Pseudo-unitary transformations and solvable models of vector non-linear Schrodinger equation with balanced loss and gain.

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outline

1. Nonlinear Schrödinger Equation(NLSE)
2. $\mathcal{PT}$ Symmetry inspired generalized NLSE
   - NLSE in $\mathcal{PT}$ symmetric external complex potential
   - Non-local NLSE (Ablowitz & Musslimani; Sinha & Ghosh)
   - ✓NLSE with balanced loss-gain(BLG)
3. Formalism: N-component vector NLSE with BLG
4. Mapping via pseudo-unitary transformation
5. Example: Two-component VNLSE
6. Generalizations
7. Epilogue

$$i\psi_t = -\psi_{xx} + \delta \underbrace{(\psi^* \psi)\,\psi}_{\textit{nonlinearity}}, \quad \delta \in \Re$$

$\psi(x,t)$ is a complex scalar field and $\psi_t \equiv \frac{\partial \psi}{\partial t}$, $\psi_{xx} \equiv \frac{\partial^2 \psi}{\partial x^2}$

- Relevance: Optics, Bose-Einstein Condensation(BEC), plasma physics, gravity waves, α-helix protein dynamics
- Mathematical Structure:
  - Hamiltonian: $\mathcal{H} = \psi_x^* \psi_x + \frac{\delta}{2} (\psi^* \psi)^2$
  - Classical System: Integrable, Solvable, Solitons (Ref.: Zakharov & Shabat, Sov. Phys. JETP 34, 62 (1972))
  - Quantum System: δ-function Bose gas, Integrable, Lieb-Liniger model, Exact correlation functions, Tonks-Girardeu gas

$$i\psi_t = -\psi_{xx} + \underbrace{V(x,t)}_{potential}\psi + \delta\left(\psi^*\psi\right)\psi$$

- Complex potential $V(x,t)$ is $\mathcal{PT}$ symmetric

$$\mathcal{P}: x \to -x, \mathcal{T}: t \to -t, \mathcal{PT}: V(x,t) \to V(x,t)$$

- $\delta = 0$: Quantum bound and scattering states for several V.
- $\delta \neq 0$: Stable Nonlinear modes exist
- Novel features: Self-trapping, unidirectional soliton flow, non-reciprocity, exceptional point, $\mathcal{PT}$-phase transitions, ... (Ref.: Konotop et. al., Rev. Mod. Phys. 88, 035002 (2016))

$$i\psi_t = -\psi_{xx} + \delta \underbrace{\{\psi^*(-x,t)\psi(x,t)\}}_{non-locality} \psi$$

- Stationery state: Self-induced potential is $\mathcal{PT}$-symmetric

$$\mathcal{PT}\{\psi^*(-x,0)\psi(x,0)\} = \{\psi^*(-x,0)\psi(x,0)\}$$

- Both bright & dark solitons for $\delta < 0$
- $\psi(x,t), \psi^*(-x,t)$ are treated as independent fields

$$\{\psi(x,t), \psi^*(-y,t)\} = -i\delta(x-y) \quad (1)$$

$$\mathcal{H} = \psi_x^*(-x,t)\psi_x(x,t) + \frac{\delta}{2}\left(\psi^*(-x,t)\psi(x,t)\right)^2 \quad (2)$$

- $\mathcal{H}$ is complex-valued, while $H = \int dx \mathcal{H}$ is real valued
- Integrable, solutions via inverse-scattering transformation
- Corresponding discrete NLSE as well as multicomponent generalization of NLSE are integrable

NLSE with gain/loss

$$i\psi_t + \underbrace{i\Gamma\psi}_{loss/gain} = -\psi_{xx} + \delta\left(\psi^*\psi\right)\psi$$

- Density $\rho = \psi^*\psi$ and Current $J = -i\left(\psi^*\psi_x - \psi\psi_x^*\right)$
- Continuity equation: $\boxed{\frac{\partial\rho}{\partial t} + J_x = -2\Gamma\rho}$
- For well behaved field ψ and power $N \equiv \int dx\rho$

$$\frac{dN}{dt} = -2\Gamma N \Rightarrow N = N_0 e^{-2\Gamma t}$$

- N decays for $\Gamma > 0$ and grows for $\Gamma < 0$

- Complex scalar fields $\psi_1(x,t), \psi_2(x,t)$ and $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \Gamma > 0$

$$i\psi_{1,t} - \underbrace{i\Gamma\psi_1}_{gain} - \underbrace{\beta^*\psi_2}_{LC} = -\psi_{1,xx} - \underbrace{\delta\left(\Psi^\dagger M_1 \Psi\right)}_{NLC}\psi_1$$

$$i\psi_{2,t} + \underbrace{i\Gamma\psi_2}_{loss} - \underbrace{\beta\psi_1}_{LC} = -\psi_{2,xx} - \underbrace{\delta\left(\Psi^\dagger M_2 \Psi\right)}_{NLC}\psi_2,$$

