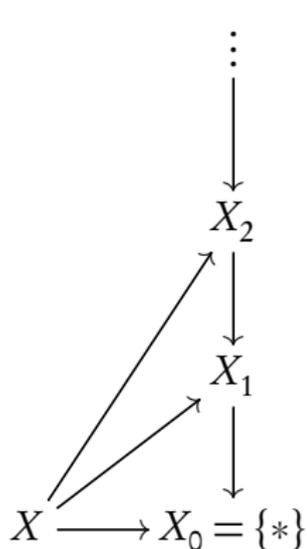# Dualities in Topology & Algebra

ICTS (1-14th February, 2021)

Rational Homotopy Groups

POSTNIKOV TOWER

A Postnikov tower for a (connected) CW complex is the following

- $\pi_i(X_n) = 0$ for $i > n$
- $X \to X_n$ induces isomorphism

$$\pi_i(X) \xrightarrow{\cong} \pi_i(X_n),\ i \le n$$

- $X_n \to X_{n-1}$ is a fibration, with fibre $K(\pi_n(X), n)$

If $X = K(A, n)$, then $X_i \simeq *,\ i < n$ and $X_i \simeq K(A, n),\ i \ge n$.

$$\vdots$$

$$X_2$$

$$X_1$$

$$X \longrightarrow X_0 = \{*\}$$

If X is simply connected, then $X_n \to X_{n-1}$ is a principal $K(\pi_n(X), n)$-fibration.

A space X is a *twisted* product of $K(\pi_n(X), n)$'s. It is known that

$$H^\bullet(K(\mathbb{Z}, n); \mathbb{Q}) = \left\{ \begin{array}{rl} \mathbb{Q}[\alpha] & \text{if } n \text{ is even and } |\alpha| = n \\ \mathbb{Q}[\alpha]/(\alpha^2) & \text{if } n \text{ is odd and } |\alpha| = n \end{array} \right.$$

This also agrees with $H^\bullet(K(\mathbb{Q}, n); \mathbb{Z})$. Thus, $K(\mathbb{Q}, n)$ is the rationalization of $K(\mathbb{Z}, n)$.

- Odd spheres: $\pi_k(S^n) \otimes \mathbb{Q} \cong \mathbb{Q}$ if and only if $k = n$ (Serre). Thus,

$$S^n \hookrightarrow K(\mathbb{Z}, n) \to K(\mathbb{Q}, n)$$

is a rationalization of S^n.

- Even spheres: $\pi_k(S^n) \otimes \mathbb{Q} \cong \mathbb{Q}$ if and only if $k = n, 2n - 1$ (Serre). Thus, $S^n_{\mathbb{Q}}$ is a twisted product of $K(\mathbb{Q}, n)$ and $K(\mathbb{Q}, 2n - 1)$.

PRINCIPAL FIBRATIONS

Let $E \to B$ be a principal $K(\pi, n)$-fibration. Serre spectral sequence with $\mathbb{Q}$-coefficients imply

$$E_2^{p,q} = H^p(B; H^q(K(\pi, n); \mathbb{Q})).$$

The structure is determined by the first non-zero differential

$$d_{n+1} : \mathrm{Hom}(\pi, \mathbb{Q}) = H^n(K(\pi, n); \mathbb{Q}) \to H^{n+1}(B; \mathbb{Q}).$$

Thus, d_{n+1} determines an element $[d_{n+1}] \in H^{n+1}(B; \mathbb{Q}) \otimes_{\mathbb{Z}} \pi$.

- $H^{n+1}(B; \mathbb{Q}) \otimes_{\mathbb{Z}} \pi \cong H^{n+1}(B; \mathbb{Q} \otimes_{\mathbb{Z}} \pi)$
- If $\dim_{\mathbb{Q}} \pi < \infty$, then $\mathbb{Q} \otimes_{\mathbb{Z}} \pi = \pi$

Such a fibration is determined by π and $[d_{n+1}] \in H^{n+1}(B; \pi)$.

- If $E = X_3$ in the Postnikov tower of S^2, then the fibration is determined by a generator of $H^4(\mathbb{CP}^\infty; \mathbb{Z})$.

HIRSCH EXTENSIONS

Given π such that $\dim_{\mathbb{Q}} \pi < \infty$ and $[\alpha] \in H^{n+1}(B; \pi)$ we construct a Hirsch extension of $A_{PL}(B)$ as follows:

$$\left(A_{PL}(B) \otimes \Lambda(\pi^*), d \right), \ d|_{A_{PL}(B)} \equiv d$$

and π^* is placed in degree n. Moreover,

$$d : \pi^* \rightarrow A_{PL}^{n+1}(B)$$

has image in closed forms and induces $[\alpha]$.

● Two Hirsch extensions associated with maps to closed forms, representing the same element in cohomology, are isomorphic.

● There is a bijection between principal fibrations over B with fiber $K(\pi, n)$ and Hirsch extensions of $A_{PL}(B)$ with new generators in degree n.

Given a principal fibration $K(\pi, n) \hookrightarrow E \xrightarrow{f} B$, consider the data:

- π is an abelian group such that $V = \pi \otimes_{\mathbb{Z}} \mathbb{Q}$ is f.d.
- $\rho_B : \mathfrak{M}_B \to A_{PL}(B)$ is a minimal model
- $\xi \in H^{n+1}(B; \pi)$ is the invariant of this fibration

Theorem *Let $\mathfrak{M}_E = (\mathfrak{M}_B \otimes \Lambda(V^*), d)$ be a Hirsch extension corresponding to the data above. There is a map $\rho_E : \mathfrak{M}_E \to A_{PL}(E)$ such that*

(1) $\mathfrak{M}_E$ is a minimal model for E via ρ_E.

