RANDOM MATRIX PRODUCTS + QUANTUM SIMULATION

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IN MEMORIAM: ELIZABETH MECKES

"The Random Matrix Theory of the Classical Compact Groups"

CAMBRIDGE 2019.

COLLABORATORS:

arXiv 2003.05437
with HUANG De, Niles-Weed, Ward

arXiv 2008.11751
with CHEN Chi-Fan, Kueng, HUANG Hsin-Yuan

AGENDA:

1. Products of random matrices
2. Products, martingales and geometry
3. Application: Quantum simulation.

① PRODUCTS OF RANDOM MATRICES

- Let $Y_j \in \mathbb{C}^{d \times d}$ indep, rdm, bdd.

  Often: $Y_j = I + X_j$

- Form the product

$$Z_n = Y_n Y_{n-1} \cdots Y_1 \in \mathbb{C}^{d \times d}$$

- QUESTIONS: For a norm $|||\cdot|||$ on matrices,

  GROWTH: $\mathbb{E}\, |||Z_n||| = \ldots$

  CONCENTRATION: $\mathbb{E}\, |||Z_n - \mathbb{E} Z_n||| \le \ldots$

  Keep in mind: $\mathbb{E} Z_n = (\mathbb{E} Y_n)(\mathbb{E} Y_{n-1}) \cdots (\mathbb{E} Y_1)$

ORIGINS:

- Stochastic dynamical systems (ergodic theory, control, ...)
  - ↳ dimension d fixed, factors $n \longrightarrow \infty$
- Random matrix theory (HD stat, free prob.)
  - ↳ factors n fixed, dimension $d \to \infty$.
- Recent interest: Iterative randomized algorithms
  - EXAMPLES: SGD, Oja's, R. Kaczmarz
  - Dimensions d, n large but not growing
  - Nonasymptotic analysis! Convergence rates

TODAY: Simple geometric perspective

② PRODUCTS AND MARTINGALES

- As above, $Z_n = Y_n \cdots Y_1$ for $Y_j \in \mathbb{C}^{d \times d}$ indep.
- Concentration.

$$Z_n - \mathbb{E} Z_n = Y_n \cdots Y_1 - (\mathbb{E} Y_n) \cdots (\mathbb{E} Y_1)$$

$$= \sum_{j=1}^{n} (\mathbb{E} Y_n) \cdots (\mathbb{E} Y_{j+1}) (Y_j - \mathbb{E} Y_j) \times (Y_{j-1} \cdots Y_1)$$

$$=: \sum_{j=1}^{n} \Delta_j$$

- By indep., cond. on $Y_1, \ldots, Y_{j-1}$, Δ_j is zero-mean.
- Δ_j is a matrix martingale sequence.

- Let $\|\cdot\|_p$ be the Schatten p-norm
- In particular, $\|\cdot\|_2$ = Frobenius ; $\|\cdot\|_\infty$ = operator.

$$\mathbb{E}\|Z_n - \mathbb{E}Z_n\|_2^2 = \mathbb{E}\Big\|\sum_{j=1}^n \Delta_j\Big\|_2^2$$

$$= \sum_{j=1}^n \mathbb{E}\|\Delta_j\|_2^2 \qquad \text{orthog.}$$

$$\le \sum_{j=1}^n \|\mathbb{E}\, Y_n \cdots Y_{j+1}\|_\infty^2 \cdot \mathbb{E}\|Y_j - \mathbb{E}Y_j\|_2^2 \cdot \mathbb{E}\|Y_{j-1}\cdots Y_1\|_\infty^2$$

$$= \sum_{j=1}^n (\text{predicted growth}) \times (\text{deviation } j\text{th term}) \times \text{expectation of } \|Z_{j-1}\|_\infty^2 .$$

- For many applications, need spectral norm

$\rightarrow$ Need another idea...

- IDEA: Uniform smoothness. For $2 \leq p < \infty$,

$$\mathbb{E} \left\| \sum_{j=1}^{n} \Delta_j \right\|_p^2 \leq C_p \cdot \sum_{j=1}^{n} \mathbb{E} \|\Delta_j\|_p^2$$

where $C_p = p-1$.

- Approximate the spectral norm by Schatten p for $p \doteq \log d$

$$\|A\|_\infty \leq \|A\|_p \leq d^{1/p} \|A\|_\infty$$

For $p = \log d$, $\qquad \leq e \|A\|_\infty$.

