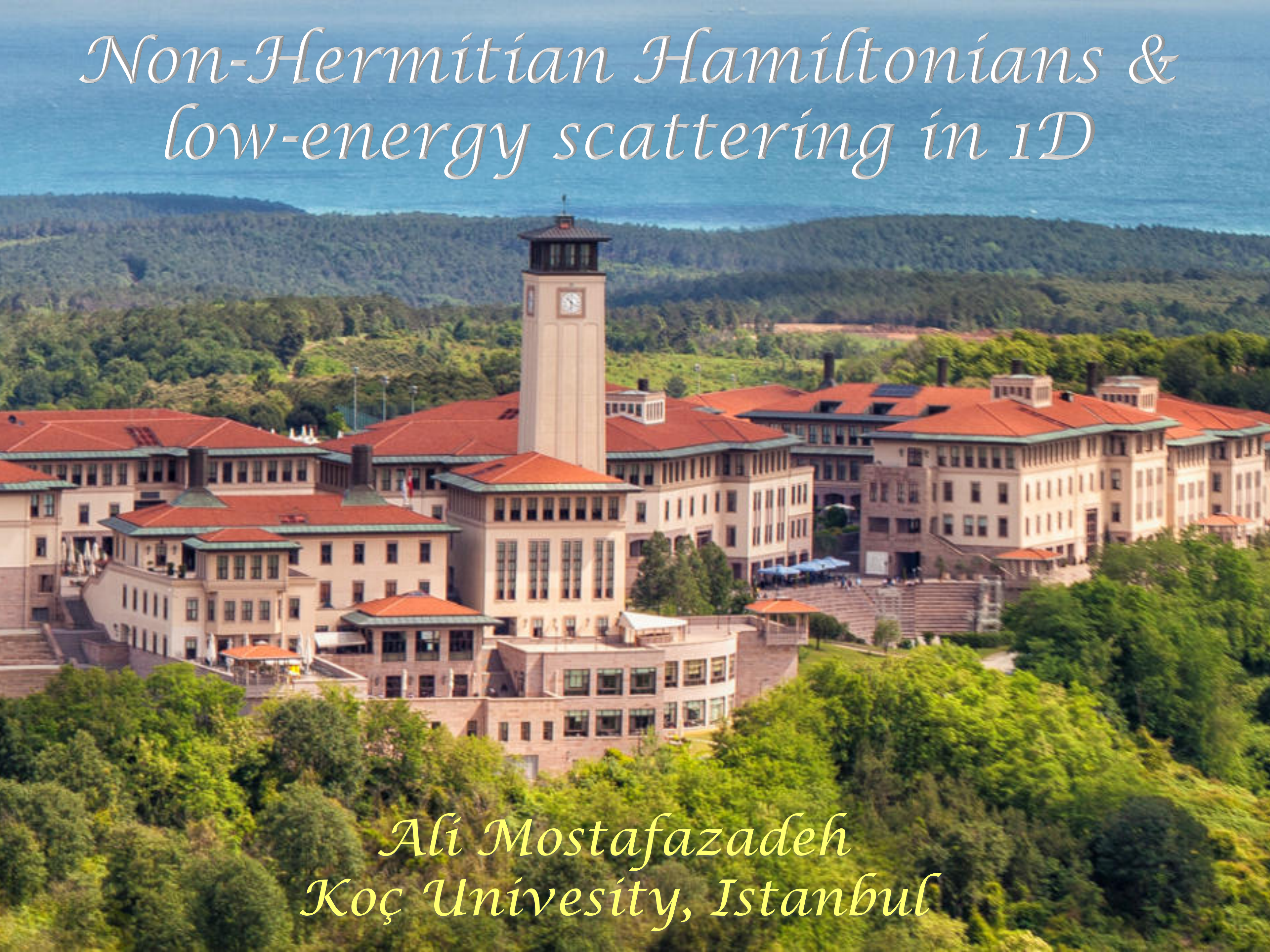

Non-Hermitian Hamiltonians & low-energy scattering in 1D

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Outline:

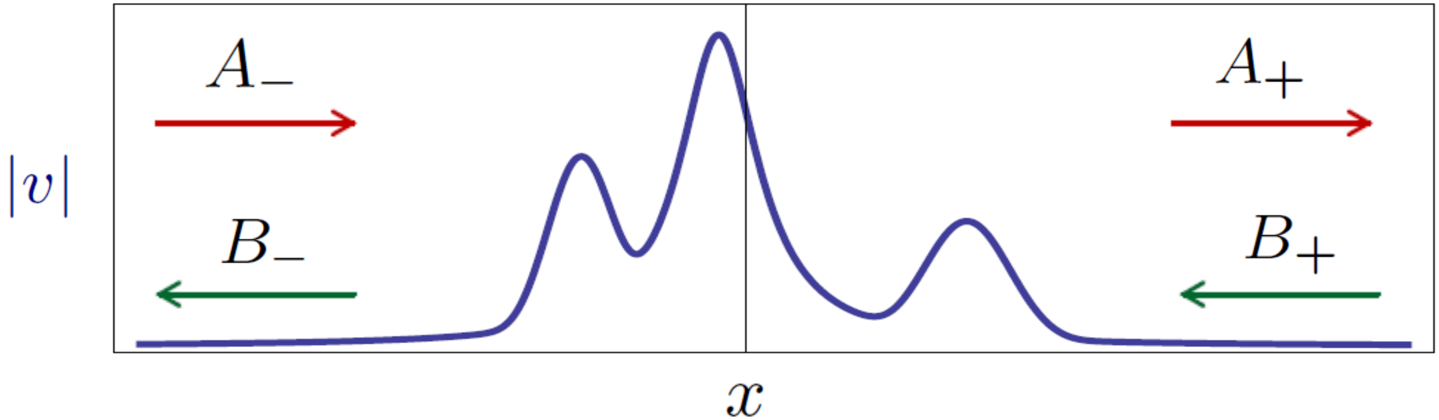
- Stationary scattering in 1D and its **dynamical formulation**
- Low-energy expansion of the transfer matrix
- Transfer matrix for zero-energy Schrödinger equation
- Low-energy scattering in the half-line
- Application: Transmission of scalar wave through a wormhole
- Conclusions

Scattering in 1D $\quad \Psi(x,t) = e^{-i\omega t}\psi(x)$

- Time-Indep. Schrödinger Eq.: $-\psi(x)'' + v(x)\psi(x) = k^2\psi(x)$

$$v \in L^1_1(\mathbb{R}), \qquad L^1_\sigma := \left\{ f : \mathbb{R} \to \mathbb{C} \,\middle|\, \int_{-\infty}^{\infty} dx(1+|x|^\sigma)|v(x)| < \infty \right\}.$$

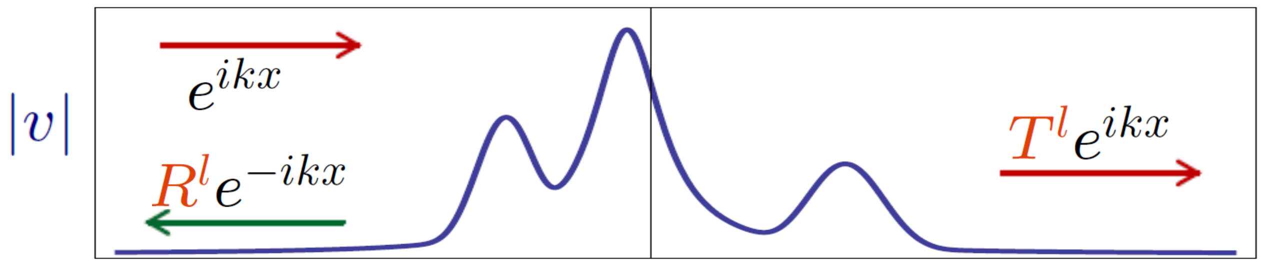
$$\Rightarrow \quad \psi(x) \to \begin{cases} A_- e^{ikx} + B_- e^{-ikx} & \text{for } x \to -\infty \\ A_+ e^{ikx} + B_+ e^{-ikx} & \text{for } x \to +\infty \end{cases}$$