(2) There is a commutative diagram

$$
\begin{array}{ccc}
\mathfrak{M}_B & \xrightarrow{\rho_B} & A_{PL}(B) \\
\downarrow & & \downarrow{f^*} \\
\mathfrak{M}_E & \xrightarrow{\rho_E} & A_{PL}(E)
\end{array}
$$

- If $\mathfrak{M}_B = (\Lambda V, d)$ is a minimal model for B, where $\pi_1(B) = 0$ and $H_j(B; \mathbb{Q})$ is finite dimensional for each j, then V and $\pi_\bullet(B) \otimes_{\mathbb{Z}} \mathbb{Q}$ are dual to each other.

- In a minimal cdga $(\Lambda V, d)$, the differential is determined by $d : V \to \Lambda V$. We write $d = d_1 + d_2 + \cdots$, where

$$d_i : V \to \Lambda V = \Lambda^2 V \oplus \Lambda^3 V \oplus \cdots \to \Lambda^{i+1} V.$$

For degree reasons, $d^2 = 0$ splits into

$$d_1^2 = 0, \quad d_1 d_2 + d_2 d_1 = 0, \quad d_2^2 + d_1 d_3 + d_3 d_1 = 0, \quad \cdots$$

Thus, $(\Lambda V, d_1)$ is another minimal cdga. Dualizing $d_1 : V \to \Lambda^2 V$, we obtain

$$d_1^* : V^* \otimes V^* \to V^*.$$

The Whitehead bracket of $\alpha \in \pi_k(X)$ and $\beta \in \pi_l(X)$

$$[\alpha, \beta]: S^{k+l-1} \xrightarrow{u} S^k \vee S^l \xrightarrow{\alpha \vee \beta} X$$

where u is the attaching map of D^{k+l} to create $S^k \times S^l$.

- If $k = l = 1$, then $[\,,\,]$ is the commutator.
- If $k = 1$, then this is related to $\pi_1(X)$-action on $\pi_l(X)$.
- Commutativity: $[\alpha, \beta] = (-1)^{kl}[\beta, \alpha]$ if $k, l \geq 2$.
- Jacobi identity: If $\gamma \in \pi_m(X)$ and $k, l, m \geq 2$, then

$$(-1)^{km}[\alpha, [\beta, \gamma]] + (-1)^{lk}[\beta, [\gamma, \alpha]] + (-1)^{ml}[\gamma, [\alpha, \beta]] = 0.$$

- $[\alpha, \beta] = 0$ if and only if it is a restriction of a map $S^k \times S^l \to X$.

RATIONAL LIE ALGEBRA

The rational homotopy groups $\pi_k(X) \otimes \mathbb{Q}$, placed in degree $k-1$, form a Lie algebra over $\mathbb{Q}$. Transferring the Whitehead bracket via the identification $\pi_k(\Omega X) \cong \pi_{k+1}(X)$, we obtain a Lie algebra $\mathscr{L}_X = \pi_{\bullet}(\Omega X) \otimes \mathbb{Q}$.

- $[id, id] = 2\eta$, where $id : S^2 \to S^2$ and $\eta : S^3 \to S^2$ is the Hopf map. In general, $[id, id]$ generates a $\mathbb{Z}$ if $id : S^{2k} \to S^{2k}$.

- If $\alpha : S^2 \hookrightarrow \mathbb{CP}^n$ generates π_2, then for $n \geq 2$, $[\alpha, \alpha] = 0$ since $\pi_3(\mathbb{CP}^n) = 0$.

- Let G be a Lie group. As $\pi_1(G)$ is abelian, the commutator is zero. More generally, given $\alpha : S^k \to G$, $\beta : S^l \to G$ form the product

$$\alpha \times \beta : S^k \times S^l \longrightarrow G \times G \xrightarrow{m} G.$$

This implies that $[\,,\,] \equiv 0$.

EXAMPLES

- Spheres: Choose $\omega \in A_{PL}(S^3)$ such that $[\omega]$ generates $H^3(S^3; \mathbb{Q})$. Define the map

$$\varphi : (\Lambda(x), 0) \to \big(A_{PL}(S^3), d\big), \ x \mapsto \omega.$$

This is quasi-isomorphism and $(\Lambda(x), 0)$ is the minimal model for S^3. For S^2 consider

$$\varphi : (\Lambda(x, y), dx = 0, dy = x^2) \to \big(A_{PL}(S^2), d\big), \ x \mapsto \omega, \ y \mapsto \eta$$

where $[\omega]$ generates $H^2(S^2; \mathbb{Q})$ and $d\eta = \omega^2$.

- Lie groups: Let G be a compact connected Lie group. By Hopf's work, $H^\bullet(G; \mathbb{R})$ is a free algebra generated by odd generators $x_1, \ldots, x_k$. Thus, the minimal model looks like $(\Lambda V^{\mathrm{odd}}, 0)$. For instance, the minimal model for $U(n)$ and $SU(n)$ are given by

$$(\Lambda(x_1, x_3, \ldots, x_{2n-1}), 0), \ (\Lambda(x_3, \ldots, x_{2n-1}), 0).$$