- $M_i^T = M_i, \forall\, i$; Optics: NLC contains self-phase modulation, cross-phase modulation and four-wave mixing terms
- $\mathcal{P}: \Psi \to \sigma_1 \Psi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi$, $\mathcal{T}: t \to -t,\ i \to -i$
- $\delta = 0$: $\mathcal{PT}$ symmetric
- $\delta \neq 0$: $\mathcal{PT}$ symmetric for $M_2 = \sigma_1 M_1 \sigma_1$
- Features: Solitons, Breathers, Power oscillations, solitionic switch, exceptional points

- Compact notation for Two-component NLSE with BLG for $M_1 = M_2 \equiv M$; $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, $\sigma_{1,2,3}$ = Pauli matrices

$$i\Psi_t - [\underbrace{(\beta\sigma_- + \beta^*\sigma_+)}_{\substack{B \\ (LC)}} + i\underbrace{\Gamma\sigma_3}_{\substack{C \\ (BLG)}}]\Psi = -\Psi_{xx} - \delta \underbrace{\left(\Psi^\dagger M \Psi\right)\Psi}_{NLC}$$

- Consider $N \times N$ matrices $B, C, A := B + iC$ and M : $B^\dagger = B, C^\dagger = C, M^\dagger = M$, $A^\dagger \neq A$, $Tr(C) = 0$
- B, C, M may be expressed in terms of $SU(N)$ generators
- N-component VNLSE with BLG:

$$i\underbrace{(I\partial_t + iA)}_{D_0}\Psi = -\Psi_{xx} - \delta\left(\Psi^\dagger M\Psi\right)\Psi$$

- D_0 resembles temporal component of covariant derivative with complex gauge potential A

Hamiltonian Formalism

- $\mathcal{H}$ is complex-valued $\Rightarrow$ Non-hermitian Quantum $\mathcal{H}$

$$\boxed{\mathcal{H} = \underbrace{\Psi_x^\dagger M \Psi_x - \frac{\delta}{2}\left(\Psi^\dagger M \psi\right)^2}_{real\ \ valued} + \underbrace{\Psi^\dagger M A \Psi}_{complex\ \ valued}}$$

- The effect of BLG is contained in the term $\mathcal{M} \equiv \Psi^\dagger M A \Psi$
  - Non-hermiticity is introduced via mass term for $\mathcal{PT}$ symmetric relativistic Field Theory also
- $\mathcal{M}$ and hence, $\mathcal{H}$ is real valued for M-pseudo-hermitian A
  - With suitable quantization condition hermitian quantum $\mathcal{H}$
- $E = \int dtdx\mathcal{H}$ may not be bounded from below even for $\delta < 0$.
- $E \geq 0$ for M-pseudo-hermitian A and positive-definite M

Brief Review: Pseudo-unitary Matrices

- η-Pseudo-hermitian matrix $\hat{O}$: $\hat{O}^\dagger = \eta \hat{O} \eta^{-1}$
  - $\hat{O}$ admits entirely real eigenvalues for a positive-definite η_+
  - Quasi-hermiticity: $\hat{\mathcal{O}} = \rho \hat{O} \rho^{-1}, \ \rho := \sqrt{\eta_+}$
  - $\hat{\mathcal{O}}$ is hermitian in a vector space endowed with the metric η_+
- Pseudo-unitary matrix:

$$\boxed{U^\dagger \eta U = \eta}$$

  - Representation of U:

$$\boxed{U := e^{i\hat{O}}, \hat{O}^\dagger = \eta \hat{O} \eta^{-1}}$$

  - U is non-unitary as well as non-pseudo-unitary, whenever $\hat{O}$ is non-pseudo-hermitian

- Non-unitary transformation :

$$\boxed{\Psi = U\Phi(t,x), \quad U(t) = e^{-iAt}}$$

- Equation satisfied by Φ

$$\boxed{i\Phi_t = -\Phi_{xx} - \delta\left(\Phi^\dagger G \Phi\right)\Phi, \ G = U^\dagger M U}$$

- Pseudo-unitary transformation:
  - $U^\dagger M U = M \Rightarrow A^\dagger = M A M^{-1}$
  - Solvable Manakov system of VNLSE up to a unitary rotation followed by a scaling

$$\boxed{i\Phi_t = -\Phi_{xx} - \delta\left(\Phi^\dagger M \Phi\right)\Phi} \qquad (3)$$

What if A is NOT M-pseudo-hermitian?