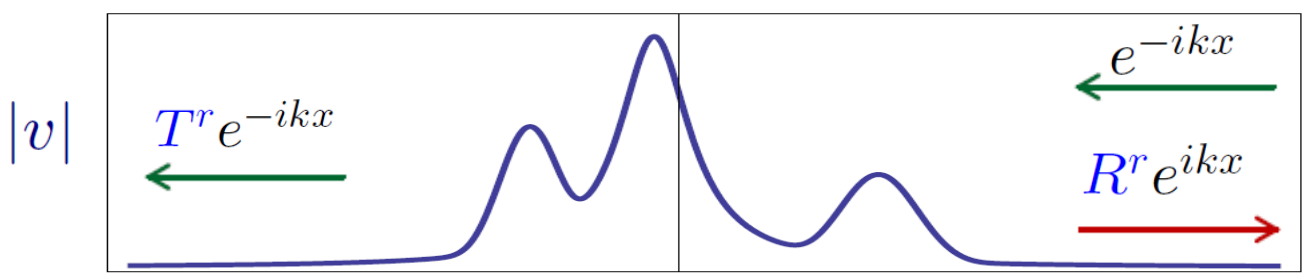
- Transfer matrix: $\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = \mathbf{M} \begin{bmatrix} A_- \\ B_- \end{bmatrix}.$

- Scattering from the left and right:

$$\psi^{\text{left}}(x) = \begin{cases} e^{ikx} + R^l e^{-ikx} & \text{for} \quad x \to -\infty \\ T^l e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$



$$\psi^{\text{right}}(x) = \begin{cases} T^r e^{-ikx} & \text{for} \quad x \to -\infty \\ e^{-ikx} + R^r e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$

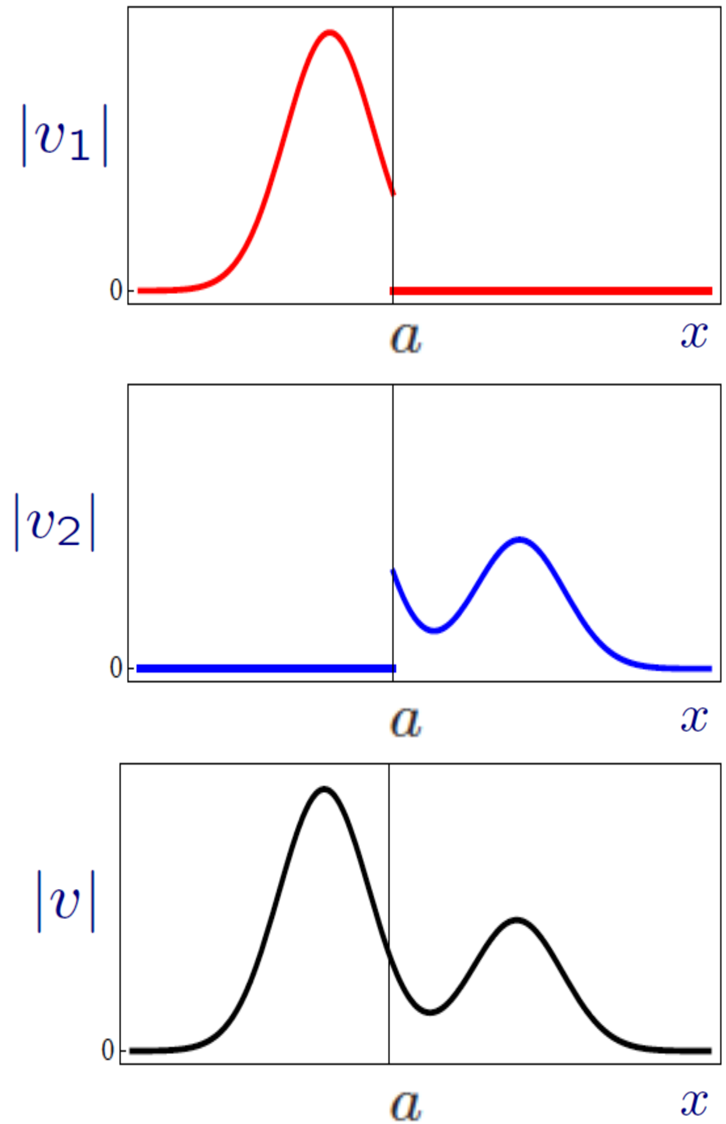


$$R^l = -\frac{M_{21}}{M_{22}}, \qquad R^r = \frac{M_{12}}{M_{22}}, \qquad T^l = T^r =: T = \frac{1}{M_{22}}.$$

Composition Property of $\mathbf{M}$

Let v_1 and v_2 be scattering potentials such that

$$v_1(x) = 0 \quad \text{for} \quad x > a,$$
$$v_2(x) = 0 \quad \text{for} \quad x < a$$
$$v(x) = v_1(x) + v_2(x).$$



$\mathbf{M}_1$: Transfer matrix of v_1

$\mathbf{M}_2$: Transfer matrix of v_2

$\mathbf{M}$: Transfer matrix of $v = v_1 + v_2$

Then $\mathbf{M} = \mathbf{M}_2\,\mathbf{M}_1$.

Composition Property of M

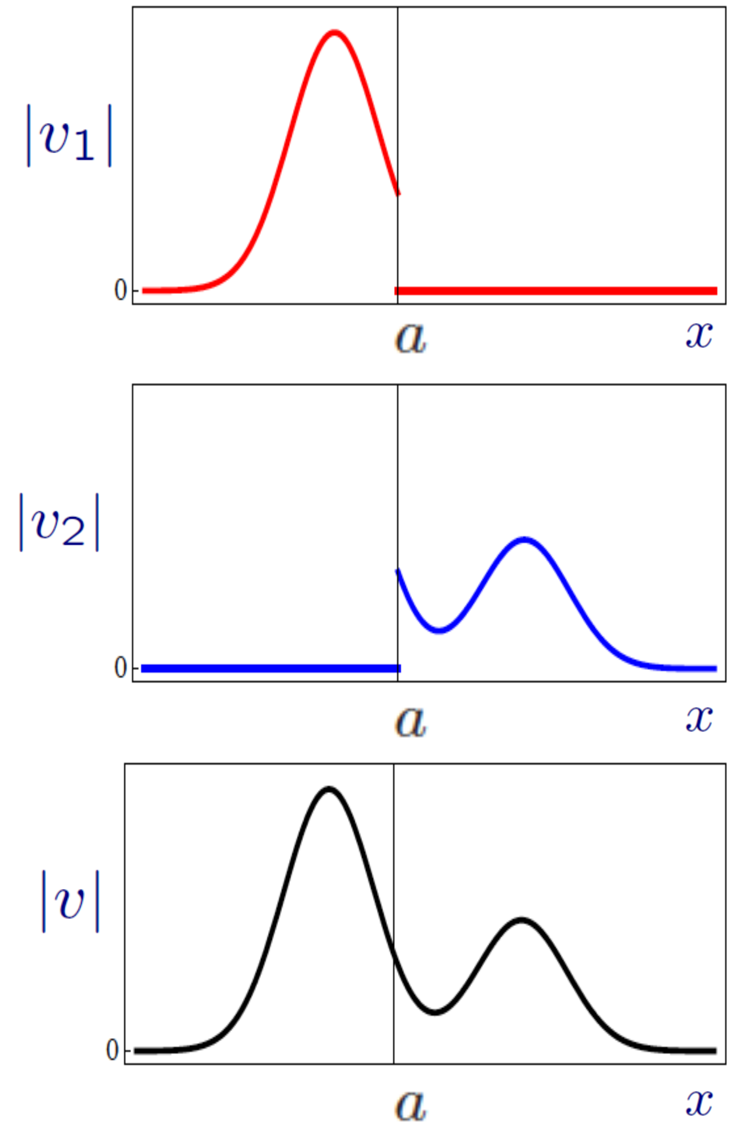
$\mathrm{M} = \mathrm{M}_2\,\mathrm{M}_1$

