Dualities in Topology & Algebra

ICTS (1-14th February, 2021)

Dichotomy & Poincaré Duality

ELLIPTIC VS. HYPERBOLIC

All spaces considered are simply connected with finite rational cohomology, i.e., $\sum_j \dim H^j(X;\mathbb{Q}) < \infty$.

The maximum integer n such that $H^n(X;\mathbb{Q}) \neq 0$ is called the dimension of X.
A space X is called rationally elliptic if $\sum_j \dim \pi_j(X) \otimes \mathbb{Q} < \infty$. It is called rationally hyperbolic otherwise.

- Spheres and complex projective spaces are rationally elliptic.
- Lie groups and homogeneous spaces are rationally elliptic.
- If $F \hookrightarrow E \to B$ is a fibration with B, F rationally elliptic, so is E.
- Any closed, simply connected 4-manifold is rationally elliptic if and only if $b_2 = \dim H_2(X;\mathbb{Q}) \leq 2$.

Dichotomy Theorem I

Theorem *Let X be a rationally hyperbolic space of dimension n.*
(1) The sequence $\sum_{j\leq m} \dim \pi_j(X)\otimes\mathbb{Q}$ has exponential growth, i.e., there exists $\lambda > 0$ and $C > 1$ such that for m large enough

$$\sum_{j\leq m} \dim \pi_j(X)\otimes\mathbb{Q} \geq \lambda C^m.$$

(2) There are no large gaps in the sequence $\{\pi_j(X)\otimes\mathbb{Q}\}$, i.e., given any k, there exists $k < l < k+n$ such that $\pi_l(X)\otimes\mathbb{Q} \neq 0$.

- A minimal model for $\mathbb{CP}^2\#\mathbb{CP}^2\#\mathbb{CP}^2$ is given by closed generators x_1, x_2, x_3 in degree 2, y in degree 3 such that

$$dy = x_1^2 + x_2^2 + x_3^2,\ du = x_1x_2,\ dv = x_2x_3,\ dw = x_1x_3.$$

But $x_1v - x_3u$ is closed and has to be killed by a new generator in degree 4.

DICHOTOMY THEOREM II

Theorem *Let X be a rationally elliptic space of dimension n.*
(1) Rational homotopy groups vanish beyond $2n-1$, i.e., $\pi_j(X)\otimes\mathbb{Q}=0$ for $j\geq 2n$.
(2) Rational cohomology $H^\bullet(X;\mathbb{Q})$ satisfies Poincaré duality.
(3) The total dimension $\dim H^\bullet(X;\mathbb{Q})\leq 2^n$.

There are further strong constraints imposed on X. This forces generic simply connected spaces to be rationally hyperbolic.

A commutative graded algebra A (over $\mathbb{Q}$) is a Poincaré duality algebra of dimension n if

(i) each A^j is of finite dimension, with $A^n\cong\mathbb{Q}$, $A^{>n}=0$;
(ii) the multiplication induces a nondegenerate bilinear pairing

$$A^j\otimes A^{n-j}\to A^n \text{ for } 0\leq j\leq n.$$

Poincaré duality models

A Poincaré duality model for a closed, simply connected n-dimensional manifold M is a Poincaré duality algebra (A, d) of dimension n that satisfies the following:

(i) There are quasi-isomorphisms

$$(A, d) \xleftarrow{\simeq} \mathfrak{M}_M \xrightarrow{\simeq} A_{PL}(M).$$

(ii) $A^0 = \mathbb{Q}$ and $A^1 = 0$.

- Spheres: For odd spheres, the minimal model $(\Lambda(x), 0)$ is itself a Poincaré duality model. For even spheres, consider

$$(A, d) = (H^\bullet(S^{2k}; \mathbb{Q}), 0).$$

The map $(\Lambda(x, y), dy = x^2) \to (A, d)$ defined by $x \mapsto [\omega], y \mapsto 0$ is a quasi-isomorphism.

