi}_n(x)\right)+\tilde{V}(x)\tilde{\psi}_n(x)=\tilde{E}_n\tilde{\psi}_n(x). \tag{19}$$

Where

$$\tilde{V}(x)=V_1(x). \tag{20}$$

- **Let the equation (19) is equivalent to the following Schrödinger equation for constant mass**

$$-\frac{\hbar^2}{2}\frac{d^2}{dy^2}\psi_n(y)+V(y)\psi_n(y)=E_n\psi_n(y). \tag{21}$$

We have set the constant mass to unity.

Eigenstate of the equivalent Hamiltonian

- **If the transformation rules from the old PDEM system to the new constant mass system are given by**

$$y = f(x), \tag{22}$$

$$\psi_n(y) = g(x)\tilde{\psi}_n(x), \tag{23}$$

- **The compatibility of** (21) **and** (19) **gives**

$$f'(x) = \sqrt{M(x)}, \tag{24}$$

$$g(x) = m^{-\frac{1}{4}}(x), \tag{25}$$

$$E_n = \tilde{E}_n, \tag{26}$$

$$\tilde{V}(x) = V(f(x)) + \frac{\hbar^2}{8M}\left[\frac{M''}{M} - \frac{7}{4}\left(\frac{M'}{M}\right)^2\right]. \tag{27}$$

- **Let us consider that the constant mass case is a simple harmonic oscillator with potential**

$$V(y) = \frac{1}{2}\beta^2 y^2, \tag{28}$$

where β is a constant.

- **In order to get the exact form of original potential of our system, we have to confine ourselves for a particular choice of PDEM.**
- **Let us consider**

$$M(x) = \frac{m_0}{(1+\beta x)^{\frac{4}{3}}}. \tag{29}$$

- $M(x)$ **can be reduced to the constant mass** (m_0) **case for the limit** $\beta \to 0$**.**

Eigenstate of the equivalent Hamiltonian

- **The choice of mass** (29)**, gives**

$$f(x) = \frac{3}{\beta}\sqrt{m_0}(1+\beta x)^{\frac{1}{3}}. \tag{30}$$

- **From** (27)**, one can write**

$$\tilde{V}(x) = \frac{9}{2}m_0(1+\beta x)^{\frac{2}{3}}. \tag{31}$$

- **Comparing** (27) **with** (18)**, we get**

$$V_0(x) = \frac{9m_0}{2\beta^2}(1+\beta x)^{\frac{2}{3}} - \frac{2\nu\beta x}{3\hbar(1+\beta x)} - \frac{m_0\nu^2 x^2}{2\hbar^2(1+\beta x)^{\frac{4}{3}}}. \tag{32}$$

Eigenstate of the equivalent Hamiltonian

- **The PDEM system is equivalent to the constant mass harmonic oscillator (21).**
- **The normalized eigen-states for constant mass harmonic oscillator (21) are given by**

$$\psi_n(y) = \left(\frac{\beta}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\frac{1}{\sqrt{\hbar}}\sqrt{\beta} y) e^{-\frac{\beta}{2\hbar}y^2}, \tag{33}$$

with the eigenvalues

$$E_n = \left(n + \frac{1}{2}\right)\hbar\beta. \tag{34}$$

Where $H_n(\zeta)$ are the Hermite polynomials of order n.

Eigenstate of the equivalent Hamiltonian

- **from (23) and (26), we can say that the original PDEM system has the eigenfunctions**

$$\tilde{\psi}_n(x) = \frac{1}{g(x)}\psi_n(f(x)), \tag{35}$$

with the eigen-values

$$\tilde{E}_n = \left(n + \frac{1}{2}\right)\hbar\beta. \tag{36}$$

- **In order to construct the coherent states, let us first calculate the ground state $\tilde{\psi}_0(x)$, which reads**

$$\tilde{\psi}_0(x) = \left(\frac{\beta m_0}{\pi\hbar}\right)^{\frac{1}{4}}(1+\beta x)^{-\frac{1}{3}}\exp[-\frac{9m_0}{2\hbar\beta}(1+\beta x)^{\frac{2}{3}}]. \tag{37}$$