This is the same as the composition rule for **evolution operators** in QM.

$$\Psi(t) \in \mathscr{H}$$

$$\Psi(t) = \mathbf{U}(t, t_0)\Psi(t_0)$$

$$\mathbf{U}(t, t_0) : \mathscr{H} \rightarrow \mathscr{H}$$



Composition Property of $\mathbf{M}$

$\mathbf{M} = \mathbf{M}_2 \mathbf{M}_1$

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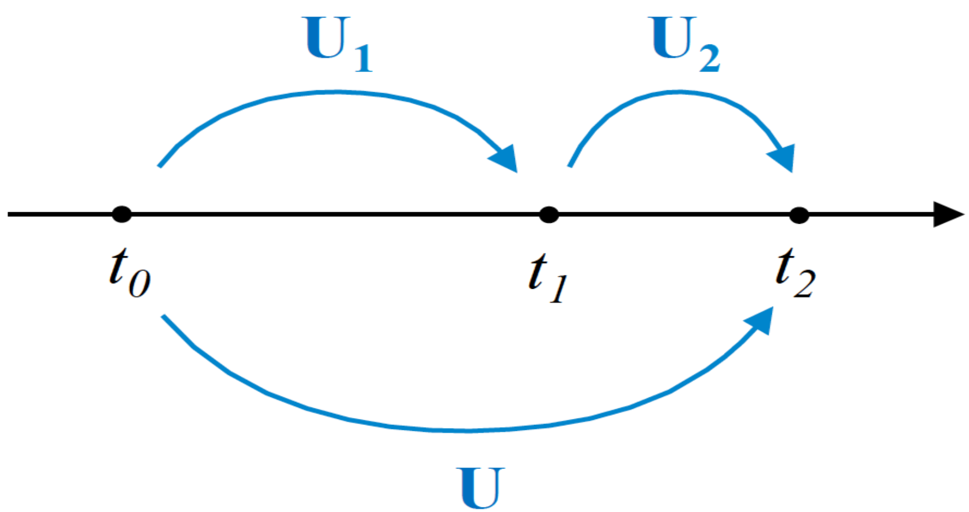
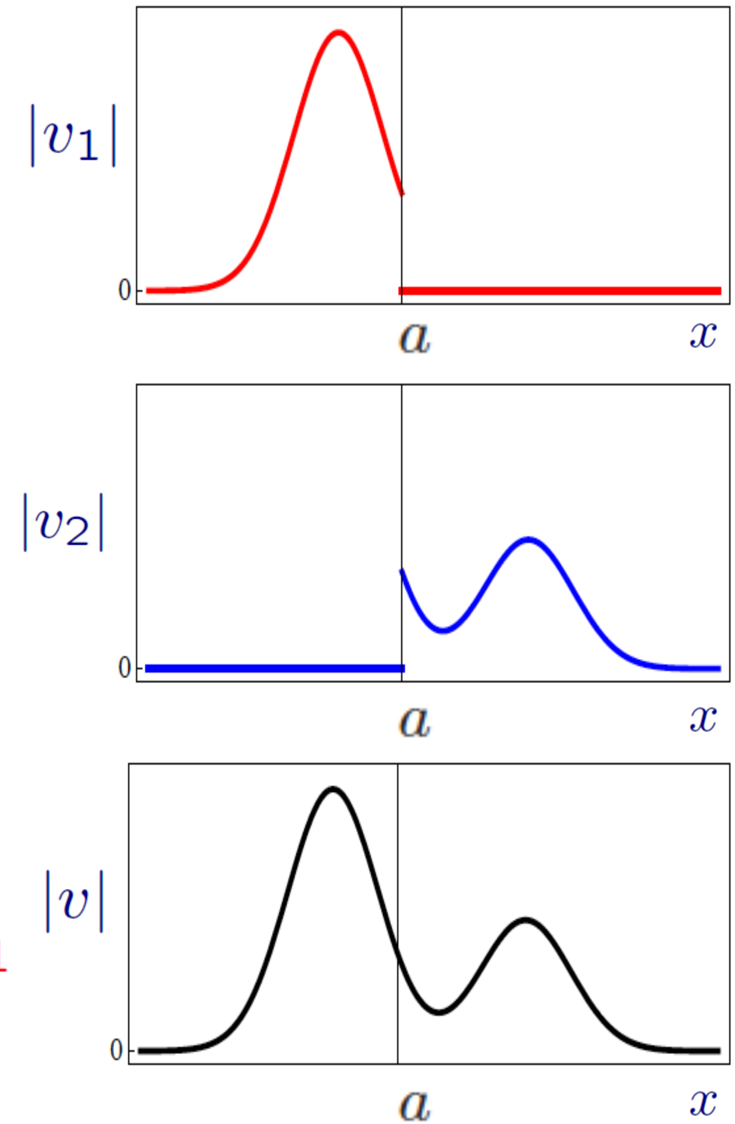


$$\left.\begin{array}{r} \mathbf{U}_1 := \mathbf{U}(t_1, t_0) \\ \mathbf{U}_2 := \mathbf{U}(t_2, t_1) \\ \mathbf{U} := \mathbf{U}(t_2, t_0) \end{array}\right\} \Rightarrow \mathbf{U} = \mathbf{U}_2 \mathbf{U}_1$$



$$\Psi(t) = \mathbf{U}(t, t_0)\Psi(t_0) \qquad \mathbf{U}(t_0, t_0) = \mathbf{I}$$

$$i\partial_t \mathbf{U}(t, t_0) = \mathbf{H}(t)\mathbf{U}(t, t_0)$$

$$\mathbf{H}(t) : \mathscr{H} \to \mathscr{H}$$

$$\mathbf{U}(t, t_0) = \mathbf{I} - i\int_{t_0}^{t} dt_1 \mathbf{H}(t_1)\mathbf{U}(t_1, t_0)$$

$$\begin{aligned} \mathbf{U}(t, t_0) &= \mathbf{I} - i\int_{t_0}^{t} dt_1 \mathbf{H}(t_1) + \cdots + \\ &(-i)^n \int_{t_0}^{t} dt_n \int_{t_0}^{t_n} dt_{n-1} \cdots \int_{t_0}^{t_2} dt_1 \mathbf{H}(t_n)\mathbf{H}(t_{n-1}) \cdots \mathbf{H}(t_1) + \cdots \\ &=: \mathscr{T} \exp \int_{t_0}^{t} -i\mathbf{H}(t)dt \end{aligned}$$

- Let $\mathbf{g}(x)$ be a 2×2 invertible matrix,

$$\mathbf{\Psi}(x) := \mathbf{g}(x) \begin{bmatrix} \psi(x) \\ \psi'(x) \end{bmatrix}, \qquad \mathbf{V}(x) := i \begin{bmatrix} 0 & 1 \\ v(x) - k^2 & 0 \end{bmatrix}.$$