Examples

- Products: Given two manifolds M and N with Poincaré duality models (A_M, d_M) and (A_N, d_N), we consider

$$A_M \otimes A_N \xleftarrow{\simeq} \mathfrak{M}_M \otimes \mathfrak{M}_N \xrightarrow{\simeq} A_{PL}(M) \otimes A_{PL}(N) \xrightarrow{\simeq} A_{PL}(M \times N).$$

Thus, $M \times N$ admits a Poincaré duality model.

- Lie groups: Like the odd spheres, the minimal model serves as the Poincaré duality model, in view of Hopf's result that $H^\bullet(G;\mathbb{Q})$ is free cga on finitely many odd generators.

Notice that $(H^\bullet(M;\mathbb{Q}), 0)$ is a Poincaré duality algebra for any closed simply connected manifold M. In all the examples, we could choose the cohomology ring as the Poincaré duality model of the manifold.

FORMALITY

A simply connected space X is called formal if there exists a quasi-isomorphism

$$\varphi : \mathfrak{M}_X = (\Lambda V, d) \xrightarrow{\simeq} (H^\bullet(X;\mathbb{Q}), 0).$$

A cdga (A, d) is called formal if there exist quasi-isomorphisms

$$(A,d) \xleftarrow{\simeq} (B_1, d_1) \xrightarrow{\simeq} \cdots \xleftarrow{\simeq} (B_k, d_k) \xrightarrow{\simeq} (H^\bullet(A,d), 0).$$

- Spheres, complex projective spaces
- Lie groups
- Products of formal spaces
- Retracts of formal spaces
- Complex Kähler manifolds [Deligne-Griffiths-Morgan-Sullivan].

Existence

Consider a minimal model $(\Lambda V, d)$, where $V = V^{\text{odd}}$ and $\dim V < \infty$. It follows that $(\Lambda V, d)$ is a Poincaré algebra. Halperin conjectured that Poincaré duality algebra models should exist for simply connected manifolds.

Theorem *Every compact simply connected manifold admits a Poincaré duality model.*

The original result of Lambrechts-Stanley is stronger; works over a field of any characteristic.

In the context of Poincaré duality, we define a Poincaré space - a space X of topological dimension n equipped with $[X] \in H_n(X)$ such that

$$[X] \cap - : H^k(X) \to H_{n-k}(X)$$

is an isomorphism.

Spivak fibre

Consider a finite n-dimensional subcomplex of $\mathbb{R}^{n+k}$. Let N be a regular neighbourhood of X with boundary ∂N. The Spivak fibre F_X is defined to be the homotopy fibre of $\partial N \hookrightarrow N$. This is, upto suspension, a homotopy invariant of X.

- Consider the standard embedding $S^2 \hookrightarrow \mathbb{R}^3$, further stabilized inside $\mathbb{R}^{k+2}$. Set N to be a tubular neighbourhood of S^2, i.e., $N \cong S^2 \times D^k$ with $\partial N \cong S^2 \times S^{k-1}$. Then

$$F_{S^2} = \{\gamma : [0,1] \to N \mid \gamma(0) = (p, \mathbf{v}), \gamma(1) = (p_0, 0)\}.$$

The second component of each γ can be deformed to the radial path joining $\mathbf{v}$ to 0. Thus, $F_{S^2} \simeq PS^2 \times S^{k-1} \simeq S^{k-1}$.

Theorem *F_X is a homotopy sphere if and only if X is a Poincaré complex.*

Gorenstein spaces

The reduced homology of F_X can be computed as follows.

Fact *For a simply connected Poincaré space X*

$$\widetilde{H}_\bullet(F_X;\Bbbk) \cong \operatorname{Ext}_{C^\bullet(X;\Bbbk)}(\Bbbk, s^{n+k-1}C^\bullet(X;\Bbbk)).$$

In particular, we conclude that

$$\dim \operatorname{Ext}_{C^\bullet(X;\Bbbk)}(\Bbbk, C^\bullet(X;\Bbbk)) = 1.$$

Based on the definition of Gorenstein rings, we define the following. A Gorenstein space at $\Bbbk$ is a simply connected space X with

$$\dim \operatorname{Ext}_{C^\bullet(X;\Bbbk)}(\Bbbk, C^\bullet(X;\Bbbk)) = 1.$$