Coherent state by SUSY

- **Let us consider the following auxiliary hamiltonian corresponding to our system in order to solve it by SUSY.**

$$\hat{H}_1 = -\frac{\hbar^2}{2}\frac{d}{dx}\left(\frac{1}{M(x)}\frac{d}{dx}\right) + V_1(x). \tag{38}$$

- **Let us relate the problem with a similar hamiltonian with different potential (partner potential)**

$$\hat{H}_2 = -\frac{\hbar^2}{2}\frac{d}{dx}\left(\frac{1}{M(x)}\frac{d}{dx}\right) + V_2(x). \tag{39}$$

- **Now we seek for an intertwining relation**

$$\hat{A}\hat{H}_1 = \hat{H}_2\hat{A}. \tag{40}$$

Coherent state by SUSY

- **Let us take the ansatz**

$$\hat{A} = \frac{1}{\sqrt{2}}\left(a(x)\frac{d}{dx}a(x) + \phi(x)\right). \tag{41}$$

- **The commutation relation between $\hat{A}$ and $\hat{A}^\dagger$ is**

$$[\hat{A}, \hat{A}^\dagger] = \frac{\phi'(x)}{\sqrt{M(x)}}. \tag{42}$$

- **From intertwining relation, we get**

$$a(x) = \frac{1}{(M(x))^{\frac{1}{4}}}, \tag{43}$$

$$V_2(x) = V_1(x) + \frac{1}{\sqrt{M(x)}}\phi'(x), \tag{44}$$

Coherent state by SUSY

- $$K' + M\left(\frac{1}{M}\right)' K - K^2 + (v - 2\lambda M) = 0. \tag{45}$$

- **Where**

$$\phi(x) = Ka^2 - aa', \tag{46}$$

$$v(x) = 2M(x)V_1(x). \tag{47}$$

λ **is an integration constant.**

- $K(x)$ **is related to the ground state** $u = \psi_0$ **by**

$$K(x) = -\frac{u'}{u}. \tag{48}$$

$$\therefore K(x) = \frac{\beta}{3(1+\beta x)} + \frac{3m_0}{\hbar(1+\beta x)^{\frac{1}{3}}}. \tag{49}$$

Coherent state by SUSY

- **The explicit form of $\phi(x)$ is**

$$\phi(x) = \frac{\beta}{3\sqrt{m_0}}\left[\frac{1}{(1+\beta x)^{\frac{1}{3}}} - \frac{1}{(1+\beta x)^{\frac{11}{6}}} + \frac{9m_0}{\hbar\beta}(1+\beta x)^{\frac{1}{3}}\right]. \quad (50)$$

One can now construct the eigen-states ($|\alpha\rangle$) of $\hat{A}$ directly.

$$|\alpha\rangle = \frac{c_1}{\zeta^{\frac{2}{3}}}\exp\left[\frac{3\sqrt{2m_0}}{\beta}\zeta^{\frac{1}{3}} - \frac{9m_0}{2\hbar\beta}\zeta^{\frac{2}{3}} - \frac{2}{9}\zeta^{-\frac{3}{2}}\right]. \quad (51)$$

Where

$$\zeta = 1 + \beta x, \quad (52)$$

and c_1 is constant.

Conclusions

- **We have seen that the coherent state structure can be constructed for position dependent effective mass (PDEM) in the ηH hamiltonian scenario.**
- **First step is to find a metric operator associated with the hamiltonian, in order to make it self-adjoint.**
- **Then the eigen-states of the system is constructed with the help of an equivalent constant mass system.**
- **Annihilation operator can be constructed for the PDEM system with the aid of supersymmetric quantum mechanics.**
- **The eigen-states of the annihilation operator are the desired coherent state.**