Then $-\psi''(x) + v(x)\psi(x) = k^2\psi(x) \quad \Leftrightarrow \quad i\mathbf{\Psi}'(x) = \mathbf{H}(x)\mathbf{\Psi}(x)$,

$$\mathbf{H}(x) := \mathbf{g}(x)\mathbf{V}(x)\mathbf{g}(x)^{-1} + i\mathbf{g}'(x)\mathbf{g}(x)^{-1}.$$

Choose : $\mathbf{g}(x) := \frac{1}{2k} \begin{bmatrix} ke^{-ikx} & -ie^{-ikx} \\ ke^{ikx} & ie^{ikx} \end{bmatrix}$

$$\Rightarrow \begin{cases} \mathbf{\Psi}(\pm\infty) = \begin{bmatrix} A_\pm \\ B_\pm \end{bmatrix} \quad \Rightarrow \mathbf{\Psi}(+\infty) = \mathbf{M}\,\mathbf{\Psi}(-\infty) \\ \mathbf{H}(x) = \frac{v(x)}{2k} \begin{bmatrix} 1 & e^{-2ikx} \\ -e^{2ikx} & -1 \end{bmatrix}. \end{cases}$$

Theorem: $\mathbf{M} = \mathbf{U}(+\infty, -\infty)$ where $\mathbf{U}(x, x_0)$ is the evolution operator for

$$\mathbf{H}(x) := \frac{v(x)}{2k}\begin{bmatrix} 1 & e^{-2ikx} \\ -e^{2ikx} & -1 \end{bmatrix}.$$

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x plays the role of "time".

[Ann. Phys. (NY), **341**, 77 (2014)]

Theorem: $\mathbf{M} = \mathbf{U}(+\infty, -\infty)$ where $\mathbf{U}(x, x_0)$ is the evolution operator for

$$\mathbf{H}(x) := \frac{v(x)}{2k}\begin{bmatrix} 1 & e^{-2ikx} \\ -e^{2ikx} & -1 \end{bmatrix},$$

i.e.,

$$\begin{aligned}
\mathbf{M} =& \mathscr{T} \exp \int_{-\infty}^{\infty} -i\mathbf{H}(x)dx \\
=& \mathbf{I} - i\int_{-\infty}^{\infty} dx_1 \mathbf{H}(x_1) \\
& + (-i)^2 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{x_2} dx_1 \mathbf{H}(x_2)\mathbf{H}(x_1) + \cdots
\end{aligned}$$

[Ann. Phys. (NY), **341**, 77 (2014)]

$$\mathbf{H}(x) := \frac{v(x)}{2k}\begin{bmatrix} 1 & e^{-2ikx} \\ -e^{2ikx} & -1 \end{bmatrix},$$

If $v(x)$ is a real-valued potential, $\mathbf{H}(x)$ is $\boldsymbol{\sigma}_3$-pseudo-Hermitian;

$$\mathbf{H}(x)^\dagger = \boldsymbol{\sigma}_3\mathbf{H}(x)\boldsymbol{\sigma}_3^{-1}$$

Otherwise, $\mathbf{H}(x)$ is $\boldsymbol{\sigma}_3$-pseudo-normal;

$$[\mathbf{H}(x), \mathbf{H}(x)^\sharp] = \mathbf{0},$$

$$\mathbf{H}(x)^\sharp := \boldsymbol{\sigma}_3^{-1}\mathbf{H}(x)^\dagger\boldsymbol{\sigma}_3.$$

- Make the k-dependence explicit:

$$\mathbf{H}(x;k) := \frac{v(x)}{2k}\begin{bmatrix} 1 & e^{-2ikx} \\ -e^{2ikx} & -1 \end{bmatrix}$$

$$\mathbf{M}(k) = \mathscr{T}\exp\left\{-i\int_{-\infty}^{\infty} dx\,\mathbf{H}(x;k)\right\}.$$

- Consider a finite-range potential v with support $[x_-, x_+]$:

$$v(x) = 0 \quad \text{for} \quad x \notin [x_-, x_+],$$

$$\mathbf{H}(x;k) = \mathbf{0} \quad \text{for} \quad x \notin [x_-, x_+]$$

$$\mathbf{M}(k) = \mathscr{T}\exp\left\{-i\int_{x_-}^{x_+} dx\,\mathbf{H}(x;k)\right\} = \mathbf{U}(x_+, x_-; k).$$

Low-energy scattering:

Find the Small-k behavior of $R^{l/r}(k)$ & $T(k)$.

[Bollé et al, J. Opt. Theory (1985)]

Find the Small-k behavior of $\mathbf{M}(k)$ using:

$$i\partial_x \mathbf{U}(x, x_-; k) = \mathbf{H}(x; k)\mathbf{U}(x, x_-; k),$$

$$\mathbf{U}(x_-, x_-; k) = \mathbf{I}, \qquad \mathbf{U}(x_+, x_-; k) = \mathbf{M}(k).$$

$$\mathbf{H}(x; k) := \frac{v(x)}{2k}\begin{bmatrix} 1 & e^{-2ikx} \\ -e^{2ikx} & -1 \end{bmatrix}$$

Introduce:

$$\mathbf{\Gamma} := \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{\Delta} := \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{K} := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},$$

$$\mathbf{D}(x, x_-; k) := i\mathbf{\Delta}\mathrm{U}(x, x_-; k),$$

$$\mathbf{G}(x, x_-; k) := k\mathbf{\Gamma}\mathrm{U}(x, x_-; k).$$

Then $\frac{1}{2}(\mathcal{K}\,\mathbf{\Gamma} + \mathcal{K}^T\mathbf{\Delta}) = \mathbf{I} \Rightarrow$

$$\begin{aligned} \mathrm{U}(x, x_-; k) &= \frac{1}{2}(\mathcal{K}\,\mathbf{\Gamma} + \mathcal{K}^T\mathbf{\Delta})\mathrm{U}(x, x_-; k) \\ &= \frac{1}{2k}\left[\mathcal{K}\mathbf{G}(x, x_-; k) - ik\mathcal{K}^T\mathbf{D}(x, x_-; k)\right]. \end{aligned}$$

$$\mathbf{U} \;=\; \frac{1}{2k}\Big(\mathcal{K}\mathbf{G} - ik\mathcal{K}^T\mathbf{D}\Big).$$

$$\partial_x \mathbf{U}(x, x_-; k) = \mathbf{H}(x; k)\mathbf{U}(x, x_-; k)$$

$$\mathbf{H}(x; k) := \frac{v(x)}{2k}\begin{bmatrix} 1 & e^{-2ikx} \\ -e^{2ikx} & -1 \end{bmatrix}$$

$$\partial_x \mathbf{D} \;=\; v(x)\big[-x\, s(kx)\, \mathbf{D} + x^2\, c(kx)\, \mathbf{G}\big],$$

$$\partial_x \mathbf{G} \;=\; v(x)\big[x\, s(kx)\, \mathbf{G} - d(kx)\, \mathbf{D}\big],$$

$$s(\tau) := \frac{\sin 2\tau}{2\tau} = 1 + \sum_{n=1}^{\infty} s_n \tau^{2n}, \qquad s_n := \frac{(-4)^n}{(2n+1)!},$$

$$c(\tau) := \frac{1 - \cos 2\tau}{2\tau^2} = 1 + \sum_{n=1}^{\infty} c_n \tau^{2n}, \qquad c_n := \frac{2(-4)^n}{(2n+2)!},$$

$$d(\tau) := \frac{1 + \cos 2\tau}{2} = 1 + \sum_{n=1}^{\infty} d_n \tau^{2n}, \qquad d_n := \frac{(-4)^n}{2[(2n)!]}.$$

$$\mathbf{U} = \frac{1}{2k}\left(\mathcal{K}\mathbf{G} - ik\mathcal{K}^T\mathbf{D}\right).$$