- Here's the argument: $\quad p = \log d$

$$\mathbb{E}\, \| Z_n - \mathbb{E} Z_n \|_\infty^2 \leq \mathbb{E}\, \| Z_n - \mathbb{E} Z_n \|_p^2$$
$$= \mathbb{E}\, \Big\| \sum_{j=1}^n \Delta_j \Big\|_p^2$$
$$\leq C_p \sum_{j=1}^n \mathbb{E}\, \| \Delta_j \|_p^2$$
$$\leq e^2 C_p \sum_{j=1}^n \mathbb{E}\, \| \Delta_j \|_\infty^2$$
$$\leq \text{Const} \cdot p \cdot \sum_{j=1}^n \| \mathbb{E}\, Y_n \cdots Y_{j+1} \|_\infty^2 \times \mathbb{E}\, \| Y_j - \mathbb{E} Y_j \|_\infty^2 \times \mathbb{E}\, \| Y_{j-1} \cdots Y_1 \|_\infty^2$$

- EXAMPLE: $\quad Y_j = I + X_j \qquad$ perturbation

$$Y_j - \mathbb{E} Y_j = X_j - \mathbb{E} X_j \qquad \text{small!}$$

③ QUANTUM SIMULATION

- Let $\psi \in \mathbb{C}^d$ be a unit vector "quantum state"
  $\hookrightarrow$ think about $d = 2^k$
- Let $\underline{H} \in \mathbb{C}^{d \times d}$ Hermitian "Hamiltonian"
- Quantum simulation problem: Compute/approximate

  $$e^{itH}\psi$$

  $t \in \mathbb{R}_+$ positive time
  i = imaginary unit
- Set $t=1$.
- In many situations, Hamiltonian has structure

  $$H = \sum_{l=1}^{L} A_l$$

  $A_l \in \mathbb{C}^{d \times d}$ Hermitian
  sparse / "easy" to implement on quantum computer

  L very large.

- $\lambda = \sum_{l=1}^{L} \|A_l\|_\infty$ "interaction strength"

- <u>IDEA</u>: Product formulas (Lie–Trotter–Suzuki)

$$e^{iH} \approx e^{iA_{j(n)}/n} \cdots e^{iA_{j(1)}/n}$$

$$=: Z_n$$

GOAL: $\|e^{iH} - Z_n\|_\infty \leq \varepsilon$.

PROBLEM: Need $n = O(L\lambda/\varepsilon)$

$\longrightarrow$ which has bad dependence on L, the # of summands.

- <u>IDEA</u>: Random products! (Campbell)

IDEA: Random product

$$e^{iH} \approx e^{iX_n/n} \cdots e^{iX_1/n} =: Z_n$$

where X_i are random "simple"

Each $X_i = \text{const}_\ell \cdot A_\ell$. random!

QUESTION: To get $\|e^{iH} - Z_n\|_\infty \leq \varepsilon$,

what number n of random factors

do we need?

ANSWER: $O(\log d \cdot \lambda^2/\varepsilon^2)$

Replace $L \longrightarrow \log d \approx k$

qDRIFT protocol: For $H = \sum_{\ell=1}^{L} A_\ell$. $\lambda = \sum \|A_\ell\|_\infty$.

$$X = \frac{\lambda}{\|A_\ell\|_\infty} A_\ell \quad \text{w.p. } \|A_\ell\|_\infty / \lambda$$

So $\mathbb{E}X = H$ and $\|X\|_\infty \leq \lambda$.

Let $V = e^{iX/n}$.

<u>Key observation</u>: $\|V - \mathbb{E}V\|_\infty \leq \frac{2\lambda}{n}$.

$\|\mathbb{E}V - e^{iH/n}\| \leq \cdots$.

Form product $Z_n = V_n \cdots V_1$

where V_j are iid copies V.

Q?? $\|e^{iH} - Z_n\|_\infty \leq$? as fn of n.

$$\text{error} = \| e^{+iH} - \mathbb{E} Z_n \|_\infty \le \lambda^2/n$$

$$+ \| \mathbb{E} Z_n - Z_n \|_\infty$$

For 2° term, we use the random product tools

$$\le (\log d) \sum_{j=1}^{n} \mathbb{E} \|\Delta_j\|_\infty^2 \qquad 1$$

$$= (\log d) \sum_{j=1}^{n} \|\mathbb{E} V_n \cdots V_{j+1}\|_\infty^2 \, \mathbb{E} \|V_j - \mathbb{E} V_j\|_\infty^2 \qquad 1$$

$$\times \, \mathbb{E} \|V_{j-1} \cdots V_1\|_\infty^2$$

$$\le (\log d) \sum_{j=1}^{n} \frac{\lambda^2}{n^2} = (\log d) \cdot \frac{\lambda^2}{n}.$$

COMBINE: $\text{Error} \le \text{const} \sqrt{\frac{\log d \cdot \lambda^2}{n}}$