- Time-independent M: NLC of the transformed equation becomes time-dependent
- Time-dependent M: time-dependence of NLC gets modified
- Specific time-dependence of M

$$M(t) = \sum_{j=0}^{N^2-1} \alpha_j U^\dagger(-t) \lambda_j U(-t)$$

$$G = U^\dagger(t) M(t) U(t) = \sum_{j=0}^{N^2-1} \alpha_j \lambda_j, \ \lambda_0 \equiv I$$

- Non-autonomous equation is mapped to solvable Manakov system

$$i\Psi_t - [\underbrace{(\beta^*\sigma_+ + \beta\sigma_-) + i\Gamma\sigma_3}_{A}]\Psi = -\Psi_{xx} - \delta\left(\Psi^\dagger M \Psi\right)\Psi$$

- Positive-definite $M = \alpha_0 I + \alpha\sigma_- + \alpha^*\sigma^+$ for $\alpha_0 > |\alpha|$
- Condition for M-pseudo-hermitian A:

$$\frac{\alpha_0}{|\alpha|} = \frac{|\beta|}{\Gamma}\sin(\theta_\alpha - \theta_\beta) > 1, \quad \alpha = |\alpha| e^{i\theta_\alpha}, \; \beta = |\beta| e^{i\theta_\beta},$$

- The pseudo-unitary operator:

$$U = I\cos(\theta t) - \frac{iA}{\theta}\sin(\theta t) \text{ for } \theta \equiv \sqrt{|\beta|^2 - \Gamma^2} \neq 0$$

- Consistency Conditions:
  - $-|\beta| < \Gamma < |\beta|$: U is periodic in time.
  - $\Gamma > 0 : 0 < \theta_\alpha - \theta_\beta < \pi$, $\Gamma < 0 : \pi < \theta_\alpha - \theta_\beta < 2\pi$

Exact One Soliton Solution for $\delta = 1$

- Power $P \equiv \Psi^\dagger \Psi$ oscillates with time for $\Gamma \neq 0$

$$P = \frac{2\kappa^2 W^\dagger W}{|W^\dagger M W|} \operatorname{sech}^2 \left[\kappa \left(x - vt\right)\right] N(t)$$
$$N(t) = 1 + N_1 \sin^2(\theta t) + N_2 \sin(2\theta t)$$

- W = arbitrary 2-component complex vector, κ, v = constant
- N_1 and N_2 are W, Γ, β dependent constants
- $N_1 = N_2 = 0$ for $\Gamma = 0$: No power oscillation without BLG
- Period of oscillation increases as $\Gamma \rightarrow |\beta|$
- P is unbounded for $\Gamma \geq |\beta|$
- Γ can be used as a controlling parameter for oscillation

- Solvable model of $2m$-component VNLSE with pair-wise BLG:

$$\boxed{A_{2m} = I_m \otimes A, M_{2m} = I_m \otimes M, U_{2m} = I_m \otimes U}$$

Each component interacts with all other components via NLC

- Time-dependent loss-gain term $A(t)$: $U(t) = e^{-i\int^t A(t')dt'}$
- Adding potential V and allowing space-time modulated NLC

$$\boxed{i\left(I\partial_t + iA\right)\Psi = -\Psi_{xx} + V(x,t)\Psi - \delta(x,t)\left(\Psi^\dagger M \Psi\right)\Psi}$$

  - Non-unitary transformation: $\Psi = U\Phi$

$$\boxed{i\partial_t \Phi = -\Phi_{xx} + V(x,t)\Phi - \delta(x,t)\left(\Phi^\dagger G \Phi\right)\Phi}$$

  - Similarity transformation (Juan Belmonte-Beitia et al, Phys.Rev.Lett. 98, 064102 (2007)) to Manakov System

Summary, Discussions & Future Directions

- Summary & Discussions
  - A class of VNLSE with BLG is mapped via pseudo-unitary transformation to the same equation without the BLG term
  - Solvable models of VNLSE with BLG may be constructed via the mapping.
  - Specific examples are constructed with analytic expression for power oscillation
  - Various generalizations are discussed
- Future Directions
  - Solvable models for $N > 2$ with non-pair-wise balancing of BLG terms
  - Similar mapping for VNLSE with BLG and Kerr nonlinearity?
  - Quantization of $\mathcal{H}$
  - Extension: Discrete VNLSE with BLG, oligomers and other lattice models