$$\partial_x \mathbf{U}(x, x_-; k) = \mathbf{H}(x; k)\mathbf{U}(x, x_-; k)$$

$$\mathbf{H}(x; k) := \frac{v(x)}{2k}\begin{bmatrix} 1 & e^{-2ikx} \\ -e^{2ikx} & -1 \end{bmatrix}$$

$$\mathrm{U}(x, x_-; k) = \sum_{m=-1}^{\infty} \mathrm{U}_m(x, x_-)\, k^m$$

$$\left.\begin{aligned} \mathbf{D}(x, x_-; k) &:= i\boldsymbol{\Delta}\mathrm{U}(x, x_-; k) \\ \mathbf{G}(x, x_-; k) &:= k\boldsymbol{\Gamma}\mathrm{U}(x, x_-; k) \end{aligned}\right. \Rightarrow \left\{ \begin{aligned} \mathbf{D}(x, x_-; k) &= \sum_{m=-1}^{\infty} \mathbf{D}_m(x)\, k^m \\ \mathbf{G}(x, x_-; k) &= \sum_{m=-1}^{\infty} \mathbf{G}_m(x)\, k^{m+1} \end{aligned} \right.$$

$$\mathbf{U}(x_-, x_-; k) = \mathbf{I} \quad \Rightarrow \left\{ \begin{aligned} \mathbf{D}_m(x_-) &= i\delta_{0m}\,\boldsymbol{\Delta} \\ \mathbf{G}_m(x_-) &= \delta_{0m}\,\boldsymbol{\Gamma} \end{aligned} \right.$$

$$\partial_x \mathbf{D} = v(x)\big[-x\, s(kx)\, \mathbf{D} + x^2\, c(kx)\, \mathbf{G}\big],$$
$$\partial_x \mathbf{G} = v(x)\big[x\, s(kx)\, \mathbf{G} - d(kx)\, \mathbf{D}\big],$$

$$\mathrm{D}(x, x_-; k) = \sum_{m=-1}^{\infty} \mathbf{D}_m(x)\, k^m,$$

$$\mathrm{G}(x, x_-; k) = \sum_{m=-1}^{\infty} \mathbf{G}_m(x)\, k^{m+1}$$

$$s(\tau) := \frac{\sin 2\tau}{2\tau} = 1 + \sum_{n=1}^{\infty} s_n \tau^{2n},$$

$$c(\tau) := \frac{1 - \cos 2\tau}{2\tau^2} = 1 + \sum_{n=1}^{\infty} c_n \tau^{2n},$$

$$d(\tau) := \frac{1 + \cos 2\tau}{2} = 1 + \sum_{n=1}^{\infty} d_n \tau^{2n}$$

$$\mathbf{D}_m(x_-) = i\delta_{0m}\, \mathbf{\Delta}$$

$$\mathbf{G}_m(x_-) = \delta_{0m}\, \mathbf{\Gamma}$$

This gives a system of 1st order ODEs for D_m and G_m that we could decouple and solve iteratively, [arXiv:2102.06084].

[F. Loran & A.M. arXiv:2102.06084]

$$\partial_x \mathbf{D} = v(x)\big[-x\, s(kx)\,\mathbf{D} + x^2\, c(kx)\,\mathbf{G}\big],$$
$$\partial_x \mathbf{G} = v(x)\big[x\, s(kx)\,\mathbf{G} - d(kx)\,\mathbf{D}\big],$$

$$\mathbf{D}(x, x_-; k) = \sum_{m=-1}^{\infty} \mathbf{D}_m(x)\, k^m,$$

$$\mathbf{G}(x, x_-; k) = \sum_{m=-1}^{\infty} \mathbf{G}_m(x)\, k^{m+1}$$

$$s(\tau) := \frac{\sin 2\tau}{2\tau} = 1 + \sum_{n=1}^{\infty} s_n \tau^{2n},$$

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This gives a system of 1st order ODEs for D_m and G_m that we could decouple and solve iteratively, [arXiv:2102.06084].

$$(\mathbf{D}_m, \mathbf{G}_m) \;\to\; (\mathbf{D}, \mathbf{G}) \;\to\; \mathbf{U}(x, x_-; k) = \frac{1}{2k}\Big(\boldsymbol{\mathcal{K}}\mathbf{G} - ik\boldsymbol{\mathcal{K}}^T\mathbf{D}\Big)$$

$$\begin{aligned}\mathbf{U}(x,x_-;k) \;=\;& -\frac{i\phi_1'(x)}{2k}\boldsymbol{\mathcal{K}} + \frac{1}{2}\Big\{\ell\phi_2'(x)(\mathbf{I}-\boldsymbol{\sigma}_1) + \\ & [\phi_1(x)-x\phi_1'(x)](\mathbf{I}+\boldsymbol{\sigma}_1)\Big\} + \\ & \frac{ik\ell}{2}\Big\{g_1(x)\boldsymbol{\mathcal{K}} + [\phi_2(x)-x\phi_2'(x)]\boldsymbol{\mathcal{K}}^T\Big\} + O(k^2)\end{aligned}$$

ℓ: Arbitrary length scale

ϕ_1 & ϕ_2 solve $-\phi''(x)+v(x)\phi(x)=0$ &

$$\phi_1(x_-)-x_-\phi_1'(x_-)=1, \qquad \phi_1'(x_-)=0,$$
$$\phi_2(x_-)-x_-\phi_2'(x_-)=0, \qquad \phi_2'(x_-)=\ell^{-1}.$$

$$\begin{aligned} g_1(x) &:= \ell^{-1}\int_{x_-}^{x} d\tilde{x}\,\partial_x\mathscr{G}(x,\tilde{x})v(\tilde{x})\varsigma(\tilde{x}), \\ \mathscr{G}(x,\tilde{x}) &:= \frac{\ell[\phi_1(x)\phi_2(\tilde{x})-\phi_2(x)\phi_1(\tilde{x})]}{\phi_1(x_-)}, \\ \varsigma(x) &:= -\frac{1}{3}\left\{x^2\left[3\phi_1(x)-x\phi_1'(x)\right] + \int_{x_-}^{x} d\tilde{x}\,\tilde{x}^3 v(\tilde{x})\phi_1(\tilde{x})\right\}.\end{aligned}$$

$$\mathbf{U}(x,x_-;k) = -\frac{i\phi_1'(x)}{2k}\mathcal{K} + \frac{1}{2}\Big\{\ell\phi_2'(x)(\mathbf{I}-\boldsymbol{\sigma}_1)+ [\phi_1(x)-x\phi_1'(x)](\mathbf{I}+\boldsymbol{\sigma}_1)\Big\} + \frac{ik\ell}{2}\Big\{g_1(x)\mathcal{K} + [\phi_2(x)-x\phi_2'(x)]\mathcal{K}^T\Big\} + O(k^2)$$

$$\mathbf{M}(k) = \mathbf{U}(x_+,x_-;k).$$

Generalization to exponentially decaying potentials:

$$\exists\,\mu>0,\ |v(x)| \le e^{-\mu|x|} \quad \text{for} \quad x\to\pm\infty$$

$$x_\pm \to \pm\infty$$

$$-\phi_j''(x) + v(x)\phi_j(x) = 0, \qquad x\in\mathbb{R}, \qquad j\in\{1,2\}$$

$$\lim_{x\to-\infty}[\phi_1(x)-x\phi_1'(x)] = 1, \qquad \lim_{x\to-\infty}\phi_1'(x) = 0,$$

$$\lim_{x\to-\infty}[\phi_2(x)-x\phi_2'(x)] = 0, \qquad \lim_{x\to-\infty}\phi_2'(x) = \ell^{-1}.$$

