8th Feb.

(I) M-cellular approximations

R ring (or a dg ring / ring spectrum)

M an R-complex (or a dg R-module / ...)

For $X \in D(R)$, there is a natural exact Δ^{fc}:

$$Cell_M(X) \longrightarrow X \longrightarrow L_M(X) \longrightarrow$$

where ① $Cell_M(X)$ is built out of M
i.e. is in $Loc_R(M)$

② $Cell_M(X) \to X$ is an M-equivalence

i.e. $\mathrm{Hom}^*_D(M, Cell_M(X)) \xrightarrow{\simeq} \mathrm{Hom}^*_D(M, X)$

Equivalently $\mathrm{Hom}_D(-, Cell_M(X)) \xrightarrow{\simeq} \mathrm{Hom}_D(-, X)$
on $Loc(M)$

Equivalently $\mathrm{Hom}^*_D(M, L_M(X)) = 0$

Morita theory: R, S rings ${}_{R,S}M$ bimodule.

$$\operatorname{Hom}_R(M,-) : \operatorname{Mod} R \rightleftarrows \operatorname{Mod} S^{\mathrm{op}} : - \otimes_S M$$

$$\begin{array}{ccc} \cup & & \cup \\ \operatorname{add} M & \equiv & \operatorname{add} S^{\mathrm{op}} = \operatorname{proj} S^{\mathrm{op}} \end{array}$$

This and its variants: Switching roles of Hom and $\otimes$, or changing variance of Hom,

appear all over mathematics: Koszul duality, Fourier-Mukai transforms, RHT, ...

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R ring, M an R-complex

$$E := \operatorname{RHom}_R(M,M) = \operatorname{Hom}_R(pM, pM)$$

where $pM \xrightarrow{\simeq} M$ is a cofibrant replacement (i.e. semi-projective resln.)

- View E as a dg algebra.

- product on E = composition of maps.

M is then a R, E bimodule, with E-acting by evaluation:

$$\underset{R}{\mathrm{Hom}}(pM, pM) \otimes pM \longrightarrow pM$$
$$\alpha \otimes x \longrightarrow \alpha(x)$$

Then we get an adjoint pair

$$R\underset{R}{\mathrm{Hom}}(M, -) : D(R) \rightleftarrows D(E^{op}) : - \overset{L}{\underset{E}{\otimes}} M$$

Thm (Dwyer, Greenlees) when M is small,

these restrict to an equivalence:

$$\mathrm{Loc}_R(M) \equiv D(E^{op})$$

And the counit map

$$R\underset{R}{\mathrm{Hom}}(M, X) \overset{L}{\underset{E}{\otimes}} M \longrightarrow X \qquad X \in D(R)$$

is the M-cellular approximation:

$$Cell_M(X) = \underset{R}{RHom}(M, X) \overset{L}{\underset{E}{\otimes}} M$$

- This a "concrete" model for cellular approx.

Sketch of proof: Any $Y \in D(E^{op})$ is in $Loc(E)$

$$\therefore \quad Y \overset{L}{\underset{E}{\otimes}} M \in \underset{R}{Loc}(E \overset{L}{\underset{E}{\otimes}} M) = Loc(M)$$

So we do have that the functors restrict to:

$$\underset{R}{RHom}(M, -) : \underset{R}{Loc}(M) \rightleftarrows D(E^{op}) : - \overset{L}{\underset{E}{\otimes}} M$$

To check those are equivalences, we have to check that the natural maps:

$$\varepsilon_X \quad \underset{R}{RHom}(M, X) \overset{L}{\underset{E}{\otimes}} M \longrightarrow X \quad \text{and}$$

$$\eta_Y : \quad Y \longrightarrow \underset{R}{RHom}(M, Y \overset{L}{\underset{E}{\otimes}} M)$$

are isomorphisms $\forall\ X \in \mathrm{Loc}(M)$ and $Y \in D(E^{op})$

- ξ_X is an iso. for $X = M$:

$$\mathrm{RHom}_R(M, M) \overset{L}{\otimes}_E M \xrightarrow{\simeq} M$$

- The subcategory $\mathcal{L} = \{X \in D(R) \mid \xi_X \text{ an iso.}\}$ is localizing; i.e. a Δ^{loc} subcategory closed under all coproducts.

. Main point is to check this is closed under arbitrary sums.

. This is where we need that M is small.

$\Rightarrow \quad \mathrm{Loc}(M) \subseteq \mathcal{L}$ as desired.

Finally one verifies η_Y is an iso. on $D(E^{op})$

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A strengthening: Observe $\mathrm{Cell}_M(-)$ only depends on $\mathrm{Loc}(M)$ not M itself.

We say M is proxy-small if $\exists\ W \in D(R)$ s.t.

1. W is small
2. $\mathrm{Loc}(M) = \mathrm{Loc}(W)$
3. $W \in \mathrm{Thick}(M)$ i.e. W is finitely built out of M

Then again $\mathrm{Cell}_M(X) = \mathrm{RHom}_R(M, X) \overset{L}{\otimes}_{R\mathrm{End}(M)} M$

$\cong R$

$\mathrm{RHom}_R(M, M)$.

- This turns out to be rather useful.

(II) Commutative noetherian rings

R one such; $I \subsetneq R$ ideal

$V := V(I) \subseteq \operatorname{Spec} R$ closed subset.
(with Zariski top.)