$$\mathbf{M}(k) = -\frac{i\mathfrak{b}_1}{2k\ell}\boldsymbol{\mathcal{K}} + \frac{1}{2}\left[\mathfrak{b}_2(\mathbf{I}-\boldsymbol{\sigma}_1) + \mathfrak{a}_1(\mathbf{I}+\boldsymbol{\sigma}_1)\right] + \frac{ik\ell}{2}\left(\mathfrak{g}_1\boldsymbol{\mathcal{K}} + \mathfrak{a}_2\boldsymbol{\mathcal{K}}^T\right) + O(k^2).$$

$$\mathfrak{a}_j = \lim_{x\to+\infty}[\phi_j(x) - x\phi_j'(x)], \qquad \mathfrak{b}_j = \ell \lim_{x\to+\infty}\phi_j'(x),$$

$$\mathfrak{g}_1 = \ell^{-1}\int_{-\infty}^{\infty} d\tilde{x}\, \partial_x \mathscr{G}(x,\tilde{x})v(\tilde{x})\varsigma(\tilde{x}),$$

$$\varsigma(x) = -\frac{1}{3}\left\{x^2\left[3\phi_1(x) - x\phi_1'(x)\right] + \int_{-\infty}^{x} d\tilde{x}\,\tilde{x}^3 v(\tilde{x})\phi_1(\tilde{x})\right\},$$

$$\mathscr{G}(x,\tilde{x}) = \frac{\ell[\phi_1(x)\phi_2(\tilde{x}) - \phi_2(x)\phi_1(\tilde{x})]}{\lim_{x_-\to-\infty}\phi_1(x_-)}.$$

$$R^l = -\frac{M_{21}}{M_{22}}, \qquad R^r = \frac{M_{12}}{M_{22}}, \qquad T = \frac{1}{M_{22}}.$$

[F. Loran & A.M. arXiv:2102.06084]

If $\mathfrak{b}_1 := \ell \lim_{x\to+\infty} \phi_1'(x) \neq 0$,

$$\begin{aligned}
R^l(k) &= -1 - \frac{2i\mathfrak{b}_2 k\ell}{\mathfrak{b}_1} + \frac{2(\mathfrak{b}_2^2+1)(k\ell)^2}{\mathfrak{b}_1^2} + O(k^3),\\
R^r(k) &= -1 - \frac{2i\mathfrak{a}_1 k\ell}{\mathfrak{b}_1} + \frac{2(\mathfrak{a}_1^2+1)(k\ell)^2}{\mathfrak{b}_1^2} + O(k^3),\\
T(k) &= -\frac{2ik\ell}{\mathfrak{b}_1} + \frac{2(\mathfrak{a}_1+\mathfrak{b}_2)(k\ell)^2}{\mathfrak{b}_1^2} + \\
&\quad \frac{2i(\mathfrak{a}_1^2+\mathfrak{b}_2^2+\mathfrak{a}_1\mathfrak{b}_2-\mathfrak{b}_1\mathfrak{g}_1+1)(k\ell)^3}{\mathfrak{b}_1^3} + O(k^4).
\end{aligned}$$

If $\mathfrak{b}_1 = 0$,

$$\begin{aligned}
R^l(k) &= \frac{\mathfrak{b}_2^2-1}{\mathfrak{b}_2^2+1} + \frac{2i\mathfrak{b}_2(\mathfrak{b}_2^2\mathfrak{g}_1-\mathfrak{a}_2)k\ell}{(\mathfrak{b}_2^2+1)^2} + O(k^2),\\
R^r(k) &= -\frac{\mathfrak{b}_2^2-1}{\mathfrak{b}_2^2+1} + \frac{2i\mathfrak{b}_2(\mathfrak{g}_1-\mathfrak{a}_2\mathfrak{b}_2^2)k\ell}{(\mathfrak{b}_2^2+1)^2} + O(k^2),\\
T(k) &= \frac{2\mathfrak{b}_2}{\mathfrak{b}_2^2+1} + \frac{2i\mathfrak{b}_2^2(\mathfrak{a}_2+\mathfrak{g}_1)k\ell}{(\mathfrak{b}_2^2+1)^2} + O(k^2).
\end{aligned}$$

Zero-energy resonance

Generalization to $v \in L^1_\sigma(\mathbb{R})$, i.e., $\int_{-\infty}^{\infty} dx(1+|x|^\sigma)|v(x)| < \infty$

[Newton 1986, Aktosun & Klaus 2001]

- For $\sigma \geq 1$, $\mathbf{M}(k)$ is continuous in $\mathbb{R} \setminus \{0\}$.
- For $\sigma = 2(n+\epsilon)$, $n \in \mathbb{Z}^+$, $0 \leq \epsilon < 1$,

$$\mathbf{M}(k) = \sum_{m=-1}^{n-1} \mathbf{m}_m (k\ell)^m + o(k^{n-1+\epsilon}) \qquad \lim_{k\to 0} \frac{o(k^\mu)}{k^\mu} = 0$$

Our method gives $\mathbf{m}_m$.

$$\mathbf{M}(k) = -\frac{i\mathfrak{b}_1}{2k\ell}\mathcal{K} + \frac{1}{2}\left[\mathfrak{b}_2(\mathbf{I}-\boldsymbol{\sigma}_1) + \mathfrak{a}_1(\mathbf{I}+\boldsymbol{\sigma}_1)\right] + \frac{ik\ell}{2}\left(\mathfrak{g}_1\mathcal{K} + \mathfrak{a}_2\mathcal{K}^T\right) + O(k^2).$$

$$\mathfrak{g}_1 = \ell^{-1}\int^{\infty} d\tilde{x}\, \partial_x \mathscr{G}(x,\tilde{x}) v(\tilde{x}) \varsigma(\tilde{x})$$

$$\varsigma(x) = -\frac{1}{3}\left\{x^2\left[3\phi_1(x) - x\phi_1'(x)\right] + \int_{-\infty}^{x} d\tilde{x}\, \tilde{x}^3 v(\tilde{x})\phi_1(\tilde{x})\right\}$$

Zero-energy Schrödinger equation: $-\phi''(x) + v(x)\phi(x) = 0$,

For $v \in L^1_1(x)$, $\exists\, \mathfrak{a}_\pm, \mathfrak{b}_\pm \in \mathbb{C}$, $x \to \pm\infty$ $\Rightarrow$ $\begin{cases} \phi'(x) \to \mathfrak{b}_\pm/\ell \\ \phi(x) - x\phi'(x) \to \mathfrak{a}_\pm \end{cases}$

Zero-energy transfer matrix:

$$\begin{bmatrix} \mathfrak{a}_+ \\ \mathfrak{b}_+ \end{bmatrix} = \mathbf{M}_0 \begin{bmatrix} \mathfrak{a}_- \\ \mathfrak{b}_- \end{bmatrix}$$

Theorem: $\mathbf{M}_0 = \mathbf{U}_0(+\infty, -\infty)$, where $\mathbf{U}_0(x, x_0)$ is the evolution operator for the Hamiltonian

$$\mathbf{H}_0(x) := -i\, v(x) \begin{bmatrix} x & x^2/\ell \\ -\ell & -x \end{bmatrix}$$

$$\mathbf{M}_0 = \mathscr{T} \exp\left\{-i \int_{-\infty}^{\infty} dx\, \mathbf{H}_0(x)\right\}$$

[F. Loran & A.M. arXiv:2102.06084]

Zero-energy Schrödinger equation: $-\phi''(x) + v(x)\phi(x) = 0,$

For $v \in L^1_1(x)$, $\exists\, \mathfrak{a}_\pm, \mathfrak{b}_\pm \in \mathbb{C}$, $x \to \pm\infty \;\Rightarrow\; \begin{cases} \phi'(x) \to \mathfrak{b}_\pm/\ell \\ \phi(x) - x\phi'(x) \to \mathfrak{a}_\pm \end{cases}$