For $X \in \operatorname{Mod} R$

$$\Gamma_I X := \Gamma_V(X^{\sim}) \leftarrow \text{sections of } X^{\sim} \text{ supported on } V$$

I-power torsion submodule of X

$$= \{x \in X \mid x = 0 \text{ in } R_p \quad p \notin V\}$$

$R \to R_p$

$$= \ker\left(X \to \prod_{p \notin V} X_p\right) \quad \text{i.e. } p \not\supseteq I$$

$$= \{x \in X \mid I^n x = 0 \text{ for some } n \geq 1\}$$

For $X \in D(R)$, let

$R\Gamma_I(X) :=$ (right) derived functor of Γ_I evaluated on X

$$= \Gamma_I(iX) \qquad X \to iX \text{ "injective resln"}$$

$H^i_I(X) := H^i(\Gamma_I(iX))$

– local cohomology of X supported on I.

There is a natural map

$$R\Gamma_I(X) \longrightarrow X$$

the

Fact: This is R_I - cellular approx. of X

$$\mathrm{Cell}_{R_I}(X) = R\Gamma_I(X)$$

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Would like good models for $\mathrm{Cell}_{R_I}(X)$.

*Note: R_I is typically not small, or even proxy-small.

Koszul complexes: $r \in R$

$$K(r) := \mathrm{Cone}\left(R \xrightarrow{r} R\right) = 0 \to R \xrightarrow{r} R \to 0$$

(degree 1, deg 0)

Given $\underline{r} := r_1, \dots, r_n$ in R

$$K(\underline{r}) = \bigotimes_i K(r_i)$$

Choose $\underline{x}$ l.f. $I = (\underline{x})$. Then

$K := K(\underline{x})$ is such that

① K is small (i.e. a perfect (x.)

② $\mathrm{Tor}(K) = \mathrm{Tor}(R/I)$

Thus $\mathrm{Cell}_{R/I}(-) = \mathrm{Cell}_{K}(-) = \mathrm{RHom}_{R}(K,-) \overset{\mathbb{L}}{\otimes}_{E} K$

$E = \mathrm{End}_{R}(K)$

Finite Matrix ring / R

+ differential

Thus

$$R\Gamma_I(-) := \mathrm{RHom}_{R}(K,-) \overset{\mathbb{L}}{\otimes}_{E} K$$

Ex: $\mathfrak{p} \in \mathrm{Spec}\, R$ s.t. $R/\mathfrak{p}$ is regular (as a ring)

i.e. $\mathrm{gldim}(R/\mathfrak{p}) < \infty$.

Then R/p is proxy-small in $D(R)$.

- In fact $K(\underline{r})$ where $p = (\underline{r})$ is a small proxy for R/p : i.e. $K(r) \in \mathrm{Thick}(R/p)$ in $D(R)$.

This applies whenever $m \in \mathrm{Max}(R)$ ← maximal ideal.

- Thus $k := R/m$ is proxy-small.

$$R\Gamma_m(-) = \mathrm{RHom}_R(k, -) \overset{L}{\otimes}_{\mathrm{REnd}_R(k)} k$$

- Special case: (R, m, k) local ring, $k = R/m$.
  ↑ maximal ideal

Note: $H^\bullet(\mathrm{REnd}_R(k)) = \mathrm{Ext}^\bullet_R(k, k)$ ← graded ring.

$$R\Gamma_m(R) = \underbrace{R/\mathrm{Hom}_R(k,R)}_{R\mathrm{End}_R(k)} \otimes^{\mathbb{L}} k$$

its cohomology is $\mathrm{Ext}^*_R(k,R)$

Defn: R Gorenstein if $\mathrm{Ext}^*_R(k,R) \cong \Sigma^d k$.

Proxy-smallness: R ring (comm. noeth.)

- Interesting to know when R/I proxy-small $\forall\ I \subseteq R$.

Thm: R is locally complete intersection $\iff$ each $X \in D^b(\mathrm{mod}\,R)$ is proxy-small.

(DGI, Pollitz, Letz)

Compare: R regular $\iff$ each $X \in D^b(\mathrm{mod}\,R)$ is small.

(Auslander, Buchsbaum, Serre)

Back to R ring, M dg R-complex.

$$D(R) \underset{\mathrm{RHom}_R(M,-)}{\overset{-\otimes^{\mathbb{L}}_E M}{\rightleftarrows}} D(R\mathrm{End}_R(M)^{op})$$

M small

$$\mathrm{RHom}_R(M^{\#},-)$$

$$M^{\#} = \mathrm{RHom}_R(M,R) \quad \text{right dg } E\text{-module.}$$

The $\forall\ X \in D(R)$

$$X \longrightarrow \mathrm{RHom}_R(M^{\#}, \mathrm{RHom}_R(M,X))$$

is "derived completion" in local algebra.

DG:

$$D(R) \underset{\mathrm{RHom}_R(M,-)}{\overset{-\otimes^{\mathbb{L}}_E M}{\rightleftarrows}} D(E^{op}) \underset{\mathrm{RHom}_E(M^{\#},-)}{\overset{\mathrm{RHom}_R(M,-)}{\rightleftarrows}} D(R)$$

$\cup$ | | |

$(M\text{-cellular}) \equiv D(E^{op}) \equiv D(M\text{-complete})$