Zero-energy transfer matrix:

$$\begin{bmatrix} \mathfrak{a}_+ \\ \mathfrak{b}_+ \end{bmatrix} = \mathbf{M}_0 \begin{bmatrix} \mathfrak{a}_- \\ \mathfrak{b}_- \end{bmatrix}$$

Theorem: $\mathbf{M}_0 = \mathbf{U}_0(+\infty, -\infty)$, where $\mathbf{U}_0(x, x_0)$ is the evolution operator for the Hamiltonian

$$\mathbf{H}_0(x) := -i\, v(x) \begin{bmatrix} x & x^2/\ell \\ -\ell & -x \end{bmatrix}$$

- $\mathbf{H}_0(x)$ is $\boldsymbol{\eta}(x)$-pseudo-normal for $\boldsymbol{\eta}(x) := \begin{bmatrix} 1 & 0 \\ 0 & -x^2/\ell^2 \end{bmatrix}$.
- If v is a real potential, $i\mathbf{H}_0(x)$ is $\boldsymbol{\eta}(x)$-pseudo-Hermitian.

[F. Loran & A.M. arXiv:2102.06084]

Zero-energy Schrödinger equation: $-\phi''(x)+v(x)\phi(x)=0,$

For $v\in L^1_1(x)$, $\exists\,\mathfrak{a}_\pm,\mathfrak{b}_\pm\in\mathbb{C}$, $x\to\pm\infty$ $\Rightarrow$ $\begin{cases}\phi'(x)\to\mathfrak{b}_\pm/\ell\\ \phi(x)-x\phi'(x)\to\mathfrak{a}_\pm\end{cases}$

Zero-energy transfer matrix:

$$\begin{bmatrix}\mathfrak{a}_+\\ \mathfrak{b}_+\end{bmatrix}=\mathbf{M}_0\begin{bmatrix}\mathfrak{a}_-\\ \mathfrak{b}_-\end{bmatrix}$$

Theorem: $\mathbf{M}_0=\mathbf{U}_0(+\infty,-\infty)$, where $\mathbf{U}_0(x,x_0)$ is the evolution operator for the Hamiltonian

$$\mathbf{H}_0(x):=-i\,v(x)\begin{bmatrix}x & x^2/\ell\\ -\ell & -x\end{bmatrix}$$

$$\Rightarrow\quad\begin{cases}\phi_1(x)=U_{0\,11}(x,-\infty)+\ell^{-1}x\,U_{0\,21}(x,-\infty),\\ \phi_2(x)=U_{0\,12}(x,-\infty)+\ell^{-1}x\,U_{0\,22}(x,-\infty).\end{cases}$$

Corollary: v has a zero-energy resonance iff $M_{0\,21}=0$.

[F. Loran & A.M. arXiv:2102.06084]

Low-energy scattering in the half-line:

$$-\partial_x^2\psi(x;k) + \mathcal{V}(x)\psi(x;k) = k^2\psi(x;k), \qquad x \in \mathbb{R}^+,$$

$$\mathcal{V} \in L_1^1(\mathbb{R}^+)$$

$$\alpha(k)\psi(0;k) + k^{-1}\beta(k)\partial_x\psi(0;k) = 0, \qquad (\text{BC})$$

$$|\alpha(k)| + |\beta(k)| \neq 0.$$

$$\psi(x;k) \to A_+(k)e^{ikx} + B_+(k)e^{-ikx} \quad \text{for} \quad x \to +\infty.$$

Reflection amplitude: $\mathcal{R}(k) := \dfrac{A_+(k)}{B_+(k)}.$

$$-\partial_x^2\psi(x;k)+\mathcal{V}(x)\psi(x;k)=k^2\psi(x;k),\qquad x\in\mathbb{R}^+,$$
$$\alpha(k)\psi(0;k)+k^{-1}\beta(k)\partial_x\psi(0;k)=0,\qquad (\mathrm{BC})$$

$$\psi(x;k)\to A_+(k)e^{ikx}+B_+(k)e^{-ikx}\quad\text{for}\quad x\to+\infty.$$

- Consider the scattering in $\mathbb{R}$ by the potential

$$v(x):=\begin{cases}0 & \text{for}\quad x\leq 0\\ \mathcal{V}(x) & \text{for}\quad x>0\end{cases}$$

$$-\partial_x^2\psi(x;k)+v(x)\psi(x;k)=k^2\psi(x;k)\qquad x\in\mathbb{R}$$
$$\psi(x;k)=A_-(k)e^{ikx}+B_-(k)e^{-ikx}\quad\text{for}\quad x\leq 0.$$

- Choose A_- and B_- so that (BC) holds, & recall $\begin{bmatrix}A_+\\B_+\end{bmatrix}=\mathbf{M}\begin{bmatrix}A_-\\B_-\end{bmatrix}$

$$\Rightarrow\quad \mathcal{R}(k)=\frac{M_{11}(k)-\gamma(k)M_{12}(k)}{M_{21}(k)-\gamma(k)M_{22}(k)},$$
$$\gamma(k):=\frac{\alpha(k)+i\beta(k)}{\alpha(k)-i\beta(k)}.$$

[Ann. Phys. **411**, 167980 (2019); arXiv: 1910.07382]

$$-\partial_x^2\psi(x;k) + \mathcal{V}(x)\psi(x;k) = k^2\psi(x;k), \qquad x \in \mathbb{R}^+,$$

$$\alpha(k)\psi(0;k) + k^{-1}\beta(k)\partial_x\psi(0;k) = 0, \qquad (\mathsf{BC})$$

$$\psi(x;k) \to A_+(k)e^{ikx} + B_+(k)e^{-ikx} \quad \text{for} \quad x \to +\infty.$$

- Consider the scattering in $\mathbb{R}$ by the potential

$$v(x) := \begin{cases} 0 & \text{for} \quad x \le 0 \\ \mathcal{V}(x) & \text{for} \quad x > 0 \end{cases}$$

$$-\partial_x^2\psi(x;k) + v(x)\psi(x;k) = k^2\psi(x;k) \quad x \in \mathbb{R}$$

$$\psi(x;k) = A_-(k)e^{ikx} + B_-(k)e^{-ikx} \quad \text{for} \quad x \le 0.$$

- Choose A_- and B_- so that (BC) holds, & recall $\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = \mathbf{M} \begin{bmatrix} A_- \\ B_- \end{bmatrix}$

$$\Rightarrow \quad \mathcal{R}(k) = \frac{M_{11}(k) - \gamma(k)M_{12}(k)}{M_{21}(k) - \gamma(k)M_{22}(k)},$$

$$\gamma(k) := \frac{\alpha(k) + i\beta(k)}{\alpha(k) - i\beta(k)}.$$

$\Rightarrow$ low-energy expansion of $\mathcal{R}(k)$

For $\beta = 0$ (Dirichlet BC):

$$\mathcal{R}(k) = \begin{cases} 1 + O(k) & \text{for} \quad \mathfrak{b}_2 = 0, \\ -1 - \dfrac{2i\mathfrak{a}_2\, k\ell}{\mathfrak{b}_2} + O(k^2) & \text{for} \quad \mathfrak{b}_2 \neq 0. \end{cases}$$

For $\beta \neq 0$:

$$\mathcal{R}(k) = \begin{cases} -1 - \dfrac{2i\mathfrak{a}_1\, k\ell}{\mathfrak{b}_1} + \dfrac{2(\mathfrak{a}_1^2 + i\rho)(k\ell)^2}{\mathfrak{b}_1^2} + O(k^3) & \text{for} \quad \mathfrak{b}_1 \neq 0, \\ -\dfrac{\rho\, \mathfrak{b}_2^2 - i}{\rho\, \mathfrak{b}_2^2 + i} - \dfrac{2i\mathfrak{b}_2(\rho^2\mathfrak{a}_2\mathfrak{b}_2^2 + \mathfrak{g}_1)k\ell}{(\rho\, \mathfrak{b}_2^2 + i)^2} + O(k^2) & \text{for} \quad \mathfrak{b}_1 = 0. \end{cases}$$

Theorem: $\mathcal{V}$ with Dirichlet BC lead to a zero-energy resonance iff $\mathfrak{b}_2 = M_{0\,22} = 0$.

[F. Loran & A.M. arXiv:2102.06084]

Application: Wormhole scattering

Static spherically symmetric wormhole:

$$ds^2 = -p(r)^2 dt^2 + q(r)^2\, dr^2 + r^2 d\Omega^2,$$

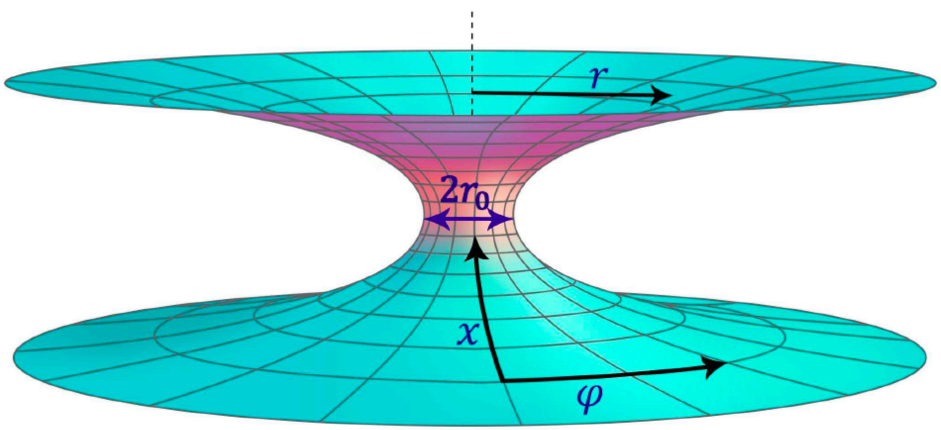
$$d\Omega^2 = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2$$

Introduce $x \in \mathbb{R}$, $r = r(x)$ such that $dx^2 = q(r)^2 dr^2 \Rightarrow$

$$ds^2 = -p(r)^2 dt^2 + dx^2 + r^2 d\Omega^2.$$

$$r(x) \to \pm x \text{ for } x \to \pm\infty.$$

Ellis wormhole: $p(r) = 1$ & $r(x) = \sqrt{x^2 + r_0^2}$ for some $r_0 > 0$.



Φ: A scalar field of mass m

$$(-g)^{-\frac{1}{2}}\partial_\mu\left[(-g)^{\frac{1}{2}}g^{\mu\nu}\partial_\nu\Phi\right]-m^2\Phi=0$$

For $p(r)=1$, $\exists$ time-harmonic solutions:

$$\Phi(t,x,\vartheta,\varphi)=e^{-i\omega t}Y_l^m(\vartheta,\varphi)\frac{\psi(x)}{r(x)},$$

$$-\psi''(x)+v(x)\psi(x)=k^2\psi(x),$$

$$v(x):=\frac{l(l+1)}{r(x)^2}+\frac{r''(x)}{r(x)},$$

$$k:=\sqrt{\omega^2-m^2}$$

Stability of the wormhole $\Rightarrow$ study low-energy waves.

$\Rightarrow$ only spherical waves ($l=0$) can tunnel through v;

$$v(x):=\frac{r''(x)}{r(x)}.$$

Suppose that $\exists \epsilon \in \mathbb{R}^+$, $\exists\, c_\pm, a_n \in \mathbb{R}$,

$$r(x) = \pm x + c_\pm + \frac{1}{x^\epsilon} \sum_{n=0}^{\infty} \frac{a_n}{x^n} \quad \text{as} \quad x \to \pm\infty,$$

$$v(x) = \frac{r''(x)}{r(x)} \;\Rightarrow\; v \in L^1_2(\mathbb{R}) \text{ \& our method gives}$$

$$\mathbf{M}(k) = \mathbf{m}_{-1} k^{-1} + \mathbf{m}_0 + o(k^0)$$

where $\mathbf{m}_{-1}$ and $\mathbf{m}_0$ are given by ϕ_1 of ϕ_2.

$$v(x) = \frac{r''(x)}{r(x)} \Rightarrow r(x) \text{ solves } -\phi''(x) + v(x)\phi(x) = 0.$$

$$R^l(k) \;=\; -1 - 2i\left(c_- - s^{-1}\right) k + o(k^1),$$

$$R^r(k) \;=\; -1 - 2i\left(c_+ - s^{-1}\right) k + o(k^1),$$

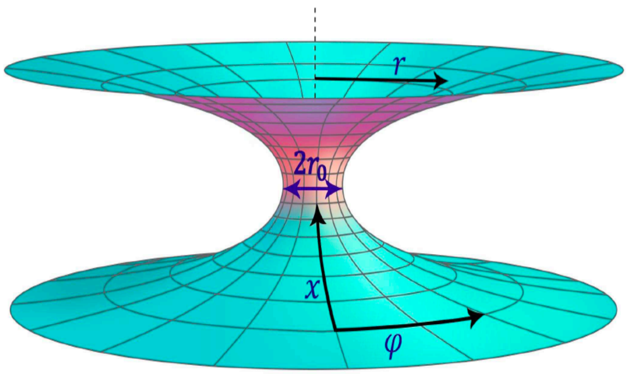
$$T(k) \;=\; -2is^{-1}k + 2s^{-1}(c_+ + c_- - 2s^{-1})k^2 + o(k^2)$$

$$s = \int_{-\infty}^{\infty} \frac{dx}{r(x)^2}$$

$$T(k) = -2is^{-1}k + 2s^{-1}(c_+ + c_- - 2s^{-1})k^2 + o(k^2).$$

$$s = \int_{-\infty}^{\infty} \frac{dx}{r(x)^2}, \qquad c_\pm = \lim_{x\to\pm\infty}[r(x) - |x|],$$

Absorption cross section: $\sigma = \dfrac{\pi |T(k)|^2}{k^2}$



Low-energy absorprtion cross section:

Wormholes with $p = 1$: $\boxed{\sigma = A + o(k)}$

$A := 4\pi\rho^2, \qquad \rho := \dfrac{1}{s}.$

For Schwarzschild blackholes [Unruh 1976]:

Massless scalar field: $\sigma = A + \dfrac{\sqrt{\pi}}{2} A^{3/2} k + O(k^2).$

Massive scalar field: $\sigma = \dfrac{\sqrt{\pi}\, m^3 A^{3/2}}{2k^2}\left[1 + o(k^\infty)\right].$

Conclusions:

- Potential scattering admits a dynamical formulation where the scattering phenomenon is described in terms of an effective quantum system with a non-Hermitian Hamiltonian $\mathbf{H}(x)$. For real scattering potential $\mathbf{H}(x)$ is pseudo-Hermitian.

- The formulation has many interesting applications and admits generalizations to higher dimensions and EM scattering.

- It offers a method for constructing low-energy expansion of the transfer matrix. This requires the study of the zero-energy Schrödinger equation, which may be done using an appropriate zero-energy transfer matrix.

- The results have applications in various areas including the study of the transmission of scalar waves through a wormhole.

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Thank you for your attention.